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Bernoulli numbers, convolution sums and congruences of coefficients for certain generating functions

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Abstract

In this paper, we study the convolution sums involving restricted divisor functions, their generalizations, their relations to Bernoulli numbers, and some interesting applications.

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1 Introduction

The Bernoulli polynomials $B_k(x)$, which are usually defined by the exponential generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!},$$

play an important role in different areas of mathematics, including number theory and the theory of finite differences. The Bernoulli polynomials satisfy the following well-known identities :

$$B_k(x+1) - B_k(x) = kx^{k-1}, \quad k \geq 1,$$

$$\frac{d}{dx} B_k(x) = kB_{k-1}(x), \quad k \geq 1,$$

and

$$\sum_{j=0}^N j^k = \frac{B_{k+1}(N+1) - B_{k+1}(0)}{k+1}, \quad k \geq 1.$$

We set $B_k = B_k(0)$. It is obvious from the way the polynomials $B_k(x)$ are constructed that all the B_k are rational numbers. It can be shown that $B_{2k+1} = 0$ for $k \geq 1$, and is alternatively positive and negative for even k . The B_k are called Bernoulli numbers.

Throughout the paper, we use the following arithmetical functions and q -series (sometimes defined by product expressions). For any integer $N \geq 1$, $l, s \in \mathbb{N} \cup \{0\}$, we define

$$\begin{aligned}\sigma_l(N) &:= \sum_{d|N} d^l, & \sigma_s^*(N; p) &:= \sum_{\substack{d|N \\ \frac{N}{d} \not\equiv 0 \pmod{p}}} d^s, & \sigma_s^*(N) &:= \sigma_s^*(N; 2) = \sum_{\substack{d|N \\ N/d \text{ odd}}} d^s, \\ \tilde{\sigma}_s(N) &:= \sum_{d|N} (-1)^{d-1} d^s, & S_i(N) &:= \sum_{k=1}^{N-1} k^i.\end{aligned}$$

For $q \in \mathbb{C}$ with $|q| < 1$, we consider the q -series:

$$\begin{aligned}\mathcal{A}(q) &:= \sum_{N=1}^{\infty} \sigma_1^*(N) q^N, & \mathcal{B}(q) &:= \sum_{N=1}^{\infty} \sigma_3^*(N) q^N, & \mathcal{C}(q) &:= \sum_{N=1}^{\infty} \sigma_5^*(N) q^N, \\ \sum_{N=1}^{\infty} a(N) q^N &:= q \prod_{N=1}^{\infty} (1 - q^N)^2 (1 - q^{2N})^2 (1 - q^{3N})^2 (1 - q^{6N})^2, \\ \sum_{N=1}^{\infty} b(N) q^N &:= q \prod_{N=1}^{\infty} (1 - q^N)^8 (1 - q^{2N})^8, \\ \sum_{N=1}^{\infty} c(N) q^N &:= q \prod_{N=1}^{\infty} (1 - q^N)^{16} (1 - q^{2N})^4, & \sum_{N=1}^{\infty} \tau(N) q^N &:= q \prod_{N=1}^{\infty} (1 - q^N)^{24}, \\ \sum_{N=1}^{\infty} l(N) q^N &:= q^2 \prod_{N=1}^{\infty} (1 - q^N)^8 (1 - q^{2N})^4 (1 - q^{4N})^8.\end{aligned}$$

The exact evaluation of the basic convolution sum

$$\sum_{k=1}^{N-1} \sigma_1(k) \sigma_1(N-k)$$

first appeared in a letter from Besge to Liouville in 1862. The evaluation of such sums also appear in the works of Glaisher, Lahiri, Lehmer, Ramanujan, and Skoruppa. For instance, Ramanujan [1] obtained

$$\begin{aligned}\sum_{k=1}^{N-1} \sigma_1(k) \sigma_1(N-k) &= \frac{1}{12} (5\sigma_3(N) + (1-6N)\sigma(N)) \quad \text{and} \\ \sum_{k=1}^{N-1} \sigma_1(k) \sigma_3(N-k) &= \frac{1}{240} [21\sigma_5(N) + (10-30N)\sigma_3(N) - \sigma_1(N)]\end{aligned}\tag{1}$$

using only elementary arguments. For $a, b, N \in \mathbb{N}$, Ramanujan showed that the sum

$$S_{a,b}(N) := \sum_{m=1}^{N-1} \sigma_a(m) \sigma_b(N-m)$$

can be evaluated in terms of the quantities

$$\sigma_{a+b+1}(N), \quad \sigma_{a+b-1}(N), \quad \dots, \quad \sigma_3(N), \quad \sigma_1(N)$$

for the nine pairs $(a, b) \in \mathbb{N}^2$ satisfying

$$a + b = 2, 4, 6, 8, 12, \quad a \leq b, a \equiv b \equiv 1 \pmod{2}.$$

For explicit evaluations of $S_{a,b}(N)$ for different pairs $(a, b) \in \mathbb{N}^2$ satisfying the above conditions stated, we refer to the papers of Ramanujan [1], [2], Huard *et al.* [3], Lahiri [4], and Glaisher [5], respectively. Levit [6] showed that the nine arithmetic evaluations of $S_{a,b}(N)$ (with a, b both odd) are the only ones using the theory of modular forms. In [4], Lahiri has given 37 sums of the form

$$\sum_{\substack{(m_1, \dots, m_r) \\ m_1 + \dots + m_r = N}} m_1^{a_1} \cdots m_r^{a_r} \sigma_{b_1}(m_1) \cdots \sigma_{b_r}(m_r),$$

where $a_1, \dots, a_r \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $b_1, \dots, b_r \in \mathbb{N}$, each of which can be expressed as a finite linear combination of $\sigma_1(N), \sigma_3(N), \dots, \sigma_{b_1+b_2+\dots+b_r+r-1}(N)$ with coefficients which are polynomials in N of degree at most $a_1 + \dots + a_r + r - 1$ with rational coefficients.

In 2002, Huard *et al.* [3] extended Melfi's [7] result to

$$\begin{aligned} & \sum_{k < N/2} \sigma_1(k) \sigma_3(N - 2k) \\ &= \frac{1}{48} \sigma_5(N) + \frac{1}{15} \sigma_5\left(\frac{N}{2}\right) + \frac{(2 - 3N)}{48} \sigma_3(N) - \frac{1}{240} \sigma_1\left(\frac{N}{2}\right), \\ & \sum_{k < N/2} \sigma_3(k) \sigma_1(N - 2k) \\ &= \frac{1}{240} \sigma_5(N) + \frac{1}{12} \sigma_5\left(\frac{N}{2}\right) + \frac{(1 - 3N)}{24} \sigma_3\left(\frac{N}{2}\right) - \frac{1}{240} \sigma_1(N), \end{aligned} \quad (2)$$

where N is an arbitrary positive integer.

Glaisher [5, 8, 9] extended Besge's formula by replacing $\sigma_1(N)$ in the convolution sum in (1) by other arithmetical functions; for example, he obtained

$$\begin{aligned} 24 \sum_{k=1}^{N-1} \sigma_1^*(k) \sigma_1^*(N - k) &= 6\sigma_3(N) - 6\sigma_3\left(\frac{N}{2}\right) - 6N\sigma_1(N) + 6N\sigma_1\left(\frac{N}{2}\right) \\ &= 6\{\sigma_3^*(N) - N\sigma_1^*(N)\} \end{aligned} \quad (3)$$

and

$$\sum_{k=1}^{N-1} \sigma_1^*(k) \sigma_3^*(N - k) = \frac{1}{16} (\sigma_5^*(N) - N\sigma_3^*(N)). \quad (4)$$

Recently, Hahn [10] showed that

$$16 \sum_{k < N} \tilde{\sigma}_1(k) \tilde{\sigma}_3(N - k) = -\tilde{\sigma}_5(N) + 2(N - 1)\tilde{\sigma}_3(N) + \tilde{\sigma}_1(N).$$

It is also interesting to note that the arithmetical functions (for example, $\tau(n)$ and $a(n)$), coming out as coefficients of the q -series expansion of its corresponding q -products, do

appear in the explicit evaluation of certain convolution sums. For instance, Lahiri (see [11]) proved that

$$\sum_{k=1}^{N-1} k(N-k)\sigma_3(k)\sigma_3(N-k) = \frac{1}{540}(N^2\sigma_7(N) - \tau(N))$$

and from Alaca and Williams (see [12]), we observe that

$$\begin{aligned} & \sum_{\substack{(k,m) \in \mathbb{N}^2 \\ 2k+3m=N}} \sigma_1(k)\sigma_1(m) \\ &= \frac{1}{120}\sigma_3(N) + \frac{1}{30}\sigma_3\left(\frac{N}{2}\right) + \frac{3}{40}\sigma_3\left(\frac{N}{3}\right) \\ & \quad + \frac{3}{10}\sigma_3\left(\frac{N}{6}\right) + \left(\frac{1}{24} - \frac{N}{12}\right)\sigma_1\left(\frac{N}{2}\right) + \left(\frac{1}{24} - \frac{N}{8}\right)\sigma_1\left(\frac{N}{3}\right) - \frac{1}{120}a(N). \end{aligned} \quad (5)$$

In [13], Simsek (and also in [14, (2.17)] Simsek along with Ozden and Cagul) has studied other aspects of the arithmetical function $\sigma_1(n)$ in connection with the classical Jacobi and Euler functions. We also refer to Kim and Lee [15, Lemma 2.1] and [16].

Thus the study of convolution sums and their applications is classical and they play an important role in number theory. The aim of this article is to first extend and generalize Glaishers formulas (stated in (3) and (4)). We indeed study the sums (in Section 3)

$$\begin{aligned} & \sum_{k=1}^{N-1} \sigma_1^*(2^m k) \sigma_1^*(2^n(N-k)), \quad \sum_{k=1}^{N-1} \sigma_1^*(2^m k) \sigma_3^*(2^n(N-k)), \\ & \sum_{k=1}^{N-1} \sigma_3^*(2^m k) \sigma_1^*(2^n(N-k)) \quad \text{and} \quad \sum_{r+s+t=N} \sigma_1^*(2^m r) \sigma_1^*(2^n s) \sigma_1^*(2^l t). \end{aligned}$$

Then, we study and evaluate (in Section 4) sums of the type

$$\sum_{k=1}^{N-1} \sigma_1^*(3^m k; 3) \sigma_1^*(3^n(N-k); 3) \quad \text{and} \quad \sum_{k=1}^{N-1} \sigma_1^*(2^m k; 2) \sigma_1^*(3^n(N-k); 3).$$

As applications to our study and evaluations of convolution sums, we show that $\mathcal{A}(q)$, $\mathcal{B}(q)$, and $\mathcal{C}(q)$ are connected by a second-order differential equation (see Theorem 3.8).

In Section 5, we also prove some interesting congruence relations involving the coefficients of modular-like functions and divisor functions (see Theorem 5.3). As a sample, we obtain that if $N \equiv 1 \pmod{8}$, then the congruence

$$\tau(N) \equiv \sigma_{11}(N) \pmod{1,415,168}$$

holds.

In Section 6, we present a generalization of Besge's formula by considering certain combinatorial convolution sums (see Theorem 6.3). It should be noted that Proposition 6.1, Theorem 6.3, and Remark 6.4 exhibit amply the connection between the convolution sums and the Bernoulli numbers. Finally, we record special values of $a(N)$, $\tau(N)$, $b(N)$, $c(N)$, and $l(N)$ (for $1 \leq N \leq 45$) and some convolution formulas in the Appendix.

2 Some weighted convolution sums

Let $\widehat{f}(a)\widehat{g}(b) := \frac{1}{2}\{f(a)g(b) + f(b)g(a)\}$. It is easily checked that

$$\sum_{k=1}^{N-1} \widehat{f}(k)\widehat{g}(N-k) = \sum_{k=1}^{N-1} f(k)g(N-k).$$

Lemma 2.1 For $f, g: \mathbb{N} \rightarrow \mathbb{C}$ and $N \in \mathbb{N}$, we have

$$\sum_{k=1}^{N-1} k\widehat{f}(k)\widehat{g}(N-k) = \frac{N}{2} \sum_{k=1}^{N-1} \widehat{f}(k)\widehat{g}(N-k).$$

Proof We observe that

$$\begin{aligned} \sum_{k=1}^{N-1} k\widehat{f}(k)\widehat{g}(N-k) &= \frac{1}{2} \sum_{k=1}^{N-1} k\{f(k)g(N-k) + f(N-k)g(k)\} \\ &= \frac{1}{2} \sum_{k=1}^{N-1} (N-k)\{f(N-k)g(k) + f(k)g(N-k)\} \\ &= \sum_{k=1}^{N-1} (N-k)\widehat{f}(k)\widehat{g}(N-k). \end{aligned}$$

Hence

$$2 \sum_{k=1}^{N-1} k\widehat{f}(k)\widehat{g}(N-k) = N \sum_{k=1}^{N-1} \widehat{f}(k)\widehat{g}(N-k)$$

and thus

$$\sum_{k=1}^{N-1} k\widehat{f}(k)\widehat{g}(N-k) = \frac{N}{2} \sum_{k=1}^{N-1} \widehat{f}(k)\widehat{g}(N-k).$$

□

Lemma 2.2 For $f, g: \mathbb{N} \rightarrow \mathbb{C}$ and $N \in \mathbb{N}$, we have

$$\sum_{k=1}^{N-1} k^3 \widehat{f}(k)\widehat{g}(N-k) = -\frac{1}{4}N^3 \sum_{k=1}^{N-1} \widehat{f}(N-k)\widehat{g}(k) + \frac{3N}{2} \sum_{k=1}^{N-1} k^2 \widehat{f}(N-k)\widehat{g}(k).$$

Proof We note that

$$\begin{aligned} \sum_{k=1}^{N-1} k^3 \widehat{f}(k)\widehat{g}(N-k) &= \sum_{k=1}^{N-1} (N-k)^3 \widehat{f}(N-k)\widehat{g}(k) \\ &= N^3 \sum_{k=1}^{N-1} \widehat{f}(N-k)\widehat{g}(k) - 3N^2 \sum_{k=1}^{N-1} k\widehat{f}(N-k)\widehat{g}(k) \\ &\quad + 3N \sum_{k=1}^{N-1} k^2 \widehat{f}(N-k)\widehat{g}(k) - \sum_{k=1}^{N-1} k^3 \widehat{f}(N-k)\widehat{g}(k). \end{aligned} \tag{6}$$

Then, for the second term on the right-hand side of (6), we replace k by $N - k$ in Lemma 2.1 and obtain

$$\sum_{k=1}^{N-1} (N-k) \widehat{f}(N-k) \widehat{g}(k) = \frac{N}{2} \sum_{k=1}^{N-1} \widehat{f}(N-k) \widehat{g}(k).$$

This shows that

$$\sum_{k=1}^{N-1} k \widehat{f}(N-k) \widehat{g}(k) = \frac{N}{2} \sum_{k=1}^{N-1} \widehat{f}(N-k) \widehat{g}(k).$$

So, (6) can be written as

$$\begin{aligned} & \sum_{k=1}^{N-1} k^3 \{ \widehat{f}(k) \widehat{g}(N-k) + \widehat{f}(N-k) \widehat{g}(k) \} \\ &= -\frac{N^3}{2} \sum_{k=1}^{N-1} \widehat{f}(N-k) \widehat{g}(k) + 3N \sum_{k=1}^{N-1} k^2 \widehat{f}(N-k) \widehat{g}(k). \end{aligned} \quad (7)$$

Here the left-hand side of (7) is

$$\begin{aligned} & \sum_{k=1}^{N-1} k^3 \{ \widehat{f}(k) \widehat{g}(N-k) + \widehat{f}(N-k) \widehat{g}(k) \} \\ &= \sum_{k=1}^{N-1} k^3 \left[\frac{1}{2} \{ f(k)g(N-k) + f(N-k)g(k) \} + \frac{1}{2} \{ f(N-k)g(k) + f(k)g(N-k) \} \right] \\ &= \sum_{k=1}^{N-1} k^3 \{ f(k)g(N-k) + f(N-k)g(k) \} = 2 \sum_{k=1}^{N-1} k^3 \widehat{f}(k) \widehat{g}(N-k). \end{aligned}$$

Therefore we have

$$2 \sum_{k=1}^{N-1} k^3 \widehat{f}(k) \widehat{g}(N-k) = -\frac{1}{2} N^3 \sum_{k=1}^{N-1} \widehat{f}(N-k) \widehat{g}(k) + 3N \sum_{k=1}^{N-1} k^2 \widehat{f}(N-k) \widehat{g}(k).$$

This completes the proof. $\square$

Remark 2.3 In general, we can express

$$\sum_{k=1}^{N-1} k^{2l+1} \widehat{f}(k) \widehat{g}(N-k) \quad \text{for } k \in \mathbb{N} \cup \{0\}$$

as a combination in terms of the sum

$$\sum_{k=1}^{N-1} k^{2j} \widehat{f}(k) \widehat{g}(N-k) \quad \text{with } j = 0, 1, 2, \dots, l$$

with their coefficients being polynomials in N of degree at most $(2l+1)$.

Lemma 2.4 For $l, N \in \mathbb{N}$ and $f, g : \mathbb{N} \rightarrow \mathbb{C}$, we have

$$\{1 + (-1)^{l+1}\} \sum_{k=1}^{N-1} k^l \widehat{f}(k) \widehat{g}(N-k) = \sum_{k=1}^{N-1} \widehat{f}(N-k) \widehat{g}(k) \left\{ \sum_{j=0}^{l-1} \binom{l}{j} (-1)^j k^j N^{l-j} \right\}.$$

Proof We note that (for $l \in \mathbb{N}$),

$$\begin{aligned} \sum_{k=1}^{N-1} k^l \widehat{f}(k) \widehat{g}(N-k) &= \sum_{k=1}^{N-1} (N-k)^l \widehat{f}(N-k) \widehat{g}(k) \\ &= \sum_{k=1}^{N-1} \widehat{f}(N-k) \widehat{g}(k) \left\{ \sum_{j=0}^l \binom{l}{j} (-1)^j k^j N^{l-j} \right\} \\ &= (-1)^l \sum_{k=1}^{N-1} k^l \widehat{f}(N-k) \widehat{g}(k) + \sum_{k=1}^{N-1} \widehat{f}(N-k) \widehat{g}(k) \left\{ \sum_{j=0}^{l-1} \binom{l}{j} (-1)^j k^j N^{l-j} \right\}. \end{aligned}$$

Observing the fact that

$$\sum_{k=1}^{N-1} k^l \widehat{f}(k) \widehat{g}(N-k) = \sum_{k=1}^{N-1} k^l \widehat{f}(N-k) \widehat{g}(k)$$

for any $l \in \mathbb{N}$, we obtain

$$\{1 + (-1)^{l+1}\} \sum_{k=1}^{N-1} k^l \widehat{f}(k) \widehat{g}(N-k) = \sum_{k=1}^{N-1} \widehat{f}(N-k) \widehat{g}(k) \left\{ \sum_{j=0}^{l-1} \binom{l}{j} (-1)^j k^j N^{l-j} \right\}.$$

This proves Lemma 2.4. □

Remark 2.5 By iteration process, in principle, the convolution sum $\sum_{k=1}^{N-1} k^l \widehat{f}(k) \widehat{g}(N-k)$ can be evaluated for any odd $l \in \mathbb{N}$.

Let

$$\begin{aligned} \widehat{f}(a) \widehat{g}(b) \widehat{h}(c) &:= \frac{1}{6} \{ f(a)g(b)h(c) + f(a)g(c)h(b) + f(b)g(a)h(c) \\ &\quad + f(b)g(c)h(a) + f(c)g(a)h(b) + f(c)g(b)h(a) \}. \end{aligned}$$

Lemma 2.6 For $f : \mathbb{N} \rightarrow \mathbb{C}$ and $N \in \mathbb{N}$. Then we have

$$\sum_{\substack{(r,s,t) \in \mathbb{N}^3 \\ r+s+t=N}} r \widehat{f}(r) \widehat{f}(s) \widehat{f}(t) = \frac{N}{3} \sum_{\substack{(r,s,t) \in \mathbb{N}^3 \\ r+s+t=N}} \widehat{f}(r) \widehat{f}(s) \widehat{f}(t).$$

Proof From the cyclic transformation, $r \rightarrow s \rightarrow t \rightarrow r$, we observe that

$$\sum_{\substack{(r,s,t) \in \mathbb{N}^3 \\ r+s+t=N}} r \widehat{f}(r) \widehat{f}(s) \widehat{f}(t) = \sum_{\substack{(r,s,t) \in \mathbb{N}^3 \\ r+s+t=N}} s \widehat{f}(r) \widehat{f}(s) \widehat{f}(t) = \sum_{\substack{(r,s,t) \in \mathbb{N}^3 \\ r+s+t=N}} t \widehat{f}(r) \widehat{f}(s) \widehat{f}(t).$$

Therefore, we have

$$\sum_{\substack{(r,s,t) \in \mathbb{N}^3 \\ r+s+t=N}} N \hat{f}(r) \hat{f}(s) \hat{f}(t) = \sum_{\substack{(r,s,t) \in \mathbb{N}^3 \\ r+s+t=N}} (r+s+t) \hat{f}(r) \hat{f}(s) \hat{f}(t) = 3 \sum_{\substack{(r,s,t) \in \mathbb{N}^3 \\ r+s+t=N}} r \hat{f}(r) \hat{f}(s) \hat{f}(t).$$

Now Lemma 2.6 follows. $\square$

Lemma 2.7 For $f : \mathbb{N} \rightarrow \mathbb{C}$ and $N \in \mathbb{N}$. Then we have (with $g(m) = mf(m)$ for all $m \in \mathbb{N}$)

$$3 \sum_{\substack{(r,s,t) \in \mathbb{N}^3 \\ r+s+t=N}} r^2 f(r) f(s) f(t) = N^2 \sum_{\substack{(r,s,t) \in \mathbb{N}^3 \\ r+s+t=N}} f(r) f(s) f(t) + 6 \sum_{t=1}^{N-2} \sum_{\substack{(r,s) \in \mathbb{N}^2 \\ r+s=N-t}} g(r) g(s).$$

Proof As in Lemma 2.6, again by the cyclic transformation $r \rightarrow s \rightarrow t \rightarrow r$, we observe that

$$\sum_{\substack{(r,s,t) \in \mathbb{N}^3 \\ r+s+t=N}} r^2 f(r) f(s) f(t) = \sum_{\substack{(r,s,t) \in \mathbb{N}^3 \\ r+s+t=N}} s^2 f(r) f(s) f(t) = \sum_{\substack{(r,s,t) \in \mathbb{N}^3 \\ r+s+t=N}} t^2 f(r) f(s) f(t) \quad (8)$$

and

$$\sum_{\substack{(r,s,t) \in \mathbb{N}^3 \\ r+s+t=N}} r s f(r) f(s) f(t) = \sum_{\substack{(r,s,t) \in \mathbb{N}^3 \\ r+s+t=N}} s t f(r) f(s) f(t) = \sum_{\substack{(r,s,t) \in \mathbb{N}^3 \\ r+s+t=N}} t r f(r) f(s) f(t). \quad (9)$$

Therefore, we get (from (8) and (9))

$$\begin{aligned} \sum_{\substack{(r,s,t) \in \mathbb{N}^3 \\ r+s+t=N}} n^2 f(r) f(s) f(t) &= 3 \sum_{\substack{(r,s,t) \in \mathbb{N}^3 \\ r+s+t=N}} r^2 f(r) f(s) f(t) + 6 \sum_{\substack{(r,s,t) \in \mathbb{N}^3 \\ r+s+t=N}} r s f(r) f(s) f(t) \\ &= 3 \sum_{\substack{(r,s,t) \in \mathbb{N}^3 \\ r+s+t=N}} r^2 f(r) f(s) f(t) + 6 \sum_{t=1}^{N-2} f(t) \sum_{\substack{(r,s) \in \mathbb{N}^2 \\ r+s=N-t}} g(r) g(s), \end{aligned}$$

where $g(m) = mf(m) : \mathbb{N} \rightarrow \mathbb{C}$ for all $m \in \mathbb{N}$. This proves Lemma 2.7. $\square$

3 The convolution sum $\sum_{k=1}^{N-1} \sigma_1^*(k) \sigma_1^*(N-k)$ and its extensions

Proposition 3.1 (a) [17, Theorem 15.1, p.184], we have

$$\begin{aligned} \sum_{k < N/2} \sigma(k) \sigma(N-2k) &= \frac{1}{24} \{ 2\sigma_3(N) + (1-3N)\sigma(N) + 8\sigma_3(N/2) \\ &\quad + (1-6N)\sigma(N/2) \}. \end{aligned}$$

(b) [3, Theorem 3], [17, Theorem 15.3, p.188], we have

$$\begin{aligned} \sum_{k < N/3} \sigma_1(k) \sigma_1(N-3k) \\ = \frac{1}{24} \left\{ \sigma_3(N) + (1-2N)\sigma_1(N) + 9\sigma_3\left(\frac{N}{3}\right) + (1-6N)\sigma_1\left(\frac{N}{3}\right) \right\}. \end{aligned}$$

Proposition 3.2 *Let $s, N \in \mathbb{N}$. Then we obtain*

$$\sigma_s^*(N; p) = \sigma_s(N) - \sigma_s(N/p).$$

In particular, we have

$$\sigma_s^*(N) = \sigma_s(N) - \sigma_s(N/2),$$

which can also be seen in [17, p.27].

Proof We can know that

$$\sigma_s^*(N; p) := \sum_{\substack{d|N \\ \frac{N}{d} \not\equiv 0 \pmod{p}}} d^s = \sum_{d|N} d^s - \sum_{\substack{d|N \\ \frac{N}{d} \equiv 0 \pmod{p}}} d^s = \sigma_s(N) - \sum_{d|\frac{N}{p}} d^s = \sigma_s(N) - \sigma_s(N/p). \quad \square$$

Proposition 3.3 *Let N be any positive integer. For $m, n \in \mathbb{N} \cup \{0\}$, where $0 \leq m \leq n$, we have*

$$\begin{aligned} (a) \quad & \sum_{k=1}^{N-1} \sigma_1^*(2^m k) \sigma_1^*(2^n(N-k)) = 2^{n+m-2} (\sigma_3^*(N) - N\sigma_1^*(N)). \\ (b) \quad & \sum_{k=1}^{N-1} \sigma_1^*(2^m k) \sigma_3^*(2^n(N-k)) = 2^{3n+m-4} (\sigma_5^*(N) - N\sigma_3^*(N)). \\ (c) \quad & \sum_{k=1}^{N-1} \sigma_3^*(2^m k) \sigma_1^*(2^n(N-k)) = 2^{n+3m-4} (\sigma_5^*(N) - N\sigma_3^*(N)). \\ (d) \quad & \sum_{k=1}^{N-1} \sigma_1^*(2^m k) \{ \sigma_3^*(2^{n+1}(N-k)) - k\sigma_1^*(2^{3n+2}(N-k)) \} \\ & = 2^{3n+m-1} (\sigma_5^*(N) - N^2\sigma_1^*(N)). \end{aligned}$$

Proof For the sake of completeness, we just hint the proof of (b). We note that $\sigma_k^*(2^a) = 2^{ak}$. Since σ_k^* is a 2^k -scalar multiplicative function (i.e., $\sigma_k^*(2N) = \sigma_k^*(2) \cdot \sigma_k^*(N)$), we have

$$\sum_{k=1}^{N-1} \sigma_1^*(2^m k) \sigma_3^*(2^n(N-k)) = 2^{m+3n} \sum_{k=1}^{N-1} \sigma_1^*(k) \sigma_3^*(N-k) = 2^{3n+m-4} (\sigma_5^*(N) - N\sigma_3^*(N))$$

by (4). The proofs of (a), (c), and (d) are similar to (b). $\square$

Remark 3.4 Using Eq. (3) and Lemma 2.1, we obtain

$$\sum_{k=1}^{N-1} k\sigma_1^*(k) \sigma_1^*(N-k) = \frac{N}{2} \sum_{k=1}^{N-1} \sigma_1^*(k) \sigma_1^*(N-k) = \frac{N}{8} \{ \sigma_3^*(N) - N\sigma_1^*(N) \}. \quad (10)$$

Theorem 3.5 *Let $N (\geq 3)$ be any integer with $m, n, l \in \mathbb{N} \cup \{0\}$. Then we have*

$$\sum_{r+s+t=N} \sigma_1^*(2^m r) \sigma_1^*(2^n s) \sigma_1^*(2^l t) = \frac{2^{m+n+l}}{64} (\sigma_5^*(N) - 3N\sigma_3^*(N) + 2N^2\sigma_1^*(N)).$$

Proof Let $r_1 l_1 = p^{m_1} N$ and $l_1 \not\equiv 0 \pmod{p}$ for some $r_1, l_1 \in \mathbb{N}$ and prime p . Since p does not divide l_1 , we can write $r_1 = p^{m_1} d$ for some $d \in \mathbb{N}$. Therefore, we have

$$\begin{aligned}\sigma_s^*(p^{m_1} N; p) &= \sum_{\substack{r_1 | p^{m_1} N \\ \frac{p^{m_1} N}{r_1} \not\equiv 0 \pmod{p}}} r_1^s = \sum_{\substack{d | N \\ \frac{N}{d} \not\equiv 0 \pmod{p}}} (p^{m_1} d)^s \\ &= p^{m_1 s} \sum_{\substack{d | N \\ \frac{N}{d} \not\equiv 0 \pmod{p}}} d^s = p^{m_1 s} \sigma_s^*(N; p).\end{aligned}\quad (11)$$

We note that

$$\begin{aligned}&\sum_{r+s+t=N} \sigma_1^*(2^m r) \sigma_1^*(2^n s) \sigma_1^*(2^l t) \\ &= 2^{m+n+l} \sum_{t=1}^{N-2} \sigma_1^*(t) \sum_{r+s=N-t} \sigma_1^*(r) \sigma_1^*(s) \\ &= 2^{m+n+l} \sum_{t=1}^{N-2} \sigma_1^*(t) \left\{ \frac{1}{4} (\sigma_3^*(N-t) - (N-t) \sigma_1^*(N-t)) \right\} \quad (\text{by (3)}) \\ &= \frac{2^{m+n+l}}{4} \left\{ \sum_{t=1}^{N-2} \sigma_1^*(t) \sigma_3^*(N-t) - \sum_{t=1}^{N-2} (N-t) \sigma_1^*(t) \sigma_1^*(N-t) \right\} \\ &= \frac{2^{m+n+l}}{4} \left\{ \sum_{t=1}^{N-2} \sigma_1^*(t) \sigma_3^*(N-t) - \sum_{k=1}^{N-2} k \sigma_1^*(N-k) \sigma_1^*(k) \right\}\end{aligned}$$

and hence, we use Eq. (1) and Eq. (10) to obtain the result. $\square$

Corollary 3.6 *Let*

$$\begin{aligned}f(p) &\in \left\{ \sum_{k=1}^{p-1} \sigma_1^*(2^m k) \sigma_1^*(2^n (p-k)), \sum_{k=1}^{p-1} \sigma_1^*(2^m k) \sigma_3^*(2^n (p-k)), \right. \\ &\quad \left. \sum_{r+s+t=p} \sigma_1^*(2^m r) \sigma_1^*(2^n s) \sigma_1^*(2^l t) \right\},\end{aligned}$$

where $p = 2q + 1$ is an odd prime. Then

$$f(p) = aS_4(q+1) + bS_3(q+1) + cS_2(q+1) + dS_1(q+1),$$

where the coefficients a, b, c, d are listed in Table 1.

Table 1 Coefficients for $p = 2q + 1$

$f(p) :=$	a	b	c	d
$\sum_{k=1}^{p-1} \sigma_1^*(2^m k) \sigma_1^*(2^n (p-k))$	0	0	$3 \cdot 2^{m+n+1}$	-2^{m+n+1}
$\sum_{k=1}^{p-1} \sigma_1^*(2^m k) \sigma_3^*(2^n (p-k))$	$5 \cdot 2^{m+3n+1}$	-2^{m+3n+2}	$5 \cdot 2^{m+3n}$	-2^{m+3n}
$\sum_{r+s+t=p} \sigma_1^*(2^m r) \sigma_1^*(2^n s) \sigma_1^*(2^l t)$	$5 \cdot 2^{m+n+l-1}$	$-3 \cdot 2^{m+n+l}$	$2^{m+n+l+1}$	$-2^{m+n+l-1}$

Table 2 Convolution formulas for prime $p = 2q + 1$

Convolution sum	$f(2q + 1) :=$
$\sum_{k=1}^{p-1} \sigma_1(k) \sigma_1(p-k)$	$\frac{1}{3}q(10q^2 + 9q - 1)$
$\sum_{k=1}^{p-1} \sigma_1^*(k) \sigma_1^*(p-k)$	$2q^2(q + 1)$
$\sum_{k=1}^{p-1} \sigma_1(k) \sigma_3(p-k)$	$\frac{1}{15}q(42q^4 + 75q^3 + 50q^2 + 15q - 2)$
$\sum_{k=1}^{p-1} \sigma_1^*(k) \sigma_3^*(p-k)$	$q^2(q + 1)(2q^2 + 2q + 1)$
$\sum_{r+s+t=p} \sigma_1(r) \sigma_1(s) \sigma_1(t)$	$\frac{1}{12}q^2(14q^3 + 5q^2 - 8q + 1)$
$\sum_{r+s+t=p} \sigma_1^*(r) \sigma_1^*(s) \sigma_1^*(t)$	$\frac{1}{2}q^4(q + 1)$

Table 3 Convolution formulas

Convolution sum	
$A(N) := \sum_{k=1}^{N-1} \sigma_1(k) \sigma_1(N-k)$	$\frac{1}{12}(5\sigma_3(N) + (1-6N)\sigma_1(N))$
$B(N) := \sum_{k=1}^{N-1} \sigma_1^*(k) \sigma_1^*(N-k)$	$\frac{1}{4}(\sigma_3^*(N) - N\sigma_1^*(N))$
$C(N) := \sum_{r+s+t=N} \sigma_1(r) \sigma_1(s) \sigma_1(t)$	$\frac{7}{192}\sigma_5(N) + (\frac{5}{96} - \frac{5}{32}N)\sigma_3(N) + (\frac{1}{192} - \frac{1}{16}N + \frac{1}{8}N^2)\sigma_1(N)$
$D(N) := \sum_{r+s+t=N} \sigma_1^*(r) \sigma_1^*(s) \sigma_1^*(t)$	$\frac{1}{64}(\sigma_5^*(N) - 3N\sigma_3^*(N) + 2N^2\sigma_1^*(N))$

Remark 3.7 If $p = 2q + 1$ is any odd prime, then

$$\begin{aligned} \sum_{k=1}^{2q} \sigma_1^*(k) \sigma_1^*(2q+1-k) &\equiv \sum_{k=1}^{2q} \sigma_1^*(k) \sigma_3^*(2q+1-k) \\ &\equiv \sum_{r+s+t=2q+1} \sigma_1^*(r) \sigma_1^*(s) \sigma_1^*(t) \\ &\equiv 0 \pmod{S_1(q+1)} \end{aligned}$$

from Table 2.

Now, we compare the values of the convolution sums in Table 3 [17, p.148]. For almost all $N \in \mathbb{N}$, we find that $A(N) > B(N)$ and $C(N) > D(N)$ (Figure 1). As an application to Theorem 3.5, we have the following.

Theorem 3.8 The q -series $\mathcal{A}(q)$, $\mathcal{B}(q)$, $\mathcal{C}(q)$ are connected by the differential equation,

$$64\mathcal{A}^3(q) = \mathcal{C}(q) - 3q \frac{d\mathcal{B}(q)}{dq} + 2q^2 \frac{d^2\mathcal{A}(q)}{dq^2} + 2q \frac{d\mathcal{A}(q)}{dq}, \quad (12)$$

where $q \frac{d\mathcal{B}(q)}{dq} = \sum_{N=1}^{\infty} N\sigma_3^*(N)q^N$ and $q^2 \frac{d^2\mathcal{A}(q)}{dq^2} = \sum_{N=1}^{\infty} N(N-1)\sigma_1^*(N)q^N$.

Proof We note that

$$\begin{aligned} 64\mathcal{A}^3(q) &= 64 \left(\sum_{l=1}^{\infty} \sigma_1^*(l)q^l \right) \left(\sum_{m=1}^{\infty} \sigma_1^*(m)q^m \right) \left(\sum_{n=1}^{\infty} \sigma_1^*(n)q^n \right) \\ &= \sum_{N=3}^{\infty} \left(64 \sum_{l+m+n=N} \sigma_1^*(l) \sigma_1^*(m) \sigma_1^*(n) \right) q^N \\ &= \sum_{N=3}^{\infty} \{ \sigma_5^*(N) - 3N\sigma_3^*(N) + 2N^2\sigma_1^*(N) \} q^N \end{aligned} \quad (13)$$

Figure 1 $A(N)$, $B(N)$, $C(N)$, and $D(N)$.

from Theorem 3.5. We note that $\sigma_5^*(N) - 3N\sigma_3^*(N) + 2N^2\sigma_1^*(N)$ is zero for $N = 1$ and $N = 2$. Thus the differential Eq. (12) follows. $\square$

Remark 3.9 We also note that from Eq. (1) and Eq. (3), we can determine the equations

$$16\mathcal{A}(q)\mathcal{B}(q) = \mathcal{C}(q) - q \frac{d\mathcal{B}(q)}{dq} \quad (14)$$

and

$$4\mathcal{A}^2(q) = \mathcal{B}(q) - q \frac{d\mathcal{A}(q)}{dq}. \quad (15)$$

From (14) and (15), Eq. (12) can also be deduced. Using [18, (28)] and (15), we also get

$$\wp''\left(\frac{\tau}{2}, \tau\right) = 32\pi^4 \left(4\mathcal{A}^2(q) + q \frac{d\mathcal{A}(q)}{dq} \right),$$

where

$$\wp(z; \Lambda_\tau) = \frac{1}{z^2} + \sum_{\substack{\omega \in \Lambda_\tau, \\ \omega \neq 0}} \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\}$$

and $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$ ($\tau \in \mathcal{H}$ the complex upper-half plane) is a lattice and $z \in \mathbb{C}$.

4 The convolution sum $\sum_{k=1}^{N-1} \sigma_1^*(k; 3) \sigma_1^*(N-k; 3)$ and its extensions

Theorem 4.1 *Let N be a positive integer. And let $m, n \in \mathbb{N} \cup \{0\}$. Then we have*

$$\sum_{k=1}^{N-1} \sigma_1^*(3^m k; 3) \sigma_1^*(3^n(N-k); 3) = 3^{n+m-1} \{ \sigma_3^*(N; 3) - N \sigma_1^*(N; 3) \}.$$

Proof From Proposition 3.2 and Eq. (11), we can deduce that

$$\begin{aligned} & \sum_{k=1}^{N-1} \sigma_1^*(3^m k; 3) \sigma_1^*(3^n(N-k); 3) \\ &= 3^{n+m} \sum_{k=1}^{N-1} \left\{ \sigma_1(k) - \sigma_1\left(\frac{k}{3}\right) \right\} \left\{ \sigma_1(N-k) - \sigma_1\left(\frac{N-k}{3}\right) \right\} \\ &= 3^{n+m} \left\{ \sum_{k=1}^{N-1} \sigma_1(k) \sigma_1(N-k) - \sum_{k=1}^{N-1} \sigma_1(k) \sigma_1\left(\frac{N-k}{3}\right) \right. \\ & \quad \left. - \sum_{k=1}^{N-1} \sigma_1\left(\frac{k}{3}\right) \sigma_1(N-k) + \sum_{k=1}^{N-1} \sigma_1\left(\frac{k}{3}\right) \sigma_1\left(\frac{N-k}{3}\right) \right\}. \end{aligned}$$

Now, we note that

$$\begin{aligned} \sum_{k=1}^{N-1} \sigma_1(k) \sigma_1\left(\frac{N-k}{3}\right) &= \sum_{t < N/3} \sigma_1(N-3t) \sigma_1(t), \\ \sum_{k=1}^{N-1} \sigma_1\left(\frac{k}{3}\right) \sigma_1(N-k) &= \sum_{t < N/3} \sigma_1(t) \sigma_1(N-3t), \\ \sum_{k=1}^{N-1} \sigma_1\left(\frac{k}{3}\right) \sigma_1\left(\frac{N-k}{3}\right) &= \sum_{t < N/3} \sigma_1(t) \sigma_1\left(\frac{N}{3} - t\right). \end{aligned}$$

Therefore, the theorem follows from Eq. (1) and Proposition 3.1(b). $\square$

Theorem 4.2 *Let N be a positive integer and let $m, n \in \mathbb{N} \cup \{0\}$. Then we have*

$$\begin{aligned} & \sum_{k=1}^{N-1} \sigma_1^*(2^m k; 2) \sigma_1^*(3^n(N-k); 3) \\ &= \frac{2^m \cdot 3^n}{120} \left[36 \left\{ \sigma_3^*(N) - \sigma_3^*\left(\frac{N}{3}\right) \right\} - 20N \sigma_1^*(N) - 15N \sigma_1^*(N; 3) - a(N) \right]. \end{aligned}$$

Table 4 $\sum_{k=1}^{N-1} \sigma_1^*(2^m k; 2) \sigma_1^*(3^n(N-k); 3)$ and $\sum_{k=1}^{N-1} \sigma_1(2^m k) \sigma_1(3^n(N-k))$

	Convolution formula
$\sum_{k=1}^{N-1} \sigma_1^*(2^m k; 2) \sigma_1^*(3^n(N-k); 3)$	$\frac{2^m \cdot 3^n}{120} [36\{\sigma_3^*(N) - \sigma_3^*(\frac{N}{3})\} - 20N\sigma_1^*(N) - 15N\sigma_1^*(N; 3) - a(N)]$
$\sum_{k=1}^{N-1} \sigma_1(2^m k) \sigma_1(3^n(N-k))$	$\frac{1}{240} [(2^{m+3} - 3)(3^{n+3} - 7)\sigma_3(N) - 8(2^m - 1)(3^{n+3} - 7)\sigma_3(\frac{N}{2}) - 27(2^{m+3} - 3)(3^n - 1)\sigma_3(\frac{N}{3}) + 216(2^m - 1)(3^n - 1)\sigma_3(\frac{N}{6}) - 5\{3 - 2^{m+2} - 3^{n+1}\} + 6(7 \cdot 2^m \cdot 3^n - 2 \cdot 3^n - 2^m)N\}\sigma_1(N) + 20(2^m - 1)(2 \cdot 3^{n+1}N - 1)\sigma_1(\frac{N}{2}) + 15(3^n - 1)(3 \cdot 2^{m+1}N - 1)\sigma_1(\frac{N}{3}) - 6(2^m - 1)(3^n - 1)a(N)]$

Table 5 Some convolution formulas

Convolution sum	Convolution formula
$\sum_{k=1}^{N-1} \sigma_1^*(k) \sigma_5^*(N-k)$	$\frac{1}{408} \{12\sigma_7^*(N) - 17N\sigma_5^*(N) + 5b(N)\}$
$\sum_{k=1}^{N-1} \sigma_3^*(k) \sigma_3^*(N-k)$	$\frac{1}{136} \{\sigma_7^*(N) - b(N)\}$
$\sum_{k=1}^{N-1} \sigma_1^*(k) \sigma_7^*(N-k)$	$\frac{1}{992} \{17\sigma_9^*(N) - 31N\sigma_7^*(N) + 448l(N) + 14c(N)\}$
$\sum_{k=1}^{N-1} \sigma_3^*(k) \sigma_5^*(N-k)$	$\frac{1}{496} \{\sigma_9^*(N) - 32l(N) - c(N)\}$
$\sum_{k=1}^{N-1} \sigma_5^*(k) \sigma_5^*(N-k)$	$\frac{1}{2,764} \{\sigma_{11}^*(N) - \tau(N) + 692\tau(\frac{N}{2})\}$
$\sum_{k=1}^{N-1} \sigma_1^*(k) \sigma_9^*(N-k)$	$\frac{1}{27,640} \{310\sigma_{11}^*(N) - 691N\sigma_9^*(N) + 381\tau(N) + 109,488\tau(\frac{N}{2})\}$
$\sum_{k=1}^{N-1} \sigma_3^*(k) \sigma_7^*(N-k)$	$\frac{1}{22,112} \{17\sigma_{11}^*(N) - 17\tau(N) - 13,112\tau(\frac{N}{2})\}$

Proof We note that

$$\begin{aligned}
 & \sum_{k=1}^{N-1} \sigma_1^*(2^m k; 2) \sigma_1^*(3^n(N-k); 3) \\
 &= 2^m \cdot 3^n \sum_{k=1}^{N-1} \left\{ \sigma_1(k) - \sigma_1\left(\frac{k}{2}\right) \right\} \left\{ \sigma_1(N-k) - \sigma_1\left(\frac{N-k}{3}\right) \right\} \\
 &= 2^m \cdot 3^n \left\{ \sum_{k=1}^{N-1} \sigma_1(k) \sigma_1(N-k) - \sum_{k < N/3} \sigma_1(N-3k) \sigma_1(k) \right. \\
 & \quad \left. - \sum_{k < N/2} \sigma_1(k) \sigma_1(N-2k) + \sum_{2k+3m=N} \sigma_1(k) \sigma_1(m) \right\}.
 \end{aligned}$$

Now, the theorem follows from (1), Proposition 3.1(a), (b), and Eq. (5). $\square$

Remark 4.3 We can compare the two sums

$$\sum_{k=1}^{N-1} \sigma_1^*(2^m k; 2) \sigma_1^*(3^n(N-k); 3), \quad \sum_{k=1}^{N-1} \sigma_1(2^m k) \sigma_1(3^n(N-k))$$

as follows (see Table 4). Here we find the formula for the sum $\sum_{k=1}^{N-1} \sigma_1(2^m k) \sigma_1(3^n(N-k))$ in a similar way as in Theorem 4.2 (see [16, Theorem 2.5]).

5 Congruence relations of coefficients of certain modular-like functions

Lemma 5.1 Let $n \geq 2$. Then we have Table 5.

Proof We use Table 15 in the Appendix, and the proof of Lemma 5.1 is now similar to the proof of Theorem 4.1. $\square$

Remark 5.2 It is easy to observe that

$$\sum_{k=1}^{N-1} \sigma_5^*(k) \sigma_5^*(N-k) = \frac{8}{17} \sum_{k=1}^{N-1} \sigma_3^*(k) \sigma_7^*(N-k)$$

when N is odd.

As an application to the explicit evaluation of convolution sums, using Table 5 we prove the following theorem.

Theorem 5.3 *If N is odd, then*

- (a) $b(N) \equiv N\sigma_5(N) \pmod{48}$. In particular $N \equiv 1 \pmod{8}$, we have $b(N) \equiv N\sigma_5(N) \pmod{192}$.
- (b) $b(N) \equiv \sigma_7(N) \pmod{2,176}$. In particular $N \equiv 1 \pmod{8}$, we have $b(N) \equiv \sigma_7(N) \pmod{17,408}$.
- (c) $\tau(N) \equiv \sigma_{11}(N) \pmod{176,896}$. In particular $N \equiv 1 \pmod{8}$, we have $\tau(N) \equiv \sigma_{11}(N) \pmod{1,415,168}$.

Proof Since the proofs of (a), (b), and (c) are similar, we only prove (c). When N is odd, from Table 5, we see that

$$5,528 \sum_{k=1}^{\frac{N-1}{2}} \sigma_5^*(k) \sigma_5^*(N-k) = \sigma_{11}(N) - \tau(N). \quad (16)$$

Since N is odd, $k \not\equiv N-k \pmod{2}$, and hence either k is even or $N-k$ is even. Then, by (11), $\sigma_5^*(k) = 2^5 \sigma_5^*(\frac{k}{2})$ or $\sigma_5^*(N-k) = 2^5 \sigma_5^*(\frac{N-k}{2})$. So, in general,

$$\tau(N) \equiv \sigma_{11}(N) \pmod{176,896}.$$

In particular, for $N \equiv 1 \pmod{8}$, we have Table 6 and

$$\begin{aligned} \sum_{k=1}^{\frac{N-1}{2}} \sigma_5^*(k) \sigma_5^*(N-k) &= \sum_{k \equiv 0,1 \pmod{4}}^{\frac{N-1}{2}} \sigma_5^*(k) \sigma_5^*(N-k) + \sum_{k \equiv 2,7 \pmod{8}}^{\frac{N-1}{2}} \sigma_5^*(k) \sigma_5^*(N-k) \\ &\quad + \sum_{k \equiv 3,6 \pmod{8}}^{\frac{N-1}{2}} \sigma_5^*(k) \sigma_5^*(N-k). \end{aligned} \quad (17)$$

Table 6 k and $N-k$ by mod 8

$k \pmod{8}$	0	1	2	3	4	5	6	7
$N-k \pmod{8}$	1	0	7	6	5	4	3	2

If $n \equiv 7 \pmod{8}$, then there exists a prime $p \equiv 7 \pmod{8}$ satisfying $p^{2i-1} | n$ and $p^{2i} \nmid n$. Thus

$$\begin{aligned}\sigma_5^*(n) &= \sigma_5^*\left(\frac{n}{p^{2i-1}}\right) \sigma_5^*(p^{2i-1}) \\ &= \sigma_5^*\left(\frac{n}{p^{2i-1}}\right) (1 + p^5 + p^{5 \cdot 2} + \cdots + p^{5(2i-1)}) \\ &\equiv 0 \pmod{8}.\end{aligned}\tag{18}$$

Therefore, we obtain

$$\begin{aligned}\sigma_5^*(8k+2)\sigma_5^*(8k+7) &\equiv 2^5 \sigma_5^*(4k+1)\sigma_5^*(8k+7) \\ &\equiv 0 \pmod{2^8}.\end{aligned}$$

Arguing in a similar manner as in (18), we get

$$\sigma_5^*(8k+3) \equiv 0 \pmod{4}$$

and

$$\sigma_5^*(4k+3) \equiv 0 \pmod{4}.$$

Therefore, we obtain

$$\begin{aligned}\sigma_5^*(8k+3)\sigma_5^*(8k+6) &\equiv 2^5 \sigma_5^*(8k+3)\sigma_5^*(4k+3) \\ &\equiv 0 \pmod{2^9}.\end{aligned}\tag{19}$$

Using $\sigma_5^*(4l) = 2^{10} \sigma_5^*(l)$, (18), and Table 6, we get

$$\sum_{\substack{k=1 \\ k \equiv 0,1 \pmod{4}}}^{\frac{N-1}{2}} \sigma_5^*(k)\sigma_5^*(N-k) \equiv 0 \pmod{2^{10}}\tag{20}$$

and

$$\sum_{\substack{k=1 \\ k \equiv 2,3 \pmod{4}}}^{\frac{N-1}{2}} \sigma_5^*(k)\sigma_5^*(N-k) \equiv 0 \pmod{2^8}.\tag{21}$$

Therefore, by (16), (20), and (21), we have

$$\tau(N) \equiv \sigma_{11}(N) \pmod{1,415,168}.\quad \square$$

Remark 5.4 The same result (c) of Theorem 5.3 has been obtained by Lahiri by a different approach considering the Eisenstein series and so on (see [4, (11.1)-(11.40)], [19, p.28]).

Corollary 5.5 Suppose that $N = 176,896a + b$ is a prime number with $(176,896, b) = 1$. Then $\tau(N) \neq 0$ for $b \neq -1$.

Proof Using Mathematica 8.0, we first find b with $1 \leq b < 176,896$ satisfying $(176,896, b) = 1$. Then if $N = 176,896a + b$ is a prime number, then we find that by Theorem 5.3 (c), using Mathematica 8.0,

$$\tau(N) \equiv \sigma_{11}^*(N) \equiv b^{11} + 1 \not\equiv 0 \pmod{176,896}$$

except for $b = -1$. Thus the corollary follows. $\square$

Example 5.6 Some values of $b(N)$, $N\sigma_5(N)$, $\sigma_7(N)$, $\tau(N)$, and $\sigma_{11}(N)$ are listed in Table 7.

Corollary 5.7 Let $N \in \mathbb{N}$. Then we have Table 8.

In particular, if N is odd, then we have Table 9.

Table 7 Some values of $b(N)$, $N\sigma_5(N)$, $\sigma_7(N)$, $\tau(N)$, and $\sigma_{11}(N)$

N	1	3	5	7	9
$b(N)$	1	12	-210	1,016	-2,043
$N\sigma_5(N)$	1	732	15,630	117,656	533,637
$\sigma_7(N)$	1	2,188	78,126	823,544	4,785,157
$\tau(N)$	1	252	4,830	-16,744	-113,643
$\sigma_{11}(N)$	1	177,148	48,828,126	1,977,326,744	31,381,236,757

Table 8 Some convolution formulas

Convolution sum	Convolution formula
$\sum_{k=1}^N \sigma_1(2k-1)\sigma_5(2N-(2k-1))$	$\frac{1}{408} \{768\sigma_7^*(N) + 5b(2N) - 320b(N)\}$
$\sum_{k=1}^N \sigma_3(2k-1)\sigma_3(2N-(2k-1))$	$\frac{1}{136} \{64\sigma_7^*(N) - b(2N) + 64b(N)\}$
$\sum_{k=1}^N \sigma_1(2k-1)\sigma_7(2N-(2k-1))$	$\frac{1}{496} \{2,176\sigma_9^*(N) + 224d(2N) - 57,344l(N) + 7c(2N) - 1,792c(N)\}$
$\sum_{k=1}^N \sigma_3(2k-1)\sigma_5(2N-(2k-1))$	$\frac{1}{496} \{256\sigma_9^*(N) - 32d(2N) + 8,192l(N) - c(2N) + 256c(N)\}$
$\sum_{k=1}^N \sigma_5(2k-1)\sigma_5(2N-(2k-1))$	$\frac{1}{2,764} \{1,024\sigma_{11}^*(N) - \tau(2N) + 1,716\tau(N) - 708,608\tau(\frac{N}{2})\}$
$\sum_{k=1}^N \sigma_1(2k-1)\sigma_9(2N-(2k-1))$	$\frac{1}{27,640} \{317,440\sigma_{11}^*(N) + 381\tau(2N) - 280,656\tau(N) - 112,115,712\tau(\frac{N}{2})\}$
$\sum_{k=1}^N \sigma_3(2k-1)\sigma_7(2N-(2k-1))$	$\frac{1}{22,112} \{17,408\sigma_{11}^*(N) - 17\tau(2N) + 4,296\tau(N) + 13,426,688\tau(\frac{N}{2})\}$

Table 9 Some convolution formulas for odd N

Convolution sum	Convolution formula
$\sum_{k=1}^N \sigma_1(2k-1)\sigma_5(2N-(2k-1))$	$\frac{1}{17} \{32\sigma_7(N) - 15b(N)\}$
$\sum_{k=1}^N \sigma_3(2k-1)\sigma_3(2N-(2k-1))$	$\frac{1}{17} \{8\sigma_7(N) + 9b(N)\}$
$\sum_{k=1}^N \sigma_1(2k-1)\sigma_7(2N-(2k-1))$	$\frac{1}{496} \{2,176\sigma_9(N) + 224d(2N) - 57,344l(N) + 7c(2N) - 1,792c(N)\}$
$\sum_{k=1}^N \sigma_3(2k-1)\sigma_5(2N-(2k-1))$	$\frac{1}{496} \{256\sigma_9(N) - 32d(2N) + 8,192l(N) - c(2N) + 256c(N)\}$
$\sum_{k=1}^N \sigma_5(2k-1)\sigma_5(2N-(2k-1))$	$\frac{1}{691} \{256\sigma_{11}(N) + 435\tau(N)\}$
$\sum_{k=1}^N \sigma_1(2k-1)\sigma_9(2N-(2k-1))$	$\frac{1}{691} \{7,936\sigma_{11}(N) - 7,245\tau(N)\}$
$\sum_{k=1}^N \sigma_3(2k-1)\sigma_7(2N-(2k-1))$	$\frac{1}{691} \{544\sigma_{11}(N) + 147\tau(N)\}$

Proof Since the proofs for the convolution sums are similar, we only prove $\sum_{k=1}^N \sigma_1(2k-1)\sigma_5(2N-(2k-1))$. We write $\sum_{k=1}^{2N-1} \sigma_1^*(k)\sigma_5^*(2N-k)$ as follows:

$$\begin{aligned} & \sum_{k=1}^{2N-1} \sigma_1^*(k)\sigma_5^*(2N-k) \\ &= \sum_{k=1}^N \sigma_1^*(2k-1)\sigma_5^*(2N-(2k-1)) + \sum_{k=1}^{N-1} \sigma_1^*(2k)\sigma_5^*(2N-2k) \\ &= \sum_{k=1}^N \sigma_1^*(2k-1)\sigma_5^*(2N-(2k-1)) + 2^6 \sum_{k=1}^{N-1} \sigma_1^*(k)\sigma_5^*(N-k) \end{aligned} \quad (22)$$

by (11). Also since $2k-1$ and $2N-(2k-1)$ are odd, we find that

$$\sum_{k=1}^N \sigma_1^*(2k-1)\sigma_5^*(2N-(2k-1)) = \sum_{k=1}^N \sigma_1(2k-1)\sigma_5(2N-(2k-1)).$$

So, we obtain the formula for $\sum_{k=1}^N \sigma_1(2k-1)\sigma_5(2N-(2k-1))$ by (22) and Table 5. In particular, if N is odd, then $b(2N) = b(2)b(N)$, because $b(N)$ is multiplicative. Therefore the proof is complete. $\square$

By eliminating $b(N)$, $c(N)$, and $l(N)$ in Table 5, we can obtain the following example.

Example 5.8 Let $N \in \mathbb{N}$. Then for $k = 2, 3, 4$, we have

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \left(\sum_{m=1}^{N-1} \sigma_{2k-2s-1}^*(m)\sigma_{2s+1}^*(N-m) \right) = \frac{1}{2} \{ \sigma_{2k+1}^*(N) - N\sigma_{2k-1}^*(N) \}.$$

For $k = 2$, we refer to Eq. (1).

Now we present some convolution formulas in Table 10.

6 Certain combinatorial convolution sum

The four basic theta functions are defined below following the notation of Whittaker and Watson [20, p.464]. Let $\tau \in \mathbb{C}$ be such that $\text{Im}(\tau) > 0$. Set $q = e^{\pi i \tau}$ so that $|q| < 1$. For $z \in \mathbb{C}$, we define (as in [17])

$$\theta_1(z, q) := 2 \sum_{N=1}^{\infty} (-1)^{N-1} q^{\frac{(2N-1)^2}{4}} \sin(2N-1)z,$$

$$\theta_2(z, q) := 2 \sum_{N=1}^{\infty} q^{\frac{(2N-1)^2}{4}} \cos(2N-1)z,$$

$$\theta_3(z, q) := 1 + 2 \sum_{N=1}^{\infty} q^{N^2} \cos 2Nz,$$

$$\theta_4(z, q) := 1 + 2 \sum_{N=1}^{\infty} (-1)^N q^{N^2} \cos 2Nz.$$

Table 10 Some convolution formulas

Convolution sum	Convolution formula
$\sum_{k=1}^N \sigma_1(2k)\sigma_5(2N-2k)$	$\frac{1}{8,568} \{4\{85\sigma_7(2N) - 4,032\sigma_9^*(N)\}$ $- 357(4N-1)\sigma_5(2N) + 17\sigma_1(2N)$ $- 105\{b(2N) - 64b(N)\}\}$
$\sum_{k=1}^N \sigma_3(2k)\sigma_3(2N-2k)$	$\frac{1}{2,040} \{17\sigma_7(2N) - 960\sigma_9^*(N)$ $- 17\sigma_3(2N) + 15b(2N) - 960b(N)\}$
$\sum_{k=1}^N \sigma_1(2k)\sigma_7(2N-2k)$	$\frac{1}{14,880} \{341\sigma_9(2N) - 65,280\sigma_9^*(N)$ $- 620(3N-1)\sigma_7(2N) - 31\sigma_1(2N)$ $- 6,720/(2N) + 1,720,320/(N)$ $- 210c(2N) + 53,760c(N)\}$
$\sum_{k=1}^N \sigma_3(2k)\sigma_5(2N-2k)$	$\frac{1}{156,240} \{341\sigma_9(2N) - 80,640\sigma_9^*(N)$ $- 651\sigma_5(2N) + 310\sigma_3(2N)$ $+ 10,080/(2N) - 25,80,480/(N)$ $+ 315c(2N) - 80,640c(N)\}$
$\sum_{k=1}^N \sigma_5(2k)\sigma_5(2N-2k)$	$\frac{1}{174,132} \{65\sigma_{11}(2N) - 64,512\sigma_{11}^*(N)$ $+ 691\sigma_5(2N) - 693\tau(2N)$ $- 108,108\tau(N) + 44,642,304\tau(\frac{N}{2})\}$
$\sum_{k=1}^N \sigma_1(2k)\sigma_9(2N-2k)$	$\frac{1}{912,120} \{3\{4,550\sigma_{11}(2N) - 3,491,840\sigma_{11}^*(N)\}$ $- 7,601(12N-5)\sigma_9(2N) + 3,455\sigma_1(2N)$ $+ 3\{-7,359\tau(2N) + 3,087,216\tau(N)$ $+ 1,233,272,832\tau(\frac{N}{2})\}\}$
$\sum_{k=1}^N \sigma_3(2k)\sigma_7(2N-2k)$	$\frac{1}{331,680} \{3\{91\sigma_{11}(2N) - 87,040\sigma_{11}^*(N)\}$ $- 1,382\sigma_7(2N) - 691\sigma_3(2N)$ $+ 3\{685\tau(2N) - 21,480\tau(N)$ $- 67,133,440\tau(\frac{N}{2})\}\}$

Jacobi (see [21, 22]) proved that

$$\frac{\theta_2(0, q)\theta_3(0, q)\theta_1(z, q)}{\theta_4(z, q)} = 4 \sum_{m=0}^{\infty} \frac{q^{\frac{2m+1}{2}}}{1 - q^{2m+1}} \sin(2m+1)z \quad (23)$$

and

$$\frac{\theta_2^2(0, q)\theta_3^2(0, q)\theta_1^2(z, q)}{\theta_4^2(z, q)} = 8 \sum_{m=0}^{\infty} \frac{mq^m}{1 - q^{2m}} (1 - \cos 2mz). \quad (24)$$

From (23) and (24), we deduce that

$$\sum_{m=1}^{\infty} \frac{mq^m}{1 - q^{2m}} (1 - \cos 2mz) = 2 \left(\sum_{m=0}^{\infty} \frac{q^{\frac{2m+1}{2}}}{1 - q^{2m+1}} \sin(2m+1)z \right)^2. \quad (25)$$

Equating coefficients of q^N ($N \in \mathbb{N}$) on the left- and right-hand sides of (25) (see [17, p.16]), then we obtain the arithmetical equality involving the trigonometric functions

$$\sum_{\substack{m \in \mathbb{N} \\ m|N \\ \frac{N}{m} \text{ odd}}} m(1 - \cos 2mz) = \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=2N \\ a,b,x,y \text{ odd}}} (\cos(a-b)z - \cos(a+b)z). \quad (26)$$

If we expand each cosine in powers of z using

$$1 - \cos z = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{z^{2k}}{2k!}$$

and equate coefficients of $\frac{z^{2k}}{2k!}$ ($k \in \mathbb{N}$), then we obtain

$$\sum_{\substack{m \in \mathbb{N} \\ m|N \\ \frac{N}{m} \text{ odd}}} 2^{2k} m^{2k+1} = \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=2N \\ a,b,x,y \text{ odd}}} ((a+b)^{2k} - (a-b)^{2k}), \quad k, N \in \mathbb{N}. \quad (27)$$

A generalized Besge formula due to Liouville is as follows.

Proposition 6.1 (See [17, Theorem 12.3]) *Let $k \in \mathbb{N}$ and $N \in \mathbb{N}$, where $k, N \geq 2$. Then we have*

$$\begin{aligned} & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \left(\sum_{m=1}^{N-1} \sigma_{2k-2s-1}(m) \sigma_{2s+1}(N-m) \right) \\ &= \frac{2k+3}{4k+2} \sigma_{2k+1}(N) + \left(\frac{k}{6} - N \right) \sigma_{2k-1}(N) + \frac{1}{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}(N), \end{aligned}$$

where B_{2j} is the $2j$ th Bernoulli number.

Let $\sigma_{s,oo}(N) := \sum_{\substack{d|N \\ d \text{ odd} \\ \frac{N}{d} \text{ odd}}} d^s$. Now we prove the following lemma.

Lemma 6.2 *We have*

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{2N-1} \sigma_{2k-2s-1,oo}(m) \sigma_{2s+1,oo}(2N-m) = \frac{1}{4} \sigma_{2k+1}^*(2N).$$

Proof First we consider

$$\begin{aligned} & \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=2N, \\ a \equiv b \equiv x \equiv y \equiv 1 \pmod{2}}} ((a+b)^{2k} - (a-b)^{2k}) \\ &= \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=2N, \\ a,b,x,y \text{ odd}}} \left(\sum_{r=0}^{2k} \binom{2k}{r} a^{2k-r} b^r - \sum_{r=0}^{2k} \binom{2k}{r} (-1)^r a^{2k-r} b^r \right) \\ &= 2 \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=2N, \\ a,b,x,y \text{ odd}}} \sum_{\substack{r=0, \\ r \text{ odd}}}^{2k} \binom{2k}{r} a^{2k-r} b^r = 2 \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=2N, \\ a,b,x,y \text{ odd}}} a^{2k-2s-1} b^{2s+1} \\ &= 2 \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{2N-1} \left(\sum_{\substack{a|m, \\ a \text{ odd}, \\ \frac{m}{a} \text{ odd}}} a^{2k-2s-1} \right) \left(\sum_{\substack{b|2N-m, \\ b \text{ odd}, \\ \frac{2N-m}{b} \text{ odd}}} b^{2s+1} \right) \\ &= 2 \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{2N-1} \sigma_{2k-2s-1,oo}(m) \sigma_{2s+1,oo}(2N-m). \end{aligned}$$

Now the right-hand side of (27) is

$$\begin{aligned} \sum_{\substack{d \in \mathbb{N}, \\ d|N, \\ d \nmid \frac{N}{2}}} 2^{2k} d^{2k+1} &= 2^{2k} \left\{ \sum_{d|N} d^{2k+1} - \sum_{d|\frac{N}{2}} d^{2k+1} \right\} \\ &= 2^{2k} \left\{ \sigma_{2k+1}(N) - \sigma_{2k+1}\left(\frac{N}{2}\right) \right\} = 2^{2k} \sigma_{2k+1}^*(N) = \frac{1}{2} \sigma_{2k+1}^*(2N) \end{aligned}$$

by (11). Therefore we obtain

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,oo}(m) \sigma_{2s+1,oo}(N-m) = \frac{1}{4} \sigma_{2k+1}^*(N).$$

□

As a consequence of Proposition 6.1 and Lemma 6.2, we have the following.

Theorem 6.3

$$\begin{aligned} \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1}(2m) \sigma_{2s+1}(2N-2m) \\ = \frac{2k+3}{4k+2} \sigma_{2k+1}(2N) - \frac{1}{4} \sigma_{2k+1}^*(2N) + \left(\frac{k}{6} - 2N\right) \sigma_{2k-1}(2N) \\ + \frac{1}{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}(2N), \end{aligned}$$

where B_{2j} is the $2j$ th Bernoulli number.

Proof From Proposition 6.1, we get

$$\begin{aligned} \sum_{s=0}^{k-1} \binom{2k}{2s+1} \left(\sum_{m=1}^{2N-1} \sigma_{2k-2s-1}(m) \sigma_{2s+1}(2N-m) \right) \\ = \frac{2k+3}{4k+2} \sigma_{2k+1}(2N) + \left(\frac{k}{6} - 2N\right) \sigma_{2k-1}(2N) \\ + \frac{1}{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}(2N) \end{aligned} \quad (28)$$

by replacing $N \rightarrow 2N$. From Lemma 6.2 we know that

$$\begin{aligned} \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{2N-1} \sigma_{2k-2s-1,oo}(m) \sigma_{2s+1,oo}(2N-m) &= \frac{1}{4} \sigma_{2k+1}^*(2N) \\ &= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^N \sigma_{2k-2s-1}(2m-1) \sigma_{2s+1}(2N-2m+1). \end{aligned} \quad (29)$$

By subtracting (29) from (28), the theorem follows.

□

Remark 6.4 It is well known that

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_{2j+1} = 0 \quad (j \geq 1), \dots$$

In Proposition 6.1, taking $N = 1$, we obtain

$$\begin{aligned} 0 &= \frac{2k+3}{4k+2} + \left(\frac{k}{6} - 1\right) + \frac{1}{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \\ &= \left(\frac{1}{2k+1} + \frac{1}{2}\right) + \left(\frac{k}{6} - 1\right) + \frac{1}{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \\ &= \frac{1}{2k+1} \binom{2k+1}{0} B_0 + \frac{1}{2k+1} \binom{2k+1}{1} B_1 + \frac{1}{2k+1} \binom{2k+1}{2} B_2 \\ &\quad + \frac{1}{2k+1} \sum_{j=3}^{2k} \binom{2k+1}{j} B_j \end{aligned}$$

and this implies the well-known identity involving the Bernoulli numbers B_j

$$\frac{1}{2k+1} \sum_{j=0}^{2k} \binom{2k+1}{j} B_j = 0. \quad (30)$$

In Theorem 6.3, taking $N = 1$, we obtain

$$\begin{aligned} 0 &= \left(\frac{1}{2k+1} + \frac{1}{2}\right) (1 + 2^{2k+1}) + \left(\frac{k}{6} - 1 - 1\right) (1 + 2^{2k-1}) - \frac{1}{4} \cdot 2^{2k+1} \\ &\quad + \frac{1}{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2k+1-2j} \cdot (1 + 2^{2k+1-2j}). \end{aligned}$$

Then we have

$$\begin{aligned} 0 &= \left(\frac{1}{2k+1} + \frac{1}{2}\right) + \left(\frac{k}{6} - 1\right) + \frac{1}{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2k+1-2j} \\ &\quad + \frac{1}{2k+1} 2^{2k+1} + 2^{2k} - 2^{2k-1} + \frac{k}{6} \cdot 2^{2k-1} \\ &\quad + \frac{1}{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2k+1-2j} 2^{2k+1-2j} - 1 - 2^{2k-1} - 2^{2k-1}. \end{aligned} \quad (31)$$

Using (30) and (31), we get

$$\begin{aligned} 0 &= \frac{1}{2k+1} \binom{2k+1}{0} 2^{2k+1} - \frac{1}{2k+1} \binom{2k+1}{1} 2^{2k+1-1} \left(-\frac{1}{2}\right) \\ &\quad + \frac{1}{2k+1} \binom{2k+1}{2} 2^{2k+1-2} + \frac{1}{2k+1} \sum_{j=3}^{2k} \binom{2k+1}{j} (-1)^j B_{2k+1-2j} 2^{2k+1-2j}, \end{aligned}$$

and hence again we obtain a more general relation between Bernoulli numbers and the Faulhaber sum

$$\frac{1}{2k+1} \sum_{j=0}^{2k} \binom{2k+1}{j} (-1)^j B_{2k+1-j} 2^{2k+1-j} = 1 + 2^{2k} = S_{2k}(3). \quad (32)$$

See, for example, [23].

Appendix

The values of $a(N)$, $b(N)$, $c(N)$, $l(N)$, and $\tau(N)$ (for $(1 \leq N \leq 45)$) are given in Tables 11, 12, 13, 14, and 15 respectively. We first write the corresponding product expression into a q -series sum of a finite number of terms (*i.e.*, q -series sum of N number of terms) with an error which tends to zero as $N \rightarrow \infty$. Then we use Mathematica 8.0 to compare the corresponding coefficients up to $N \leq 45$.

Evaluation of certain convolution sums: We have Table 16.

Table 11 $a(N)$ for N ($1 \leq N \leq 45$)

N	$a(N)$	N	$a(N)$	N	$a(N)$
1	1	16	16	31	-88
2	-2	17	-126	32	-32
3	-3	18	-18	33	-36
4	4	19	20	34	252
5	6	20	24	35	-96
6	6	21	48	36	36
7	-16	22	-24	37	254
8	-8	23	168	38	-40
9	9	24	24	39	-114
10	-12	25	-89	40	-48
11	12	26	-76	41	42
12	-12	27	-27	42	-96
13	38	28	-64	43	-52
14	32	29	30	44	48
15	-18	30	36	45	54

Table 12 $b(N)$ for N ($1 \leq N \leq 45$)

N	$b(N)$	N	$b(N)$	N	$b(N)$
1	1	16	4,096	31	227,552
2	-8	17	14,706	32	-32,768
3	12	18	16,344	33	13,104
4	64	19	-39,940	34	-117,648
5	-210	20	-13,440	35	-213,360
6	-96	21	12,192	36	-130,752
7	1,016	22	-8,736	37	160,526
8	-512	23	68,712	38	319,520
9	-2,043	24	-6,144	39	16,584
10	1,680	25	-34,025	40	107,520
11	1,092	26	-11,056	41	10,842
12	768	27	-50,760	42	-97,536
13	1,382	28	65,024	43	-630,748
14	-8,128	29	-102,570	44	69,888
15	-2,520	30	20,160	45	429,030

Table 13 $c(N)$ for N ($1 \leq N \leq 45$)

N	$c(N)$	N	$c(N)$	N	$c(N)$
1	1	16	-65,536	31	-5,370,592
2	-16	17	311,442	32	-1,048,576
3	100	18	-74,448	33	-1,704,144
4	-256	19	128,244	34	3,962,592
5	-154	20	-222,720	35	2,533,552
6	2,496	21	-912,352	36	-1,191,168
7	-4,536	22	898,368	37	8,258,694
8	-4,096	23	767,192	38	-5,046,080
9	23,085	24	638,976	39	-14,176,264
10	-13,920	25	-1,405,121	40	-3,563,520
11	-38,996	26	-2,849,504	41	12,739,178
12	39,936	27	2,699,496	42	-2,376,192
13	37,806	28	243,712	43	11,055,308
14	15,232	29	1,659,454	44	14,373,888
15	-146,472	30	2,171,520	45	-12,992,274

Table 14 $l(N)$ for N ($1 \leq N \leq 45$)

N	$l(N)$	N	$l(N)$	N	$l(N)$
1	0	16	4,096	31	126,912
2	1	17	-17,472	32	65,536
3	-8	18	4,653	33	326,976
4	16	19	5,848	34	-247,662
5	32	20	13,920	35	-105,056
6	-156	21	33,152	36	74,448
7	112	22	-56,148	37	-123,488
8	256	23	-17,584	38	315,380
9	-576	24	-39,936	39	-425,200
10	870	25	6,528	40	222,720
11	-536	26	178,094	41	-350,848
12	-2,496	27	-11,088	42	148,512
13	4,384	28	-15,232	43	706,728
14	-952	29	-171,872	44	-898,368
15	336	30	-135,720	45	532,512

Table 15 $\tau(N)$ for N ($1 \leq N \leq 45$)

N	$\tau(N)$	N	$\tau(N)$	N	$\tau(N)$
1	1	16	987,136	31	-52,843,168
2	-24	17	-6,905,934	32	-196,706,304
3	252	18	2,727,432	33	134,722,224
4	-1,472	19	10,661,420	34	165,742,416
5	4,830	20	-7,109,760	35	-80,873,520
6	-6,048	21	-4,219,488	36	167,282,496
7	-16,744	22	-12,830,688	37	-182,213,314
8	84,480	23	18,643,272	38	-255,874,080
9	-113,643	24	21,288,960	39	-145,589,976
10	-115,920	25	-25,499,225	40	408,038,400
11	534,612	26	13,865,712	41	308,120,442
12	-370,944	27	-73,279,080	42	101,267,712
13	-577,738	28	24,647,168	43	-17,125,708
14	401,856	29	128,406,630	44	-786,948,864
15	1,217,160	30	-29,211,840	45	-548,895,690

Table 16 Some convolution formulas

Convolution sum	Convolution formulas	Reference
$\sum_{k=1}^{N-1} \sigma(k) \sigma_5(N-k)$	$\frac{1}{504} \{20\sigma_7(N) + (21 - 42N)\sigma_5(N) + \sigma(N)\}$	[3, (3.18)]
$\sum_{k=1}^{N-1} \sigma_3(k) \sigma_3(N-k)$	$\frac{1}{120} \{\sigma_7(N) - \sigma_3(N)\}$	[3, (3.17)]
$\sum_{k=1}^{N-1} \sigma(k) \sigma_7(N-k)$	$\frac{1}{480} \{11\sigma_9(N) + (20 - 30N)\sigma_7(N) - \sigma(N)\}$	[3, (3.28)]
$\sum_{k=1}^{N-1} \sigma_3(k) \sigma_5(N-k)$	$\frac{1}{5,040} \{11\sigma_9(N) - 21\sigma_5(N) + 10\sigma_3(N)\}$	[3, (3.27)]
$\sum_{k=1}^{N-1} \sigma_5(k) \sigma_5(N-k)$	$\frac{3}{691} \{ \frac{65}{756} \sigma_{11}(N) + \frac{691}{756} \sigma_5(N) - \tau(N) \}$	[4, (21)]
$\sum_{k=1}^{N-1} \sigma_1(k) \sigma_9(N-k)$	$\frac{455}{30,404} \sigma_{11}(N) - \frac{N}{20} \sigma_9(N) + \frac{1}{24} \sigma_9(N) + \frac{1}{264} \sigma_1(N) - \frac{36}{3,455} \tau(N)$	[4, (19)]
$\sum_{k=1}^{N-1} \sigma_3(k) \sigma_7(N-k)$	$\frac{91}{110,560} \sigma_{11}(N) - \frac{1}{240} \sigma_7(N) - \frac{1}{480} \sigma_3(N) + \frac{15}{2,764} \tau(N)$	[4, (20)]
$\sum_{k < N/2} \sigma_5(k) \sigma_1(N-2k)$	$\frac{1}{2,142} \{ \sigma_7(N) + \frac{2}{51} \sigma_7(\frac{N}{2}) + \frac{1-2N}{24} \sigma_5(\frac{N}{2}) + \frac{1}{504} \sigma_1(N) - \frac{1}{408} b(N) \}$	[24, Theorem 5.2]
$\sum_{k < N/2} \sigma_1(k) \sigma_5(N-2k)$	$\frac{1}{102} \{ \sigma_7(N) + \frac{32}{1,071} \sigma_7(\frac{N}{2}) + \frac{1-N}{24} \sigma_5(N) + \frac{1}{504} \sigma_1(\frac{N}{2}) - \frac{1}{102} b(N) \}$	[24, Theorem 5.2]
$\sum_{k < N/2} \sigma_3(k) \sigma_3(N-2k)$	$\frac{1}{2,040} \{ \sigma_7(N) + \frac{2}{255} \sigma_7(\frac{N}{2}) - \frac{1}{240} \sigma_3(N) - \frac{1}{240} \sigma_3(\frac{N}{2}) + \frac{1}{272} b(N) \}$	[24, Theorem 5.2]
$\sum_{k < N/2} \sigma_7(k) \sigma_1(N-2k)$	$\frac{1}{14,880} \{ \sigma_9(N) + \frac{17}{744} \sigma_9(\frac{N}{2}) + \frac{2-3N}{48} \sigma_7(\frac{N}{2}) - \frac{1}{480} \sigma_1(N) + \frac{1}{496} c(N) + \frac{2}{31} l(N) \}$	[24, Theorem 6.2]
$\sum_{k < N/2} \sigma_1(k) \sigma_7(N-2k)$	$\frac{17}{2,976} \{ \sigma_9(N) + \frac{8}{465} \sigma_9(\frac{N}{2}) + \frac{4-3N}{96} \sigma_7(N) - \frac{1}{480} \sigma_1(\frac{N}{2}) - \frac{1}{62} c(N) - \frac{16}{31} l(N) \}$	[24, Theorem 6.2]
$\sum_{k < N/2} \sigma_5(k) \sigma_3(N-2k)$	$\frac{1}{31,248} \{ \sigma_9(N) + \frac{1}{465} \sigma_9(\frac{N}{2}) - \frac{1}{240} \sigma_5(\frac{N}{2}) + \frac{1}{504} \sigma_3(N) - \frac{1}{496} c(N) - \frac{2}{31} l(N) \}$	[24, Theorem 6.2]
$\sum_{k < N/2} \sigma_3(k) \sigma_5(N-2k)$	$\frac{1}{7,440} \{ \sigma_9(N) + \frac{4}{1,953} \sigma_9(\frac{N}{2}) - \frac{1}{240} \sigma_5(N) + \frac{1}{504} \sigma_3(\frac{N}{2}) + \frac{1}{248} c(N) + \frac{4}{31} l(N) \}$	[24, Theorem 6.2]
$\sum_{k < N/2} \sigma_5(k) \sigma_5(N-2k)$	$\frac{1}{174,132} \{ \sigma_{11}(N) + \frac{16}{43,533} \sigma_{11}(\frac{N}{2}) + \frac{1}{504} \sigma_5(N) + \frac{1}{504} \sigma_5(\frac{N}{2}) - \frac{11}{5,528} \tau(N) - \frac{88}{691} \tau(\frac{N}{2}) \}$	[24, Theorem 7.2]
$\sum_{k < N/2} \sigma_9(k) \sigma_1(N-2k)$	$\frac{1}{91,212} \{ \sigma_{11}(N) + \frac{31}{2,073} \sigma_{11}(\frac{N}{2}) + \frac{5-6N}{120} \sigma_9(\frac{N}{2}) + \frac{1}{264} \sigma_1(N) - \frac{21}{5,528} \tau(N) - \frac{282}{3,455} \tau(\frac{N}{2}) \}$	[24, Theorem 7.2]
$\sum_{k < N/2} \sigma_1(k) \sigma_9(N-2k)$	$\frac{31}{8,292} \{ \sigma_{11}(N) + \frac{256}{22,803} \sigma_{11}(\frac{N}{2}) + \frac{5-3N}{120} \sigma_9(N) + \frac{1}{264} \sigma_1(\frac{N}{2}) - \frac{141}{6,910} \tau(N) - \frac{2,688}{691} \tau(\frac{N}{2}) \}$	[24, Theorem 7.2]
$\sum_{k < N/2} \sigma_7(k) \sigma_3(N-2k)$	$\frac{1}{331,680} \{ \sigma_{11}(N) + \frac{17}{20,730} \sigma_{11}(\frac{N}{2}) - \frac{1}{240} \sigma_7(\frac{N}{2}) - \frac{1}{480} \sigma_3(N) + \frac{23}{11,056} \tau(N) + \frac{91}{1,382} \tau(\frac{N}{2}) \}$	[24, Theorem 7.2]
$\sum_{k < N/2} \sigma_3(k) \sigma_7(N-2k)$	$\frac{17}{331,680} \{ \sigma_{11}(N) + \frac{8}{10,365} \sigma_{11}(\frac{N}{2}) - \frac{1}{240} \sigma_7(N) - \frac{1}{480} \sigma_3(\frac{N}{2}) + \frac{91}{22,112} \tau(N) + \frac{368}{691} \tau(\frac{N}{2}) \}$	[24, Theorem 7.2]

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript and typed, read and approved the final manuscript.

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