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Almost increasing sequences and their new applications

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Abstract

In this paper, we generalize a known theorem dealing with $|C,1|_k$ summability factors to the $|C,\alpha|_k$ summability factors of infinite series using an almost increasing sequence. This theorem also includes some known and new results.

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1 Introduction

A positive sequence (b_n) is said to be an almost increasing sequence if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n). By t_n^α we denote the nth Cesàro mean of order α, with $\alpha > -1$, of the sequence (na_n), that is,

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{1}$$

where

$$A_n^\alpha = \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!} = O(n^\alpha), \qquad A_{-n}^\alpha = 0 \quad \text{for } n > 0. \tag{2}$$

The series $\sum a_n$ is said to be summable $|C,\alpha|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \frac{1}{n} \left|t_n^\alpha\right|^k < \infty. \tag{3}$$

If we take $\alpha = 1$, then $|C,\alpha|_k$ summability reduces to $|C,1|_k$ summability.

2 Known result

Many works dealing with an application of almost increasing sequences to the absolute Cesàro summability factors of infinite series have been done (see [3–11]). Among them, in [10], the following main theorem dealing with $|C,1|_k$ summability factors has been proved.

Theorem A *Let* (φ_n) *be a positive sequence and* (X_n) *be an almost increasing sequence. If the conditions*

$$\sum_{n=1}^{\infty} n\left|\Delta^2\lambda_n\right|X_n < \infty, \tag{4}$$

$$|\lambda_n|X_n = O(1) \quad \text{as } n \to \infty, \tag{5}$$

$$\varphi_n = O(1) \quad \text{as } n \to \infty, \tag{6}$$

$$n\Delta\varphi_n = O(1) \quad \text{as } n \to \infty, \tag{7}$$

$$\sum_{v=1}^{n} \frac{|t_v|^k}{vX_v^{k-1}} = O(X_n) \quad \text{as } n \to \infty \tag{8}$$

are satisfied, then the series $\sum a_n\lambda_n\varphi_n$ *is summable* $|C,1|_k$, $k \geq 1$.

3 The main result

The aim of this paper is to generalize Theorem A to the $|C,\alpha|_k$ summability in the following form.

Theorem *Let* (φ_n) *be a positive sequence and let* (X_n) *be an almost increasing sequence.*

If the conditions (4), (5), (6) *and* (7) *are satisfied, and the sequence* (w_n^α) *defined by* (*see* [12])

$$w_n^\alpha = \begin{cases} |t_n^\alpha|, & \alpha = 1, \\ \max_{1\leq v\leq n} |t_v^\alpha|, & 0 < \alpha < 1, \end{cases} \tag{9}$$

satisfies the condition

$$\sum_{v=1}^{n} \frac{(w_v^\alpha)^k}{vX_v^{k-1}} = O(X_n) \quad \text{as } n \to \infty, \tag{10}$$

then the series $\sum a_n\lambda_n\varphi_n$ *is summable* $|C,\alpha|_k$, $0 < \alpha \leq 1$, $(\alpha-1)k > -1$ *and* $k \geq 1$.

Remark It should be noted that if we take $\alpha = 1$, then we get Theorem A. In this case, condition (10) reduces to condition (8) and the condition '$(\alpha-1)k > -1$' is trivial.

We need the following lemmas for the proof of our theorem.

Lemma 1 [13] *If* $0 < \alpha \leq 1$ *and* $1 \leq v \leq n$, *then*

$$\left|\sum_{p=0}^{v} A_{n-p}^{\alpha-1}a_p\right| \leq \max_{1\leq m\leq v}\left|\sum_{p=0}^{m} A_{m-p}^{\alpha-1}a_p\right|. \tag{11}$$

Lemma 2 [14] *Under the conditions* (4) *and* (5), *we have*

$$nX_n|\Delta\lambda_n| = O(1) \quad \text{as } n \to \infty, \tag{12}$$

$$\sum_{n=1}^{\infty} X_n|\Delta\lambda_n| < \infty. \tag{13}$$

4 Proof of the Theorem

Let (T_n^α) be the nth (C,α) mean, with $0 < \alpha \le 1$, of the sequence $(na_n\lambda_n\varphi_n)$.

Then, by (1), we find that

$$T_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v \varphi_n. \tag{14}$$

Thus, applying Abel's transformation first and then using Lemma 1, we have that

$$\begin{aligned}
T_n^\alpha &= \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\Delta(\lambda_v\varphi_n)\sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n\varphi_n}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \\
&= \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}(\lambda_v\Delta\varphi_v + \varphi_{v+1}\Delta\lambda_v)\sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n\varphi_n}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \\
\left|T_n^\alpha\right| &\le \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}|\lambda_v\Delta\varphi_v|\left|\sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p\right| + \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}|\varphi_{v+1}\Delta\lambda_v|\left|\sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p\right| \\
&\quad + \frac{|\lambda_n\varphi_n|}{A_n^\alpha}\left|\sum_{v=1}^{v} A_{n-v}^{\alpha-1} v a_v\right| \\
&\le \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} A_v^\alpha w_v^\alpha|\lambda_v||\Delta\varphi_v| + \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} A_v^\alpha w_v^\alpha|\varphi_{v+1}||\Delta\lambda_v| + |\lambda_n||\varphi_n| w_n^\alpha \\
&= T_{n,1}^\alpha + T_{n,2}^\alpha + T_{n,3}^\alpha.
\end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-1}\left|T_{n,r}^\alpha\right|^k < \infty \quad \text{for } r = 1,2,3.$$

Now, when $k > 1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we get that

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{-1}\left|T_{n,1}^\alpha\right|^k &\le \sum_{n=2}^{m+1} n^{-1}\left(A_n^\alpha\right)^{-k}\left\{\sum_{v=1}^{n-1} A_v^\alpha w_v^\alpha|\Delta\varphi_v||\lambda_v|\right\}^k \\
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{1+\alpha k}}\sum_{v=1}^{n-1}\left(v^\alpha\right)^k\left(w_v^\alpha\right)^k|\Delta\varphi_v|^k|\lambda_v|^k\left\{\sum_{v=1}^{n-1}1\right\}^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{2+(\alpha-1)k}}\sum_{v=1}^{n-1} v^{\alpha k}\left(w_v^\alpha\right)^k|\lambda_v|^k\frac{1}{v^k} \\
&= O(1)\sum_{v=1}^{m} v^{\alpha k}\left(w_v^\alpha\right)^k v^{-k}|\lambda_v|^k\sum_{n=v+1}^{m+1}\frac{1}{n^{2+(\alpha-1)k}} \\
&= O(1)\sum_{v=1}^{m} v^{\alpha k}\left(w_v^\alpha\right)^k v^{-k}|\lambda_v|^k\int_v^\infty\frac{dx}{x^{2+(\alpha-1)k}}
\end{aligned}$$

$$\begin{aligned}
&= O(1)\sum_{\nu=1}^{m}\left(w_\nu^\alpha\right)^k|\lambda_\nu||\lambda_\nu|^{k-1}\frac{1}{\nu}\\
&= O(1)\sum_{\nu=1}^{m}\left(w_\nu^\alpha\right)^k|\lambda_\nu|\frac{1}{\nu X_\nu^{k-1}}\\
&= O(1)\sum_{\nu=1}^{m-1}\Delta|\lambda_\nu|\sum_{r=1}^{\nu}\frac{(w_r^\alpha)^k}{rX_r^{k-1}} + O(1)|\lambda_m|\sum_{\nu=1}^{m}\frac{(w_\nu^\alpha)^k}{\nu X_\nu^{k-1}}\\
&= O(1)\sum_{\nu=1}^{m}|\Delta\lambda_\nu|X_\nu + O(1)|\lambda_m|X_m = O(1) \quad \text{as } m\to\infty
\end{aligned}$$

by virtue of the hypotheses of the theorem and Lemma 2. Again, we get that

$$\begin{aligned}
\sum_{n=2}^{m+1}n^{-1}\left|T_{n,2}^\alpha\right|^k &\le \sum_{n=2}^{m+1}n^{-1}\left(A_n^\alpha\right)^{-k}\left\{\sum_{\nu=1}^{n-1}A_\nu^\alpha w_\nu^\alpha|\varphi_{\nu+1}||\Delta\lambda_\nu|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{1+\alpha k}}\left\{\sum_{\nu=1}^{n}\nu^\alpha\left(w_\nu^\alpha\right)|\Delta\lambda_\nu|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{1+\alpha k}}\sum_{\nu=1}^{n-1}\nu^{\alpha k}\left(w_\nu^\alpha\right)^k|\Delta\lambda_\nu|^k\left\{\sum_{\nu=1}^{n-1}1\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{2+(\alpha-1)k}}\sum_{\nu=1}^{n-1}\nu^{\alpha k}\left(w_\nu^\alpha\right)^k|\Delta\lambda_\nu|^k\\
&= O(1)\sum_{\nu=1}^{m}\nu^{\alpha k}\left(w_\nu^\alpha\right)^k|\Delta\lambda_\nu||\Delta\lambda_\nu|^{k-1}\sum_{n=\nu+1}^{m+1}\frac{1}{n^{2+(\alpha-1)k}}\\
&= O(1)\sum_{\nu=1}^{m}\frac{\nu^{\alpha k}(w_\nu^\alpha)^k|\Delta\lambda_\nu|}{\nu^{k-1}X_\nu^{k-1}}\int_\nu^\infty\frac{dx}{x^{2+(\alpha-1)k}}\\
&= O(1)\sum_{\nu=1}^{m}\nu|\Delta\lambda_\nu|\frac{(w_\nu^\alpha)^k}{\nu X_\nu^{k-1}}\\
&= O(1)\sum_{\nu=1}^{m}\Delta(\nu|\Delta\lambda_\nu|)\sum_{r=1}^{\nu}\frac{(w_r^\alpha)^k}{rX_r^{k-1}} + O(1)m|\Delta\lambda_m|\sum_{\nu=1}^{m}\frac{(w_\nu^\alpha)^k}{\nu X_\nu^{k-1}}\\
&= O(1)\sum_{\nu=1}^{m-1}\nu\left|\Delta^2\lambda_\nu\right|X_\nu + O(1)\sum_{\nu=1}^{m-1}X_\nu|\Delta\lambda_\nu| + O(1)m|\Delta\lambda_m|X_m\\
&= O(1) \quad \text{as } m\to\infty
\end{aligned}$$

by hypotheses of the theorem and Lemma 2. Finally, as in $T_{n,1}^\alpha$, we have that

$$\begin{aligned}
\sum_{n=1}^{m}n^{-1}\left|T_{n,3}^\alpha\right|^k &= \sum_{n=1}^{m}n^{-1}\left|\lambda_n\varphi_n w_n^\alpha\right|^k\\
&= O(1)\sum_{n=1}^{m}\frac{(w_n^\alpha)^k|\lambda_n|}{nX_n^{k-1}} = O(1) \quad \text{as } m\to\infty
\end{aligned}$$

by virtue of the hypotheses of the theorem and Lemma 2. This completes the proof of the theorem. Also, if we take $k = 1$, then we get a new result concerning the $|C, \alpha|$ summability factors of infinite series.

Competing interests
The author declares that he has no competing interests.

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