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Hyers-Ulam-Rassias stability of generalized module left (m, n)-derivations

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Abstract

The generalized Hyers-Ulam-Rassias stability of generalized module left (m, n)-derivations on a normed algebra $\mathcal{A}$ into a Banach left $\mathcal{A}$-module is established.

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Keywords: Hyers-Ulam-Rassias stability; normed algebra; Banach left $\mathcal{A}$-module; module left (m, n)-derivation; generalized module left (m, n)-derivation

1 Introduction

We say that a functional equation $\mathcal{E}$ is stable if any function f which approximately satisfies the equation $\mathcal{E}$ is near to an exact solution of $\mathcal{E}$. The problem of stability of functional equations was formulated by Ulam in 1940 for group homomorphisms (see [1, 2]). One year later, Ulam's problem was affirmatively solved by Hyers [3] for the Cauchy functional equation $f(x + y) = f(x) + f(y)$. This gave rise to the stability theory of functional equations. Later, Aoki [4] and Rassias [5] considered mappings from a normed space into a Banach space such that the norm of the Cauchy difference is bounded by the expression $\varepsilon(\|x\|^p + \|y\|^p)$ for all x, y, some $\varepsilon \geq 0$, and $0 \leq p < 1$. The terminology Hyers-Ulam-Rassias stability indeed originates from Rassias's paper.

In the last few decades, the stability problem of several functional equations has been extensively studied by many authors. For the history and various aspects of the theory, we refer to monographs [6–8]. We also refer the reader to the paper [9], where a precise description of the Hyers-Ulam-Rassias stability is given.

As we are aware, the stability of derivations was first investigated by Jun and Park [10]. During the past few years, approximate derivations were studied by a number of mathematicians (see [11–16] and references therein). The stability result concerning derivations between operator algebras was first obtained by Šemrl [17]. Later, Moslehian [14] studied approximate generalized derivations on unital Banach algebras into Banach bimodules, and in [8] Jung examined the stability of module left derivations. Recently, the author studied the generalized Hyers-Ulam-Rassias stability of a functional inequality associated with module left (m, n)-derivations [18]. The natural question here is whether we can generalize these results in the setting of generalized module left (m, n)-derivations. Theorem 2.2 answers this question in the affirmative.

In the following, $\mathcal{A}$ and $\mathcal{M}$ will represent a complex normed algebra and a Banach left $\mathcal{A}$-module, respectively. Recall that if $\mathcal{A}$ has a unit element e such that $e \cdot x = x$ for all $x \in \mathcal{M}$, then a left $\mathcal{A}$-module $\mathcal{M}$ is called unitary. Here, $\cdot$ denotes the module multiplication on $\mathcal{M}$.

We will use the same symbol $\|\cdot\|$ to represent the norms on a normed algebra $\mathcal{A}$ and a normed left $\mathcal{A}$-module $\mathcal{M}$.

Before stating our main theorem, let us write some basic definitions and known results which we will need in the sequel. First, an additive mapping $d:\mathcal{A}\to\mathcal{M}$ is called a module left derivation if $d(xy)=x\cdot d(y)+y\cdot d(x)$ holds for all $x,y\in\mathcal{A}$. Let $m,n\geq 0$ with $m+n\neq 0$ be some fixed integers. Then an additive mapping $d:\mathcal{A}\to\mathcal{M}$ is called a module left (m,n)-derivation if

$$(m+n)d(xy)=2mx\cdot d(y)+2ny\cdot d(x)$$

for all $x,y\in\mathcal{A}$. Clearly, module left (m,n)-derivations are one of the natural generalizations of module left derivations (the case $m=n$). Furthermore, an additive mapping $g:\mathcal{A}\to\mathcal{M}$ is called a generalized module left derivation if there exists a module left derivation $d:\mathcal{A}\to\mathcal{M}$ such that $g(xy)=x\cdot g(y)+y\cdot d(x)$ is fulfilled for all $x,y\in\mathcal{A}$. Motivated by this notion, we define a generalized module left (m,n)-derivation as an additive mapping $g:\mathcal{A}\to\mathcal{M}$ for which there exists a module left (m,n)-derivation $d:\mathcal{A}\to\mathcal{M}$ such that

$$(m+n)g(xy)=2mx\cdot g(y)+2ny\cdot d(x)$$

for all $x,y\in\mathcal{A}$. Obviously, if $m=n$, then every generalized module left (m,n)-derivation is a generalized module left derivation.

In the last few decades, a lot of work has been done in the field of left derivations and generalized left derivations (see, for example, [19, 20] and references therein). Recently also (m,n)-derivations and generalized (m,n)-derivations have been defined and investigated [21–24]. This motivated us to study the generalized Hyers-Ulam-Rassias stability of functional inequalities associated with generalized module left (m,n)-derivations.

2 The main result

Throughout the paper, we assume that m and n are nonnegative integers with $m+n\neq 0$. We say that an additive mapping $f:\mathcal{A}\to\mathcal{M}$ is $\mathbb{C}$-linear (or just linear) if $f(\lambda x)=\lambda f(x)$ for all $x\in\mathcal{A}$ and all scalars $\lambda\in\mathbb{C}$. In the following, Λ denotes the set of all complex units, *i.e.*,

$$\Lambda=\{\lambda\in\mathbb{C}:|\lambda|=1\}.$$

For a given additive mapping $f:\mathcal{A}\to\mathcal{M}$, Park [25] obtained the next result.

Lemma 2.1 *If $f(\lambda x)=\lambda f(x)$ for all $x\in\mathcal{A}$ and all $\lambda\in\Lambda$, then f is $\mathbb{C}$-linear.*

Our first result is a generalization of Theorem 1 in [18].

Theorem 2.2 *Let $\mathcal{A}$ be a normed algebra, $\mathcal{M}$ be a Banach left $\mathcal{A}$-module, and $F:\mathcal{A}^2\to[0,\infty)$ be a function such that $F(2x,y)=\xi F(x,y)$ and $F(x,2y)=\eta F(x,y)$ for some nonnegative scalars ξ, η with $\xi\eta<1$. Suppose that $g:\mathcal{A}\to\mathcal{M}$ is a mapping for which there exists a mapping $d:\mathcal{A}\to\mathcal{M}$ such that*

$$\left\|g(\lambda x+y)-\lambda g(x)-g(y)\right\|\leq F(x,y), \tag{1}$$

$$\left\|d(\lambda x+y)-\lambda d(x)-d(y)\right\|\leq F(x,y), \tag{2}$$

and

$$\big\|(m+n)g(xy) - 2mx \cdot g(y) - 2ny \cdot d(x)\big\| \le F(x,y), \tag{3}$$

$$\big\|(m+n)d(xy) - 2mx \cdot d(y) - 2ny \cdot d(x)\big\| \le F(x,y) \tag{4}$$

for all $x, y \in \mathcal{A}$ *and* $\lambda \in \Lambda$. *Then there exists a unique linear generalized module left* (m,n)*-derivation* $G : \mathcal{A} \to \mathcal{M}$ *such that*

$$\big\|g(x) - G(x)\big\| \le \frac{F(x,x)}{1-\xi\eta} \tag{5}$$

for all $x \in \mathcal{A}$.

Proof Taking $\lambda = 1$ in (1), we obtain

$$\big\|g(x+y) - g(x) - g(y)\big\| \le F(x,y)$$

for all x, y. Thus, by Corollary 3.2 in [26], we conclude that there exists a unique additive mapping $G : \mathcal{A} \to \mathcal{M}$ such that (5) holds for all $x \in \mathcal{A}$. Next, replacing y by 0 in (1), we get

$$\big\|g(\lambda x) - \lambda g(x)\big\| \le F(x,0) = 0$$

and, consequently, $g(\lambda x) = \lambda g(x)$ for all $x \in \mathcal{A}$ and $\lambda \in \Lambda$. Note also that for every $k \in \mathbb{N}$ and all $x \in \mathcal{A}$, we have $G(x) = 2^{-k}G(2^k x)$ since G is additive. Therefore,

$$\begin{aligned}
&\big\|G(\lambda x) - \lambda G(x)\big\| \\
&\quad \le \big\|2^{-k}G\big(2^k\lambda x\big) - 2^{-k}g\big(2^k\lambda x\big)\big\| + \big\|2^{-k}\lambda g\big(2^k x\big) - 2^{-k}\lambda G\big(2^k x\big)\big\| \\
&\quad \le 2^{-k}\left(\frac{F(2^k\lambda x, 2^k\lambda x)}{1-\xi\eta} + \frac{\lambda F(2^k x, 2^k x)}{1-\xi\eta}\right) \\
&\quad = 2^{-k}(\xi\eta)^k\left(\frac{F(\lambda x, \lambda x)}{1-\xi\eta} + \frac{\lambda F(x,x)}{1-\xi\eta}\right).
\end{aligned}$$

Letting $k \to \infty$, we conclude that $G(\lambda x) = \lambda G(x)$ for all $x \in \mathcal{A}$ and all $\lambda \in \Lambda$. According to Lemma 2.1, this yields that G is linear.

Similarly, we can show that there exists a unique linear mapping $D : \mathcal{A} \to \mathcal{M}$ such that

$$\big\|d(x) - D(x)\big\| \le \frac{F(x,x)}{1-\xi\eta} \tag{6}$$

for all $x \in \mathcal{A}$. Moreover, by Theorem 1 in [18], D is a module left (m,n)-derivation.

It remains to prove that G is a generalized module left (m,n)-derivation with an associated module left (m,n)-derivation D, *i.e.*,

$$(m+n)G(xy) = 2mx \cdot G(y) + 2ny \cdot D(x)$$

for all $x, y \in \mathcal{A}$. So, let $x, y \in \mathcal{A}$ and $k \in \mathbb{N}$. By (3), we have

$$\begin{aligned}&\big\|(m+n)2^{-2k}g\big(\big(2^kx\big)\big(2^ky\big)\big) - 2m2^{-k}x\cdot g\big(2^ky\big) - 2n2^{-k}y\cdot d\big(2^kx\big)\big\|\\&\quad\le 2^{-2k}F\big(2^kx, 2^ky\big) = 2^{-2k}(\xi\eta)^kF(x,y).\end{aligned}$$

Furthermore,

$$\begin{aligned}&\big\|(m+n)G(xy) - 2mx\cdot G(y) - 2ny\cdot D(x)\big\|\\&\quad\le \big\|(m+n)2^{-2k}G\big(\big(2^kx\big)\big(2^ky\big)\big) - (m+n)2^{-2k}g\big(\big(2^kx\big)\big(2^ky\big)\big)\big\|\\&\qquad+ \big\|(m+n)2^{-2k}g\big(\big(2^kx\big)\big(2^ky\big)\big) - 2m2^{-k}x\cdot g\big(2^ky\big) - 2n2^{-k}y\cdot d\big(2^kx\big)\big\|\\&\qquad+ \big\|2m2^{-k}x\cdot g\big(2^ky\big) - 2m2^{-k}x\cdot G\big(2^ky\big)\big\| + \big\|2n2^{-k}y\cdot d\big(2^kx\big) - 2n2^{-k}y\cdot D\big(2^kx\big)\big\|.\end{aligned}$$

Since $\mathcal{M}$ is a Banach left $\mathcal{A}$-module, there exists a positive constant C such that

$$\begin{aligned}&\big\|(m+n)G(xy) - 2mx\cdot G(y) - 2ny\cdot D(x)\big\|\\&\quad\le (m+n)2^{-2k}\big\|G\big(\big(2^kx\big)\big(2^ky\big)\big) - g\big(\big(2^kx\big)\big(2^ky\big)\big)\big\| + 2^{-2k}(\xi\eta)^kF(x,y)\\&\qquad+ 2m2^{-k}C\|x\|\big\|g\big(2^ky\big) - G\big(2^ky\big)\big\| + 2n2^{-k}C\|y\|\big\|d\big(2^kx\big) - D\big(2^kx\big)\big\|.\end{aligned}$$

This yields that

$$\begin{aligned}&\big\|(m+n)G(xy) - 2mx\cdot G(y) - 2ny\cdot D(x)\big\|\\&\quad\le (m+n)2^{-2k}\frac{F(2^{2k}xy, 2^{2k}xy)}{1-\xi\eta} + 2^{-2k}(\xi\eta)^kF(x,y)\\&\qquad+ 2m2^{-k}C\|x\|\frac{F(2^ky, 2^ky)}{1-\xi\eta} + 2n2^{-k}C\|y\|\frac{F(2^kx, 2^kx)}{1-\xi\eta}\\&\quad= (m+n)2^{-2k}(\xi\eta)^{2k}\frac{F(xy,xy)}{1-\xi\eta} + 2^{-2k}(\xi\eta)^kF(x,y)\\&\qquad+ 2m2^{-k}C\|x\|(\xi\eta)^k\frac{F(y,y)}{1-\xi\eta} + 2n2^{-k}C\|y\|(\xi\eta)^k\frac{F(x,x)}{1-\xi\eta}.\end{aligned}$$

Letting $k \to \infty$, we conclude that $(m+n)G(xy) = 2mx\cdot G(y) + 2ny\cdot D(x)$. The proof is completed. □

3 Some additional remarks

In this section we write some additional results and observations about our main theorem.

Remark 3.1 Let g and F be as in Theorem 2.2 and let $x_0 \in \mathcal{A}$. If the maps g and $x \mapsto F(x,x)$ are continuous at point x_0, then G is continuous on $\mathcal{A}$. Namely, if G was not continuous, then there would exist an integer C and a sequence $\{x_k\}_{k=0}^{\infty}$ such that $\lim_{k\to\infty} x_k = 0$ and $\|G(x_k)\| > \frac{1}{C}$ for $k \geq 0$. Write

$$\widetilde{F}(x,x) = \frac{F(x,x)}{1-\xi\eta}, \quad x \in \mathcal{A}.$$

Let $t > C(2\widetilde{F}(x_0, x_0) + 1)$. Then

$$\lim_{k\to\infty} g(tx_k + x_0) = g(x_0)$$

since g is continuous at x_0. Thus, there exists an integer k_0 such that for every $k > k_0$, we have

$$\|g(tx_k + x_0) - g(x_0)\| < 1.$$

Therefore,

$$\begin{aligned} 2\widetilde{F}(x_0, x_0) + 1 &< \frac{t}{C} < \|G(tx_k)\| = \|G(tx_k + x_0) - G(x_0)\| \\ &\le \|G(tx_k + x_0) - g(tx_k + x_0)\| + \|g(tx_k + x_0) - g(x_0)\| + \|g(x_0) - G(x_0)\| \\ &< \widetilde{F}(tx_k + x_0, tx_k + x_0) + 1 + \widetilde{F}(x_0, x_0) \end{aligned}$$

for every $k > k_0$. Letting $k \to \infty$ and using the continuity of the map $x \mapsto \widetilde{F}(x, x)$ at point x_0, we get a contradiction.

Remark 3.2 Let $\varepsilon \ge 0$ and $p, q < 0$. Then a function $F : \mathcal{A}^2 \to [0, \infty)$ defined by

$$F(x, y) = \varepsilon\|x\|^p\|y\|^q, \quad x, y \in \mathcal{A},$$

satisfies all the assumptions of Theorem 2.2. Namely, $F(2x, y) = 2^p F(x, y)$ and $F(x, 2y) = 2^q F(x, y)$ for all $x, y \in \mathcal{A}$. In this case, we can write (5) as

$$\|g(x) - G(x)\| \le \frac{\varepsilon}{1 - 2^r}\|x\|^r,$$

where $r = p + q$ and $x \in \mathcal{A}$.

Remark 3.3 Let $\mathfrak{B}_{\mathbb{C}}$ denote the family of all sets $\Gamma \subseteq \mathbb{C}$ such that each additive function $f : \mathbb{C} \to \mathcal{M}$ bounded on Γ is continuous. The question which subsets $\Gamma \subseteq \mathbb{C}$ belong to $\mathfrak{B}_{\mathbb{C}}$ has been a subject of many papers. It is known that every non-empty open subset Γ of $\mathbb{C}$ is a member of $\mathfrak{B}_{\mathbb{C}}$. Moreover, if $\Gamma \subseteq \mathbb{C}$ and $\operatorname{Int}\Gamma \neq \emptyset$, then $\Gamma \in \mathfrak{B}_{\mathbb{C}}$. The same is true when $\Gamma \subseteq \mathbb{C}$ has a positive inner Lebesgue measure or contains a subset of the second category with the Baire property. For more information and further references concerning the subject, we refer the reader to [27–29].

Let $\Gamma_0 \in \mathfrak{B}_{\mathbb{C}}$ be a bounded set. Using standard techniques, it is easy to see that every additive function $f : \mathcal{A} \to \mathcal{M}$ with the property $f(\lambda x) = \lambda f(x)$ for all $x \in \mathcal{A}$ and all $\lambda \in \Gamma_0$ must be $\mathbb{C}$-linear (see, for example, [30, Lemma 1]). Therefore, we can show that Theorem 2.2 is valid even if we replace Λ with any set $\Gamma \subseteq \mathbb{C}$ which contains a bounded subset $\Gamma_0 \in \mathfrak{B}_{\mathbb{C}}$. Namely, as in the proof of Theorem 2.2, we can show that there exists a unique generalized module left (m, n)-derivation $G : \mathcal{A} \to \mathcal{M}$ satisfying (5). Moreover, $G(\lambda x) = \lambda G(x)$ for all $x \in \mathcal{A}$ and all $\lambda \in \Gamma_0$. Using the above mentioned arguments, it follows that G is $\mathbb{C}$-linear.

4 Superstability of generalized module left (m, n)-derivations

We end this paper with some observations on superstability. We say that a functional equation $\mathcal{E}$ is superstable if each function f, satisfying the equation $\mathcal{E}$ approximately, must actually be a solution of it. The notion of superstability has appeared in connection with the investigation of stability of the exponential equation $f(x+y) = f(x)f(y)$. The first result for the superstability of this equation was proved by Bourgin [31]. Later, this problem was renewed and investigated by Baker, Lawrence, and Zorzitto [32] (for more information, see [7] and references therein). Our last result shows that this kind of properties are valid also for conditions involving generalized module left (m, n)-derivations.

In the following, $\mathcal{A}$ and $\mathcal{M}$ will be a normed algebra with a unit e and a unitary Banach left $\mathcal{A}$-module, respectively. Assume that $F : \mathcal{A}^2 \to [0, \infty)$ is a function satisfying $F(2x, y) = \xi F(x, y)$ and $F(x, 2y) = \eta F(x, y)$ for some nonnegative scalars ξ, η with $\xi, \xi\eta < 1$. Then we have the next lemma.

Lemma 4.1 [18, Lemma 1] *If $d : \mathcal{A} \to \mathcal{M}$ is a mapping satisfying* (4) *and* $\|d(x+y) - d(x) - d(y)\| \le F(x, y)$, $x, y \in \mathcal{A}$, *then $d(tx) = td(x)$ for all $x \in \mathcal{A}$ and all $t \in \mathbb{Q} \setminus \{0\}$.*

Our last result is a generalization of Theorem 2 in [18].

Theorem 4.2 *Let $\mathcal{A}$ be a normed algebra with a unit e, let $\mathcal{M}$ be a unitary Banach left $\mathcal{A}$-module, and let $F : \mathcal{A}^2 \to [0, \infty)$ be a function such that $F(2x, y) = \xi F(x, y)$ and $F(x, 2y) = \eta F(x, y)$ for some nonnegative scalars ξ, η with $\xi, \xi\eta < 1$. Suppose that $g : \mathcal{A} \to \mathcal{M}$ is a mapping for which there exists a mapping $d : \mathcal{A} \to \mathcal{M}$ such that* (3) *and* (4) *hold true for all $x, y \in \mathcal{A}$ and*

$$\|g(x+y) - g(x) - g(y)\| \le F(x, y),$$
$$\|d(x+y) - d(x) - d(y)\| \le F(x, y)$$

for all $x, y \in \mathcal{A}$. Then g is a generalized module left (m, n)-derivation.

Proof We divide the proof into several steps.

Step 1. Firstly, we show that d is a module left (m, n)-derivation on $\mathcal{A}$. By the proof of Theorem 2.2, there exists a unique module left (m, n)-derivation D on $\mathcal{A}$ satisfying (6). Moreover, according to Lemma 4.1, we have $d(2^k x) = 2^k d(x)$ for all $x \in \mathcal{A}$, $k \in \mathbb{N}$. Thus,

$$\begin{aligned} \|d(x) - D(x)\| &= \|2^{-k} d(2^k x) - 2^{-k} D(2^k x)\| \\ &\le 2^{-k} \frac{F(2^k x, 2^k x)}{1 - \xi\eta} = 2^{-k} (\xi\eta)^k \frac{F(x, x)}{1 - \xi\eta} \end{aligned}$$

for all $x \in \mathcal{A}$ and $k \in \mathbb{N}$. Letting $k \to \infty$, we conclude that $d = D$. In other words, d is a module left (m, n)-derivation on $\mathcal{A}$.

Step 2. Let $x \in \mathcal{A}$ and $t \in \mathbb{Q} \setminus \{0\}$. We claim that $g(tx) = tg(x)$. Suppose that $G : \mathcal{A} \to \mathcal{M}$ is a module left (m, n)-derivation from the proof of Theorem 2.2 and $k \in \mathbb{N}$. Recall that G is additive and therefore $G(tx) = tG(x)$. Moreover, G satisfies (5) and, by (3), we have

$$\begin{aligned} &\|(m+n)G((2^k e)(tx)) - 2mt2^k e \cdot g(x) - 2ntx \cdot d(2^k e)\| \\ &\quad \le \|(m+n)tG(2^k ex) - (m+n)tg(2^k ex)\| \end{aligned}$$

$$\begin{aligned}
&\quad + \bigl\| (m+n)tg\bigl(2^k ex\bigr) - 2mt2^k e \cdot g(x) - 2ntx \cdot d\bigl(2^k e\bigr) \bigr\| \\
&\le |t|(m+n)\frac{F(2^k ex, 2^k ex)}{1-\xi\eta} + |t|F\bigl(2^k e, x\bigr) \\
&= |t|(m+n)(\xi\eta)^k \frac{F(x,x)}{1-\xi\eta} + |t|\xi^k F(e,x).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\bigl\| (m+n)g\bigl(\bigl(2^k e\bigr)(tx)\bigr) - 2mt2^k e \cdot g(x) - 2ntx \cdot d\bigl(2^k e\bigr) \bigr\| \\
&\quad \le \bigl\| (m+n)g\bigl(\bigl(2^k e\bigr)(tx)\bigr) - (m+n)G\bigl(\bigl(2^k e\bigr)(tx)\bigr) \bigr\| \\
&\qquad + \bigl\| (m+n)G\bigl(\bigl(2^k e\bigr)(tx)\bigr) - 2mt2^k e \cdot g(x) - 2ntx \cdot d\bigl(2^k e\bigr) \bigr\| \\
&\quad \le (m+n)\frac{F(2^k tx, 2^k tx)}{1-\xi\eta} + |t|(m+n)(\xi\eta)^k \frac{F(x,x)}{1-\xi\eta} + |t|\xi^k F(e,x) \\
&\quad = (m+n)(\xi\eta)^k \frac{F(tx,tx)}{1-\xi\eta} + |t|(m+n)(\xi\eta)^k \frac{F(x,x)}{1-\xi\eta} + |t|\xi^k F(e,x).
\end{aligned}$$

This yields that

$$\begin{aligned}
&\bigl\| 2m2^k \bigl(g(tx) - tg(x)\bigr) \bigr\| \\
&\quad = \bigl\| 2m2^k e \cdot \bigl(g(tx) - tg(x)\bigr) \bigr\| \\
&\quad \le \bigl\| 2m2^k e \cdot g(tx) + 2ntx \cdot d\bigl(2^k e\bigr) - (m+n)g\bigl(\bigl(2^k e\bigr)(tx)\bigr) \bigr\| \\
&\qquad + \bigl\| (m+n)g\bigl(\bigl(2^k e\bigr)(tx)\bigr) - 2mt2^k e \cdot g(x) - 2ntx \cdot d\bigl(2^k e\bigr) \bigr\| \\
&\quad \le F\bigl(2^k e, tx\bigr) + (m+n)(\xi\eta)^k \frac{F(tx,tx)}{1-\xi\eta} + |t|(m+n)(\xi\eta)^k \frac{F(x,x)}{1-\xi\eta} + |t|\xi^k F(e,x) \\
&\quad = \xi^k F(e,tx) + (m+n)(\xi\eta)^k \frac{F(tx,tx)}{1-\xi\eta} + |t|(m+n)(\xi\eta)^k \frac{F(x,x)}{1-\xi\eta} + |t|\xi^k F(e,x) \\
&\quad = \xi^k \bigl(F(e,tx) + |t|F(e,x)\bigr) + (m+n)\left(\frac{(\xi\eta)^k}{1-\xi\eta}\right)\bigl(F(tx,tx) + |t|F(x,x)\bigr)
\end{aligned}$$

and, therefore,

$$\begin{aligned}
\bigl\| g(tx) - tg(x) \bigr\| &\le \left(\frac{1}{2m2^k}\right)\xi^k \bigl(F(e,tx) + |t|F(e,x)\bigr) \\
&\quad + (m+n)\left(\frac{1}{2m2^k}\right)\left(\frac{(\xi\eta)^k}{1-\xi\eta}\right)\bigl(F(tx,tx) + |t|F(x,x)\bigr).
\end{aligned}$$

Letting $k \to \infty$, we get $g(tx) = tg(x)$ for all $x \in \mathcal{A}$. In particular,

$$g\bigl(2^k x\bigr) = 2^k g(x)$$

for all $x \in \mathcal{A}$ and all $k \in \mathbb{N}$.

Step 3. We prove that $g = G$. Namely, using (5), we obtain

$$\bigl\| g(x) - G(x) \bigr\| = \bigl\| 2^{-k} g\bigl(2^k x\bigr) - 2^{-k} G\bigl(2^k x\bigr) \bigr\| \le 2^{-k} \frac{F(2^k x, 2^k x)}{1-\xi\eta} = 2^{-k}(\xi\eta)^k \frac{F(x,x)}{1-\xi\eta}$$

for all $x \in \mathcal{A}$ and $k \in \mathbb{N}$. Taking the limit when $k \to \infty$, we conclude that $g = G$, as desired. Therefore, g is a generalized module left (m,n)-derivation on $\mathcal{A}$. This completes the proof. □

Corollary 4.3 *Let $\mathcal{A}$ be a normed algebra with a unit e, let $\mathcal{M}$ be a unitary Banach left $\mathcal{A}$-module, and let $F : \mathcal{A}^2 \to [0,\infty)$ be a function such that $F(2x,y) = \xi F(x,y)$ and $F(x,2y) = \eta F(x,y)$ for some nonnegative scalars ξ, η with $\xi, \xi\eta < 1$. Suppose that $g : \mathcal{A} \to \mathcal{M}$ is a mapping for which there exists a mapping $d : \mathcal{A} \to \mathcal{M}$ such that* (1)-(4) *hold true for all $x, y \in \mathcal{A}$ and $\lambda \in \Lambda$. Then g is a linear generalized module left (m,n)-derivation.*

Competing interests
The author declares that she has no competing interests.

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References
1. Ulam, SM: A Collection of the Mathematical Problems. Interscience, New York (1960)
2. Ulam, SM: Problems in Modern Mathematics. Wiley, New York (1964)
3. Hyers, DH: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. USA **27**, 222-224 (1941)
4. Aoki, T: On the stability of the linear transformation in Banach spaces. J. Math. Soc. Jpn. **2**, 64-66 (1950)
5. Rassias, TM: On the stability of the linear mappings in Banach spaces. Proc. Am. Math. Soc. **72**, 297-300 (1978)
6. Czerwik, S: Functional Equations and Inequalities in Several Variables. World Scientific, Singapore (2002)
7. Hyers, DH, Isac, G, Rassias, TM: Stability of Functional Equations in Several Variables. Birkhäuser, Boston (1998)
8. Jung, Y-S: On the generalized Hyers-Ulam stability of module left derivations. J. Math. Anal. Appl. **339**, 108-114 (2008)
9. Moszner, Z: On the stability of functional equations. Aequ. Math. **77**, 33-88 (2009)
10. Jun, K-W, Park, D-W: Almost derivations on the Banach algebra $C^n[0,1]$. Bull. Korean Math. Soc. **33**, 359-366 (1996)
11. Amyari, M, Baak, C, Moslehian, MS: Nearly ternary derivations. Taiwan. J. Math. **11**, 1417-1424 (2007)
12. Badora, R: On approximate derivations. Math. Inequal. Appl. **9**, 167-173 (2006)
13. Gordji, ME, Moslehian, MS: A trick for investigation of approximate derivations. Math. Commun. **15**, 99-105 (2010)
14. Moslehian, MS: Hyers-Ulam-Rassias stability of generalized derivations. Int. J. Math. Math. Sci. **2006**, 1-8 (2006)
15. Moslehian, MS: Ternary derivations, stability and physical aspects. Acta Appl. Math. **100**, 187-199 (2008)
16. Park, C: Linear derivations on Banach algebras. Nonlinear Funct. Anal. Appl. **9**, 359-368 (2004)
17. Šemrl, P: The functional equation of multiplicative derivation is superstable on standard operator algebras. Integral Equ. Oper. Theory **18**, 118-122 (1994)
18. Fošner, A: On the generalized Hyers-Ulam stability of module left (m,n)-derivations. Aequ. Math. **2012**, 1-8 (2012). doi:10.1007/s00010-012-0124-3
19. Vukman, J: Jordan left derivations on semiprime rings. Math. J. Okayama Univ. **39**, 1-6 (1997)
20. Vukman, J: On left Jordan derivations of rings and Banach algebras. Aequ. Math. **75**, 260-266 (2008)
21. Ali, S, Fošner, A: On generalized (m,n)-derivations and generalized (m,n)-Jordan derivations in rings. Algebra Colloq. (in press)
22. Fošner, M, Vukman, J: On some functional equations arising from (m,n)-Jordan derivations and commutativity of prime rings. Rocky Mt. J. Math. **42**, 1153-1168 (2012). doi:10.1216/RMJ-2012-42-4-1153
23. Vukman, J: On (m,n)-Jordan derivations and commutativity of prime rings. Demonstr. Math. **41**, 773-778 (2008)
24. Vukman, J, Kosi-Ulbl, I: On some equations related to derivations in rings. Int. J. Math. Math. Sci. **17**, 2703-2710 (2005)
25. Park, C: Homomorphisms between Poisson JC^*-algebras. Bull. Braz. Math. Soc. **36**, 79-97 (2005)
26. Brzdęk, J: On a method of proving the Hyers-Ulam stability of functional equations on restricted domains. Aust. J. Math. Anal. Appl. **6**, 1-10 (2009)
27. Jabłoński, W: On a class of sets connected with a convex function. Abh. Math. Semin. Univ. Hamb. **69**, 205-210 (1999)
28. Jabłoński, W: Sum of graphs of continuous functions and boundedness of additive operators. J. Math. Anal. Appl. **312**, 527-534 (2005)
29. Kuczma, M: An Introduction to the Theory of Functional Equations and Inequalities: Cauchy's Equation and Jensen's Inequality, 2nd edn. Birkhäuser, Boston (2009)
30. Brzdęk, J, Fošner, A: Remarks on the stability of Lie homomorphisms. J. Math. Anal. Appl. **400**, 585-596 (2013)
31. Bourgin, DG: Approximately isometric and multiplicative transformations on continuous function rings. Duke Math. J. **16**, 385-397 (1949)
32. Baker, J, Lawrence, J, Zorzitto, F: The stability of the equation $f(x+y) = f(x)f(y)$. Proc. Am. Math. Soc. **74**, 242-246 (1979)

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