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An Orlicz extension of difference sequences on real linear n-normed spaces

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Abstract

In this paper, we present an extension of some classes of difference sequences by considering them in a base space X, a real linear n-normed space via a sequence of Orlicz functions. We investigate the spaces for linearity, existence of norms and completeness under different conditions. We also show that they are convex spaces and compute their topologically equivalent spaces. Further some results on equivalence of various norms on such extended spaces are presented.

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1 Introduction and preliminaries

Let w denote the space of all real or complex sequences. By c, c_0 and ℓ_∞, we denote the Banach spaces of convergent, null and bounded sequences $x = (x_k)$, respectively, normed by

$$\|x\| = \sup_k |x_k|.$$

An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Lindenstrauss and Tzafriri [1] used the Orlicz function and introduced the sequence space ℓ_M as follows:

$$\ell_M = \left\{(x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0\right\}.$$

They proved that ℓ_M is a Banach space normed by

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1\right\}.$$

The study of Orlicz sequence spaces was initiated with a certain specific purpose in Banach space theory. Indeed, Lindberg got interested in Orlicz spaces in connection with finding Banach spaces with symmetric Schauder bases having complementary subspaces isomorphic to c_0 or ℓ_p ($1 \le p < \infty$). Subsequently, Lindenstrauss and Tzafriri [1] studied these

Orlicz sequence spaces in more detail, and solved many important and interesting structural problems in Banach spaces. Later on, different classes of sequence spaces defined by an Orlicz function were studied by different authors. For details, one may refer to Kamthan and Gupta [2].

The notion of a difference sequence space was introduced by Kizmaz [3] who studied the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Colak [4] by introducing the spaces $\ell_\infty(\Delta^s)$, $c(\Delta^s)$ and $c_0(\Delta^s)$. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [5] who studied the spaces $\ell_\infty(\Delta_m)$, $c(\Delta_m)$ and $c_0(\Delta_m)$. Tripathy *et al.* [6] generalized the above notions and unified these as follows.

Let m, s be non-negative integers. Then, for a given sequence space Z, we have

$$Z(\Delta_m^s) = \{x = (x_k) \in w : (\Delta_m^s x_k) \in Z\},$$

where $\Delta_m^s x = (\Delta_m^s x_k) = (\Delta_m^{s-1} x_k - \Delta_m^{s-1} x_{k+m})$ and $\Delta_m^0 x_k = x_k$ for all $k \in N$, which is equivalent to the following binomial representation:

$$\Delta_m^s x_k = \sum_{\upsilon=0}^{s} (-1)^\upsilon \binom{s}{\upsilon} x_{k+m\upsilon}.$$

Taking $m = 1$, we get the spaces $\ell_\infty(\Delta^s)$, $c(\Delta^s)$ and $c_0(\Delta^s)$ studied by Et and Colak [4]. Taking $s = 1$, we get the spaces $\ell_\infty(\Delta_m)$, $c(\Delta_m)$ and $c_0(\Delta_m)$ studied by Tripathy and Esi [5]. Taking $m = s = 1$, we get the spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kizmaz [3].

Let m, s be non-negative integers. Then, for a given sequence space Z, Dutta [7] introduced the following spaces:

$$Z(\Delta_{(s)}^m) = \{x = (x_k) \in w : (\Delta_{(s)}^m x_k) \in Z\},$$

where $\Delta_{(s)}^m x = (\Delta_{(s)}^m x_k) = (\Delta_{(s)}^{m-1} x_k - \Delta_{(s)}^{m-1} x_{k-s})$ and $\Delta_{(s)}^0 x_k = x_k$ for all $k \in N$, which is equivalent to the following binomial representation:

$$\Delta_{(s)}^m x_k = \sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} x_{k-s\nu}.$$

The concept of 2-normed spaces was introduced and studied by Gähler, a German Mathematician who worked at German Academy of Science, Berlin, in a series of papers in the German language published in *Mathematische Nachrichten*; see, for example, references [8–13]. This notion, which is nothing but a two-dimensional analogue of a normed space, got the attention of a wider audience after the publication of a paper by White [14] in 1969 entitled 2-*Banach spaces*. In the same year Gähler published another paper on this theme in the same journal. Siddiqi delivered a series of lectures on this theme in various conferences in India and Iran. His joint paper with Gähler and Gupta [15] of 1975 also provided valuable results related to the theme of this paper. The notion of n-normed spaces can be found in Misiak [16]. Since then, many others have studied this concept and obtained various results; see, for instance, Gunawan [17, 18], Gunawan and Mashadi [8, 19], Dutta

[20–22] and Gürdal *et al.* [23]. For some related and recent works in this area, one may refer to Chu and Ku [24] and Tanaka and Saito [25].

Let $n \in N$ and X be a real vector space of dimension d, where $n \leq d$. A real-valued function $\|\cdot,\ldots,\cdot\|$ on X^n satisfying the following four conditions:

(N1) $\|x_1,x_2,\ldots,x_n\| = 0$ if and only if $x_1,x_2,\ldots,x_n$ are linearly dependent,

(N2) $\|x_1,x_2,\ldots,x_n\|$ is invariant under permutation,

(N3) $\|\alpha x_1,x_2,\ldots,x_n\| = |\alpha|\|x_1,x_2,\ldots,x_n\|$ for any $\alpha \in \mathbb{R}$,

(N4) $\|x+x',x_2,\ldots,x_n\| \leq \|x,x_2,\ldots,x_n\| + \|x',x_2,\ldots,x_n\|$ is called an *n-norm* on X, and the pair $(X,\|\cdot,\ldots,\cdot\|)$ is called an n-normed space.

A trivial example of an n-normed space is $X = \mathbb{R}^n$ equipped with the following Euclidean n-norm:

$$\|x_1,x_2,\ldots,x_n\|_{\mathrm{E}} = \operatorname{abs}\left(\begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}\right),$$

where $x_i = (x_{i_1},\ldots,x_{i_n}) \in \mathbb{R}^n$ for each $i = 1,2,\ldots,n$.

If $(X,\|\cdot,\ldots,\cdot\|)$ is an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1,a_2,\ldots,a_n\}$ is a linearly independent set in X, then the following function $\|\cdot,\ldots,\cdot\|_\infty$ on X^{n-1} defined by

$$\|x_1,x_2,\ldots,x_{n-1}\|_\infty = \max\bigl\{\|x_1,x_2,\ldots,x_{n-1},a_i\| : i = 1,2,\ldots,n\bigr\} \tag{1.1}$$

defines an $(n-1)$-norm on X with respect to $\{a_1,a_2,\ldots,a_n\}$, and this is known as derived $(n-1)$-norm on X.

The standard n-norm on X, which is a real inner product space of dimension $d \geq n$, is given as follows:

$$\|x_1,\ldots,x_n\|_{\mathrm{S}} = \begin{vmatrix} \langle x_1,x_1\rangle & \cdots & \langle x_1,x_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n,x_1\rangle & \cdots & \langle x_n,x_n\rangle \end{vmatrix}^{\frac{1}{2}},$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on X. If $X = \mathbb{R}^n$, then this n-norm is exactly the same as the Euclidean n-norm $\|x_1,x_2,\ldots,x_n\|_{\mathrm{E}}$ mentioned earlier. For $n = 1$, this n-norm is the usual norm $\|x_1\| = \langle x_1,x_1\rangle^{\frac{1}{2}}$.

A sequence (x_k) in an n-normed space $(X,\|\cdot,\ldots,\cdot\|)$ is said to *converge* to some $L \in X$ in the n-norm if

$$\lim_{k\to\infty} \|x_k - L,u_2,\ldots,u_n\| = 0 \quad \text{for every } u_2,\ldots,u_n \in X.$$

A sequence (x_k) in an n-normed space $(X,\|\cdot,\ldots,\cdot\|)$ is said to be *Cauchy* with respect to the n-norm if

$$\lim_{k,l\to\infty} \|x_k - x_l,u_2,\ldots,u_n\| = 0 \quad \text{for every } u_2,\ldots,u_n \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be an n-Banach space.

Now we state the following useful results on the n-norm as lemmas which were given in [26].

Lemma 1.1 *Every n-normed space is an $(n-r)$-normed space for all $r = 1,2,\dots,n-1$. In particular, every n-normed space is a normed space.*

Lemma 1.2 *A standard n-normed space is complete if and only if it is complete with respect to the usual norm $\|\cdot\|_S = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$.*

Lemma 1.3 *On a standard n-normed space X, the derived $(n-1)$-norm $\|\cdot,\dots,\cdot\|_\infty$, defined with respect to an orthonormal set $\{e_1,e_2,\dots,e_n\}$, is equivalent to the standard $(n-1)$-norm $\|\cdot,\dots,\cdot\|_S$. Precisely, we have*

$$\|x_1,\dots,x_{n-1}\|_\infty \le \|x_1,\dots,x_{n-1}\|_S \le \sqrt{n}\|x_1,\dots,x_{n-1}\|_\infty$$

for all $x_1,\dots,x_{n-1}$, where $\|x_1,\dots,x_{n-1}\|_\infty = \max\{\|x_1,\dots,x_{n-1},e_i\|_S : i = 1,\dots,n\}$.

Let $(X,\|\cdot,\dots,\cdot\|_X)$ be a real linear n-normed space and let $w(X)$ denote an X-valued sequence space. Then, for a sequence of Orlicz functions $\mathbf{M} = (M_k)$, we define the following difference sequence spaces:

$$c_0\bigl(X,\mathbf{M},\Delta^m_{(s)}\bigr) = \Biggl\{(x_k)\in w(X) : \lim_{k\to\infty} M_k\biggl(\biggl\|\frac{\Delta^m_{(s)}x_k}{\rho},z_1,\dots,z_{n-1}\biggr\|_X\biggr) = 0 \text{ for every non-zero } z_1,\dots,z_{n-1}\in X \text{ and for some } \rho>0\Biggr\},$$

$$c\bigl(X,\mathbf{M},\Delta^m_{(s)}\bigr) = \Biggl\{(x_k)\in w(X) : \lim_{k\to\infty} M_k\biggl(\biggl\|\frac{\Delta^m_{(s)}x_k}{\rho}-L,z_1,\dots,z_{n-1}\biggr\|_X\biggr) = 0 \text{ for every non-zero } z_1,\dots,z_{n-1}\in X \text{ and for some } L\in X, \rho>0\Biggr\},$$

$$\ell_\infty\bigl(X,\mathbf{M},\Delta^m_{(s)}\bigr) = \Biggl\{(x_k)\in w(X) : \sup_k M_k\biggl(\biggl\|\frac{\Delta^m_{(s)}x_k}{\rho},z_1,\dots,z_{n-1}\biggr\|_X\biggr) < \infty \text{ for every non-zero } z_1,\dots,z_{n-1}\in X \text{ and for some } \rho>0\Biggr\}.$$

Similarly, we can define $c_0(X,\mathbf{M},\Delta^m_s)$, $c(X,\mathbf{M},\Delta^m_s)$ and $\ell_\infty(X,\mathbf{M},\Delta^m_s)$.

In the above definition of spaces, the n-norm $\|\cdot,\dots,\cdot\|_X$ on X is either a standard n-norm or a non-standard n-norm. In general, we write $\|\cdot,\dots,\cdot\|_X$, and for a standard case, we write $\|\cdot,\dots,\cdot\|_S$. For a derived norm, we use $\|\cdot,\dots,\cdot\|_\infty$.

It is obvious that $c_0(X,\mathbf{M},\Delta^m_{(s)}) \subset c(X,\mathbf{M},\Delta^m_{(s)})$. Again, $c(X,\mathbf{M},\Delta^m_{(s)}) \subset \ell_\infty(X,\mathbf{M},\Delta^m_{(s)})$ follows from the following inequality:

$$M_k\biggl(\biggl\|\frac{\Delta^m_{(s)}x_k}{2\rho},z_1,\dots,z_{n-1}\biggr\|_X\biggr) \le \frac{1}{2}M_k\biggl(\biggl\|\frac{\Delta^m_{(s)}x_k-L}{\rho},z_1,\dots,z_{n-1}\biggr\|_X\biggr) + \frac{1}{2}M_k\biggl(\biggl\|\frac{L}{\rho},z_1,\dots,z_{n-1}\biggr\|_X\biggr).$$

Similarly, we have $c_0(X,\mathbf{M},\Delta_s^m) \subset c(X,\mathbf{M},\Delta_s^m) \subset \ell_\infty(X,\mathbf{M},\Delta_s^m)$. Also, it is obvious that for $Z = c, c_0$ and ℓ_∞, $Z(X,\mathbf{M},\Delta_s^i) \subset Z(X,\mathbf{M},\Delta_s^m)$, $i = 0,1,\ldots,m-1$ and $Z(X,\mathbf{M},\Delta_{(s)}^i) \subset Z(X,\mathbf{M},\Delta_{(s)}^m)$, $i = 0,1,\ldots,m-1$.

When $X = \mathbb{R}$, $m = 1$, $s = 0$ and $M_k(x) = |x|$ for all $x \in [0,\infty)$ and $k \geq 1$, the above spaces deduce to the famous and very useful spaces c, c_0 and ℓ_∞.

2 Main results

In this section we investigate some results on the n-norm as well as the main results of this article involving the sequence spaces $c_0(X,\mathbf{M},\Delta_{(s)}^m)$, $c(X,\mathbf{M},\Delta_{(s)}^m)$, $\ell_\infty(X,\mathbf{M},\Delta_{(s)}^m)$, $c_0(X,\mathbf{M},\Delta_s^m)$, $c(X,\mathbf{M},\Delta_s^m)$ and $\ell_\infty(X,\mathbf{M},\Delta_s^m)$.

The proofs of the following two propositions are easy and so they are omitted.

Proposition 2.1 *Let $n \in N$ and X be a real vector space of dimension d, where $2 \leq n \leq d$. Let β_{n-1} be the collection of linearly independent sets B with $n-1$ elements. For $B \in \beta_{n-1}$, let us define*

$$p_B(x_1) = \|x_1, x_2, \ldots, x_n\|, \quad x_1 \in X.$$

Then p_B is a seminorm on X and the family $P = \{p_B : B \in \beta_{n-1}\}$ of seminorms generates a locally convex topology on X.

Proposition 2.2 *The seminorms p_B's have the following properties*:

(i) $\ker(p_B)$ = *linear span of B*,

(ii) *for $B \in \beta_{n-1}$, $y \in B$ and $x \in X \backslash$linear span of B, we have*

$$p_{B\cup\{x\}\backslash y}(y) = p_B(x).$$

Hence we have the following proposition.

Proposition 2.3 *The seminorms defined by Proposition 2.1 satisfy the axiom of an n-norm.*

Example 2.1 Consider the linear space P_m of real polynomials of degree $\leq m$ on the interval $[0,1]$. Let $\{x_i\}_{i=0}^{nm}$ be $nm+1$ arbitrary but distinct fixed points in $[0,1]$. For $f_1, f_2, \ldots, f_n$ in P_m, let us define

$$\|f_1, f_2, \ldots, f_n\| = \begin{cases} 0, & \text{if } f_1, f_2, \ldots, f_n \text{ are linearly dependent,} \\ \sum_{i=0}^{nm} |f_1(x_i) f_2(x_i) \cdots f_n(x_i)|, & \text{if } f_1, f_2, \ldots, f_n \text{ are linearly independent.} \end{cases}$$

Then $\|\cdot,\ldots,\cdot\|$ is an n-norm on P_m.

Proof The proof is a routine verification and so it is omitted. □

Theorem 2.1 *The spaces $c_0(X,\mathbf{M},\Delta_{(s)}^m)$, $c(X,\mathbf{M},\Delta_{(s)}^m)$, $\ell_\infty(X,\mathbf{M},\Delta_{(s)}^m)$, $c_0(X,\mathbf{M},\Delta_s^m)$, $c(X,\mathbf{M},\Delta_s^m)$ and $\ell_\infty(X,\mathbf{M},\Delta_s^m)$ are linear.*

Proof The proof is a routine verification and thus it is omitted. □

Theorem 2.2 (i) $c_0(X,\mathbf{M},\Delta^m_{(s)})$, $c(X,\mathbf{M},\Delta^m_{(s)})$ *and* $\ell_\infty(X,\mathbf{M},\Delta^m_{(s)})$ *are normed linear spaces by*

$$\|x\|^0_{(s)} = \inf\left\{\rho > 0 : \sup_{k\geq 1, z_1,\ldots,z_{n-1}\in X} M_k\left(\left\|\frac{\Delta^m_{(s)}x_k}{\rho}, z_1,\ldots,z_{n-1}\right\|_X\right) \leq 1\right\}, \tag{2.1}$$

(ii) $c_0(X,\mathbf{M},\Delta^m_s)$, $c(X,\mathbf{M},\Delta^m_s)$ *and* $\ell_\infty(X,\mathbf{M},\Delta^m_s)$ *are normed linear spaces by*

$$\|x\|^0_s = \sum_{k=1}^{ms}\|x_k\|_\infty + \inf\left\{\rho > 0 : \sup_{k\geq 1, z_1,\ldots,z_{n-1}\in X} M_k\left(\left\|\frac{\Delta^m_s x_k}{\rho}, z_1,\ldots,z_{n-1}\right\|_X\right) \leq 1\right\}, \tag{2.2}$$

where $\|\cdot\|_\infty$ *is the derived* 1*-norm* (*norm*) *on* X.

Proof (i) If $x = \theta$, then clearly $\|x\|^0_{(s)} = 0$. Conversely, assume $\|x\|^0_{(s)} = 0$. Then using (2.1), we have

$$\inf\left\{\rho > 0 : \sup_{k\geq 1, z_1,\ldots,z_{n-1}\in X} M_k\left(\left\|\frac{\Delta^m_{(s)}x_k}{\rho}, z_1,\ldots,z_{n-1}\right\|_X\right) \leq 1\right\} = 0.$$

This implies that for a given $\varepsilon > 0$, there exists some ρ_ε $(0 < \rho_\varepsilon < \varepsilon)$ such that

$$\sup_{k\geq 1, z_1,\ldots,z_{n-1}\in X} M_k\left(\left\|\frac{\Delta^m_{(s)}x_k}{\rho_\varepsilon}, z_1,\ldots,z_{n-1}\right\|_X\right) \leq 1.$$

So,

$$M_k\left(\left\|\frac{\Delta^m_{(s)}x_k}{\rho_\varepsilon}, z_1,\ldots,z_{n-1}\right\|_X\right) \leq 1 \quad \text{for every } z_1,\ldots,z_{n-1}\in X \text{ and } k \geq 1.$$

Hence, for every $z_1,\ldots,z_{n-1}\in X$,

$$M_k\left(\left\|\frac{\Delta^m_{(s)}x_k}{\varepsilon}, z_1,\ldots,z_{n-1}\right\|_X\right) \leq M_k\left(\left\|\frac{\Delta^m_{(s)}x_k}{\rho_\varepsilon}, z_1,\ldots,z_{n-1}\right\|_X\right) \leq 1 \quad \text{for every } k \geq 1.$$

Suppose $\Delta^m_{(s)}x_{n_i} \neq 0$ for some i. Let $\varepsilon \to 0$, then $\|\frac{\Delta^m_{(s)}x_{n_i}}{\varepsilon}, z_1,\ldots,z_{n-1}\|_X \to \infty$.

It follows that $M_k(\|\frac{\Delta^m_{(s)}x_{n_i}}{\varepsilon}, z_1,\ldots,z_{n-1}\|_X) \to \infty$ as $\varepsilon \to 0$ for some $n_i \in N$. This is a contradiction.

So, we must have $\Delta^m_{(s)}x_k = 0$ for all $k \geq 1$. Let $k = 1$, then $\Delta^m_{(s)}x_1 = \sum_{\nu=0}^m (-1)^\nu \binom{m}{\nu} x_{1-s\nu} = 0$ and so $x_1 = 0$, by taking $x_{1-s\nu} = 0$, for $\nu = 1,\ldots,m$. Thus, taking $k = 2,\ldots,ms,\ldots$, we can easily conclude that $x_k = 0$ for all $k \geq 1$.

Thus $x = \theta$.

Let $x = (x_k)$ and $y = (y_k)$ be any two elements. Then there exist $\rho_1, \rho_2 > 0$ such that

$$\sup_{k\geq 1, z_1,\ldots,z_{n-1}\in X} M_k\left(\left\|\frac{\Delta^m_{(s)}x_k}{\rho_1}, z_1,\ldots,z_{n-1}\right\|_X\right) \leq 1$$

and

$$\sup_{k\geq 1, z_1,\ldots,z_{n-1}\in X} M_k\left(\left\|\frac{\Delta^m_{(s)}y_k}{\rho_2}, z_1,\ldots,z_{n-1}\right\|_X\right) \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then, by the convexity of M, we have

$$\begin{aligned}
&\sup_{k\geq 1, z_1,\dots,z_{n-1}\in X} M_k\left(\left\| \frac{\Delta^m_{(s)}(x_k + y_k)}{\rho}, z_1, \dots, z_{n-1}\right\|_X\right) \\
&\leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right) \sup_{k\geq 1, z_1,\dots,z_{n-1}\in X} M_k\left(\left\| \frac{\Delta^m_{(s)}x_k}{\rho_1}, z_1, \dots, z_{n-1}\right\|_X\right) \\
&\quad + \left(\frac{\rho_2}{\rho_1+\rho_2}\right) \sup_{k\geq 1, z_1,\dots,z_{n-1}\in X} M_k\left(\left\| \frac{\Delta^m_{(s)}y_k}{\rho_2}, z_1, \dots, z_{n-1}\right\|_X\right) \\
&\leq 1.
\end{aligned}$$

Now

$$\begin{aligned}
\|x+y\|^0_{(s)} &= \inf\left\{\rho : \sup_{k\geq 1, z_1,\dots,z_{n-1}\in X} M_k\left(\left\| \frac{\Delta^m_{(s)}(x_k + y_k)}{\rho}, z_1, \dots, z_{n-1}\right\|_X\right) \leq 1\right\} \\
&\leq \inf\left\{\rho_1 : \sup_{k\geq 1, z_1,\dots,z_{n-1}\in X} M_k\left(\left\| \frac{\Delta^m_{(s)}x_k}{\rho_1}, z_1, \dots, z_{n-1}\right\|_X\right) \leq 1\right\} \\
&\quad + \inf\left\{\rho_2 : \sup_{k\geq 1, z_1,\dots,z_{n-1}\in X} M_k\left(\left\| \frac{\Delta^m_{(s)}y_k}{\rho_2}, z_1, \dots, z_{n-1}\right\|_X\right) \leq 1\right\}.
\end{aligned}$$

Thus

$$\|x+y\|^0_{(s)} \leq \|x\|^0_{(s)} + \|y\|^0_{(s)}.$$

Finally, let α be any scalar. Then

$$\begin{aligned}
\|\alpha x\|^0_{(s)} &= \inf\left\{\rho > 0 : \sup_{k\geq 1, z_1,\dots,z_{n-1}\in X} M_k\left(\left\| \frac{\Delta^m_{(s)}\alpha x_k}{\rho}, z_1, \dots, z_{n-1}\right\|_X\right) \leq 1\right\} \\
&= \inf\left\{\left(|\alpha|\lambda\right) > 0 : \sup_{k\geq 1, z_1,\dots,z_{n-1}\in X} M_k\left(\left\| \frac{\Delta^m_{(s)}x_k}{\lambda}, z_1, \dots, z_{n-1}\right\|_X\right) \leq 1\right\} \\
&= |\alpha| \|x\|^0_{(s)},
\end{aligned}$$

where $\lambda = \frac{\rho}{|\alpha|}$.

This completes the proof.

(ii) The proof follows by applying similar arguments as above. □

Remark It is obvious that $(x_k) \in Z(X, \mathbf{M}, \Delta^m_{(s)})$ if and only if $(x_k) \in Z(X, \mathbf{M}, \Delta^m_s)$ for $Z = c, c_0$ and ℓ_∞. Moreover, it is clear that the norms $\|\cdot\|^0_{(s)}$ and $\|\cdot\|^0_s$ are equivalent.

Theorem 2.3 (i) *The spaces $c_0(X, \mathbf{M}, \Delta^m_{(s)})$, $c(X, \mathbf{M}, \Delta^m_{(s)})$ and $\ell_\infty(X, \mathbf{M}, \Delta^m_{(s)})$ are topologically isomorphic with the spaces $c_0(X, \mathbf{M})$, $c(X, \mathbf{M})$ and $\ell_\infty(X, \mathbf{M})$, respectively.*

(ii) *The spaces $Sc_0(X, \mathbf{M}, \Delta^m_s)$, $Sc(X, \mathbf{M}, \Delta^m_s)$ and $S\ell_\infty(X, \mathbf{M}, \Delta^m_s)$ are topologically isomorphic with the spaces $c_0(X, \mathbf{M})$, $c(X, \mathbf{M})$ and $\ell_\infty(X, \mathbf{M})$, respectively, where $SZ(X, \mathbf{M}, \Delta^m_s) = \{x = (x_k) : x \in Z(X, \mathbf{M}, \Delta^m_s), x_1 = x_2 = \cdots = x_{ms} = 0\}$ is a subspace of $Z(X, \Delta^m_s)$, $Z = c, c_0$ and ℓ_∞.*

(iii) *$c_0(X,\mathbf{M},\Delta^m_{(s)})$, $c(X,\mathbf{M},\Delta^m_{(s)})$, $\ell_\infty(X,\mathbf{M},\Delta^m_{(s)})$, $c_0(X,\mathbf{M},\Delta^m_s)$, $c(X,\mathbf{M},\Delta^m_s)$ and $\ell_\infty(X,\mathbf{M},\Delta^m_s)$ are convex sets.*

Proof (i) For $Z = c, c_0$ and ℓ_∞, let us consider the mapping $T : Z(X,\mathbf{M},\Delta^m_{(s)}) \to Z(X,\mathbf{M})$, defined by

$$Tx = y = \left(\Delta^m_{(s)}x_k\right) \quad \text{for every } x = (x_k) \in Z\left(X,\mathbf{M},\Delta^m_{(s)}\right).$$

Clearly T is linear homeomorphism.

(ii) In this case we consider the mapping $T' : SZ(X,\mathbf{M},\Delta^m_s) \to Z(X,\mathbf{M})$, defined by

$$T'x = y = \left(\Delta^m_s x_k\right) \quad \text{for every } x = (x_k) \in SZ\left(X,\mathbf{M},\Delta^m_s\right).$$

Clearly T' is a linear homeomorphism.

(iii) The proof follows by using the convexity of Orlicz functions. □

Remark Let $\{a_1, a_2, \ldots, a_n\}$ be a linearly independent set in X. Then $\|\Delta^m_{(s)}x_k, z_1, \ldots, z_{n-r-1}\|_\infty = \max\{\|\Delta^m_{(s)}x_k, z_1, \ldots, z_{n-r-1}, a_{i_1}, \ldots, a_{i_r}\|_X\}$, $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, n\}$ is a derived $(n-r)$-norm on X for each $r = 1, \ldots, n-1$ and for each $k \geq 1$.

Hence we have the following theorem.

Theorem 2.4 *Let $\{a_1, a_2, \ldots, a_n\}$ be any linearly independent set in X. Then $c_0(X,\mathbf{M},\Delta^m_{(s)})$, $c(X,\mathbf{M},\Delta^m_{(s)})$ and $\ell_\infty(X,\mathbf{M},\Delta^m_{(s)})$ are normed linear spaces by*

$$\|x\|^r_{(s)} = \inf\left\{\rho > 0 : \sup_{k\geq 1, z_1,\ldots,z_{n-r-1}\in X} M_k\left(\left\|\frac{\Delta^m_{(s)}x_k}{\rho}, z_1, \ldots, z_{n-r-1}\right\|_\infty\right) \leq 1\right\}$$
$$\text{for each } r = 1, \ldots, n-1. \tag{2.3}$$

We call these norms derived norms.

Proof Proof is similar to that of Theorem 2.2. □

Theorem 2.5 *Let X be an n-Banach space. Then $c_0(X,\mathbf{M},\Delta^m_{(s)})$, $c(X,\mathbf{M},\Delta^m_{(s)})$ and $\ell_\infty(X,\mathbf{M},\Delta^m_{(s)})$ are Banach spaces under the norm* (2.1).

Proof Let Y be any one of the spaces $c_0(X,\mathbf{M},\Delta^m_{(s)})$, $c(X,\mathbf{M},\Delta^m_{(s)})$ and $\ell_\infty(X,\mathbf{M},\Delta^m_{(s)})$. Let (x^i) be any Cauchy sequence Y. Let $x_0 > 0$ be fixed and $t > 0$ be such that for a $0 < \varepsilon < 1$, $\frac{\varepsilon}{x_0 t} > 0$ and $x_0 t \geq 1$. Then there exists a positive integer n_0 such that

$$\left\|x^i - x^j\right\|^0_{(s)} < \frac{\varepsilon}{x_0 t} \quad \text{for all } i, j \geq n_0.$$

Using (2.1), we get

$$\inf\left\{\rho : \sup_{k\geq 1, z_1,\ldots,z_{n-1}\in X} M_k\left(\left\|\frac{\Delta^m_{(s)}(x^i_k - x^j_k)}{\rho}, z_1, \ldots, z_{n-1}\right\|_X\right) \leq 1\right\} < \frac{\varepsilon}{x_0 t} \quad \text{for all } i, j \geq n_0.$$

Hence we have

$$\sup_{k\geq 1, z_1,\dots,z_{n-1}\in X} M_k\left(\left\| \frac{\Delta^m_{(s)}(x^i_k - x^j_k)}{\|x^i - x^j\|^0_{(s)}}, z_1,\dots,z_{n-1}\right\|_X\right) \leq 1 \quad \text{for all } i,j \geq n_0.$$

It follows that for every $z_1,\dots,z_{n-1} \in X$,

$$M_k\left(\left\| \frac{\Delta^m_{(s)}(x^i_k - x^j_k)}{\|x^i - x^j\|^0_{(s)}}, z_1,\dots,z_{n-1}\right\|_X\right) \leq 1 \quad \text{for each } k \geq 1 \text{ and for all } i,j \geq n_0.$$

For $t > 0$ with $M_k(\frac{tx_0}{2}) \geq 1$, for all $k \geq 1$, we have

$$M_k\left(\left\| \frac{\Delta^m_{(s)}(x^i_k - x^j_k)}{\|x^i - x^j\|^0_{(s)}}, z_1,\dots,z_{n-1}\right\|_X\right) \leq M_k\left(\frac{tx_0}{2}\right) \quad \text{for every non-zero } z_1,\dots,z_{n-1} \in X.$$

This implies that

$$\left\|\Delta^m_{(s)}\left(x^i_k - x^j_k\right), z_1,\dots,z_{n-1}\right\|_X \leq \frac{tx_0}{2}\cdot\frac{\varepsilon}{tx_0} = \frac{\varepsilon}{2} \quad \text{for every non-zero } z_1,\dots,z_{n-1} \in X.$$

Hence $(\Delta^m_{(s)}x^i_k)$ is a Cauchy sequence in X for all $k \in N$. Since X is an n-Banach space, $(\Delta^m_{(s)}x^i_k)$ is convergent in X for all $k \in N$. For simplicity, let $\lim_{i\to\infty} \Delta^m_{(s)}x^i_k = y_k$ for each $k \in N$. By taking $k = 1,2,\dots,ms,\dots$, we can conclude that

$$\lim_{i\to\infty} x^i_k = x_k \quad \text{(say) exist for each } k \in N.$$

Now we can find that

$$\lim_{j\to\infty}\left[\inf\left\{\rho : \sup_{k\geq 1, z_1,\dots,z_{n-1}\in X} M_k\left(\left\| \frac{\Delta^m_{(s)}(x^i_k - x^j_k)}{\rho}, z_1,\dots,z_2\right\|_X\right) \leq 1\right\}\right] < \varepsilon \quad \text{for all } i \geq n_0.$$

Then, using the continuity of Orlicz functions, we have

$$\lim_{j\to\infty}\left[\inf\left\{\rho : \sup_{k\geq 1, z_1,\dots,z_{n-1}\in X} M_k\left(\left\| \frac{\Delta^m_{(s)}(x^i_k - \lim_{j\to\infty} x^j_k)}{\rho}, z_1,\dots,z_{n-1}\right\|_X\right) \leq 1\right\}\right]$$
$$< \varepsilon \quad \text{for all } i \geq n_0.$$

Hence we have

$$\inf\left\{\rho : \sup_{k\geq 1, z_1,\dots,z_{n-1}\in X} M_k\left(\left\| \frac{\Delta^m_{(s)}(x^i_k - x_k)}{\rho}, z_1,\dots,z_2\right\|_X\right) \leq 1\right\} < \varepsilon \quad \text{for all } i \geq n_0.$$

It follows that $(x^i - x) \in Y$. Since $(x^i) \in Y$ and Y is a linear space, so we have $x = x^i - (x^i - x) \in Y$.

This completes the proof of the theorem. □

The following corollary is due to Lemma 1.2.

Corollary 2.6 *If X is a Banach space under the standard n-norm, then $c_0(X, \mathbf{M}, \Delta^m_{(s)})$, $c(X, \mathbf{M}, \Delta^m_{(s)})$ and $\ell_\infty(X, \mathbf{M}, \Delta^m_{(s)})$ are Banach spaces under the norm*

$$\|x\|^0_{(s)} = \inf\left\{\rho > 0 : \sup_{k\geq 1, z_1, \ldots, z_{n-1} \in X} M_k\left(\left\|\frac{\Delta^m_{(s)} x_k}{\rho}, z_1, \ldots, z_{n-1}\right\|_S\right) \leq 1\right\}.$$

For the following results, let us assume Y to be any one of the spaces $c_0(X, \mathbf{M}, \Delta^m_{(s)})$, $c(X, \mathbf{M}, \Delta^m_{(s)})$ and $\ell_\infty(X, \mathbf{M}, \Delta^m_{(s)})$.

Theorem 2.7 *If (x^i) converges to an x in Y in the norm $\|\cdot\|^0_{(s)}$ defined by* (2.1), *then (x^i) also converges to x in the derived norm $\|\cdot\|^r_{(s)}$ defined by* (2.3) *for $r = 1$.*

Proof Let (x^i) converge to x in Y in the norm $\|\cdot\|^0_{(s)}$. Then

$$\left\|x^i - x\right\|^0_{(s)} \to 0 \quad \text{as } i \to \infty.$$

Using (2.1), we get

$$\inf\left\{\rho > 0 : \sup_{k\geq 1, z_1, \ldots, z_{n-1} \in X} M_k\left(\left\|\frac{\Delta^m_{(s)}(x^i_k - x_k)}{\rho}, z_1, \ldots, z_{n-1}\right\|_X\right) \leq 1\right\} \to 0 \quad \text{as } i \to \infty.$$

So, for any linearly independent set $\{a_1, a_2, \ldots, a_n\}$, we have

$$\inf\left\{\rho > 0 : \sup_{k\geq 1, z_1, \ldots, z_{n-2} \in X} M_k\left(\left\|\frac{\Delta^m_{(s)}(x^i_k - x_k)}{\rho}, z_1, \ldots, z_{n-2}, a_j\right\|_X\right) \leq 1\right\}$$
$$\to 0 \quad \text{as } i \to \infty \text{ and for each } j = 1, \ldots, n.$$

Hence by (1.1), we get

$$\inf\left\{\rho > 0 : \sup_{k\geq 1, z_1, \ldots, z_{n-2} \in X} M_k\left(\left\|\frac{\Delta^m_{(s)}(x^i_k - x_k)}{\rho}, z_1, \ldots, z_{n-2}\right\|_\infty\right) \leq 1\right\} \to 0 \quad \text{as } i \to \infty.$$

Thus

$$\left\|x^i - x\right\|^1_{(s)} \to 0 \quad \text{as } i \to \infty.$$

Hence (x^i) converges to x in the norm $\|\cdot\|^1_{(s)}$. □

If X is equipped with the standard n-norm and the derived norm is with respect to an orthonormal set, then the converse of the above theorem is also true.

Theorem 2.8 *Let X be a standard n-normed space and the derived $(n-1)$-norm on X is with respect to an orthonormal set. Then (x^i) is convergent in Y in the norm $\|\cdot\|^0_{(s)}$ defined by* (2.1) *if and only if (x^i) is convergent in Y in the derived norm $\|\cdot\|^r_{(s)}$ defined by* (2.3) *for $r = 1$.*

Proof In view of the above theorem, it is enough to prove that (x^i) is convergent in the norm $\|\cdot\|^1_{(s)}$ implies (x^i) is convergent in the norm $\|\cdot\|^0_{(s)}$.

Let (x^i) converge to x in Y in the norm $\|\cdot\|^1_{(s)}$. Then

$$\|x^i - x\|^1_{(s)} \to 0 \quad \text{as } i \to \infty.$$

Using (2.3) for $r = 1$, we get

$$\inf\left\{\rho > 0 : \sup_{k\geq 1, z_1,\ldots,z_{n-2}\in X} M_k\left(\left\|\frac{\Delta^m_{(s)}(x^i_k - x_k)}{\rho}, z_1, \ldots, z_{n-2}\right\|_\infty\right) \leq 1\right\} \to 0 \quad \text{as } i \to \infty.$$

Now one may observe that

$$\begin{aligned}&\left\|\Delta^m_{(s)}(x^i_k - x_k), z_1, \ldots, z_{n-1}\right\|_S \\ &\quad\leq \left\|\Delta^m_{(s)}(x^i_k - x_k), z_1, \ldots, z_{n-2}\right\|_S \|z_{n-1}\|_S \quad \text{for every non-zero } z_1, \ldots, z_{n-1} \in X,\end{aligned}$$

where $\|\cdot, \ldots, \cdot\|_S$ and $\|\cdot\|_S$ on the right-hand side denote the standard $(n-1)$-norm and the usual norm on X, respectively. Since the derived $(n-1)$-norm on X is with respect to an orthonormal set, using Lemma 1.3, we have

$$\left\|\Delta^m_{(s)}(x^i_k - x), z_1, \ldots, z_{n-1}\right\|_S \leq \sqrt{n}\left\|\Delta^m_{(s)}(x^i_k - x), z_1, \ldots, z_{n-2}\right\|_\infty \|z_{n-1}\|_S$$

and in this case $\|\cdot, \ldots, \cdot\|_\infty$ on the right-hand side is the derived $(n-1)$-norm which we used to define the norm $\|\cdot\|^1_{(s)}$.

Hence

$$\begin{aligned}&\inf\left\{\rho > 0 : \sup_{k\geq 1, z_1,\ldots,z_{n-1}\in X} M_k\left(\left\|\frac{\Delta^m_{(s)}(x^i_k - x_k)}{\rho}, z_1, \ldots, z_{n-1}\right\|_S\right) \leq 1\right\} \\ &\quad\leq \inf\left\{\rho > 0 : \sup_{k\geq 1, z_1,\ldots,z_{n-2}\in X} M_k\left(\sqrt{n}\left\|\frac{\Delta^m_{(s)}(x^i_k - x_k)}{\rho}, z_1, \ldots, z_{n-2}\right\|_\infty \|z_{n-1}\|_S\right) \leq 1\right\}.\end{aligned}$$

It follows that

$$\inf\left\{\rho > 0 : \sup_{k\geq 1, z_1,\ldots,z_{n-1}\in X} M_k\left(\left\|\frac{\Delta^m_{(s)}(x^i_k - x_k)}{\rho}, z_1, \ldots, z_{n-1}\right\|_S\right) \leq 1\right\} \to 0 \quad \text{as } i \to \infty.$$

Hence $\|x^i - x\|^0_{(s)} \to 0$ as $i \to \infty$.

That is, (x^i) converges to x in Y in the norm $\|\cdot\|^0_{(s)}$. □

Using Lemma 1.3, we get the following corollary.

Corollary 2.9 *Let X be a standard n-normed space and let the derived $(n-r)$-norms on X be with respect to an orthonormal set. Then a sequence in Y is convergent in the norm $\|\cdot\|^0_{(s)}$ defined by* (2.1) *if and only if it is convergent in the derived norm $\|\cdot\|^1_{(s)}$ and, by induction, in the derived norm $\|\cdot\|^r_{(s)}$ defined by* (2.3) *for all $r = 1, \ldots, n-1$. In particular, a sequence in*

Y is convergent in the norm $\|\cdot\|^0_{(s)}$ if and only if it is convergent in the derived norm $\|\cdot\|^{n-1}_{(s)}$, defined by

$$\|x\|^{n-1}_{(s)} = \inf\left\{\rho > 0 : \sup_k M_k\left(\left\|\frac{\Delta^m_{(s)}x_k}{\rho}\right\|_\infty\right) \le 1\right\}. \tag{2.4}$$

Theorem 2.10 *Let X be a standard n-normed space and let the derived $(n-r)$-norms on X for all $r = 1, \dots, n-1$ be with respect to an orthonormal set. Then Y is complete with respect to the norm $\|\cdot\|^0_{(s)}$ defined by (2.1) if and only if it is complete with respect to the derived norm $\|\cdot\|^1_{(s)}$ defined by (2.3). By induction, Y is complete with respect to the norm $\|\cdot\|^0_{(s)}$ if and only if it is complete with respect to the derived norm $\|\cdot\|^{n-1}_{(s)}$ defined by (2.4).*

Proof By replacing the phrases '(x^i) converges to x' with '(x^i) is Cauchy' and '$x^i - x$' with '$x^i - x^j$', we see that the analogues of Theorem 2.7, Theorem 2.8 and Corollary 2.9 hold for Cauchy sequences. This completes the proof. □

Remark Analogues of Theorem 2.4, Theorem 2.5, Corollary 2.6, Theorem 2.7, Theorem 2.8, Corollary 2.9 and Theorem 2.10 hold for the spaces $c_0(X, \mathbf{M}, \Delta^m_s)$, $c(X, \mathbf{M}, \Delta^m_s)$ and $\ell_\infty(X, \mathbf{M}, \Delta^m_s)$.

Competing interests
The author declares that they have no competing interests.

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