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Relations between anisotropic Besov spaces and multivariate Bernstein-Durrmeyer operators

Guo Feng[1,2*] and Yuan Feng[3]

*Correspondence:
gfeng@tzc.edu.cn
[1]School of Mathematics and Information Engineering, Taizhou University, Taizhou, Zhejiang 317000, China
[2]School of Mathematical Sciences, Beijing Normal University, Beijing, 100875, China
Full list of author information is available at the end of the article

Abstract
In this paper, we use the multivariate Bernstein-Durrmeyer operators defined on the simplex to characterize anisotropic Besov spaces.
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1 Introduction and some notations

Let T be the simplex in $\mathbb{R}^d$ defined by

$$\mathrm{T} = \left\{ \mathrm{x} = (x_1, x_2, \dots, x_d) : x_i \geq 0, i = 1, 2, \dots, d, |\mathrm{x}| = \sum_{i=1}^{d} x_i \leq 1 \right\}.$$

Let $L_p(\mathrm{T}) := L_{\mathrm{p}}(\mathrm{T})$, $\mathrm{p} = (p_1, p_2, \dots, p_d)$, $p_1 = p_2 = \cdots = p_d = p$, $1 \leq p < \infty$, be the space consisting of all Lebesgue measurable functions f on T for which the norm $\|f\|_p := (\int_{\mathrm{T}} |f(\mathrm{x})|^p \, dx)^{1/p}$ is finite. Let $C(\mathrm{T}) := L_{\infty}(\mathrm{T})$, $\infty = (\infty, \infty, \dots, \infty)$ be the space consisting of all continuous functions f on T for which the norm $\max_{\mathrm{x} \in \mathrm{T}} |f(\mathrm{x})|$ is finite.

Let $f \in L_1(\mathrm{T})$. For each $n \in \mathbb{N}$, the multivariate Bernstein-Durrmeyer operators of f are defined by [1]

$$M_{n,\mathrm{d}}(f; \mathrm{x}) = \sum_{|\mathrm{k}| \leq n} p_{n,\mathrm{k}}(\mathrm{x}) \frac{(n+d)!}{n!} \int_{\mathrm{T}} p_{n,\mathrm{k}}(\mathrm{u}) f(\mathrm{u}) \, d\mathrm{u}, \tag{1.1}$$

where

$$p_{n,\mathrm{k}}(\mathrm{x}) = \frac{n!}{\mathrm{k}!(n - |\mathrm{k}|)!} \mathrm{x}^{\mathrm{k}} (1 - \mathrm{x})^{n - |\mathrm{k}|}, \quad \mathrm{x} \in \mathrm{T},$$

$\mathrm{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, $\mathrm{k} = (k_1, k_2, \dots, k_d) \in \mathbb{N}_0^d$, $\mathbb{N}_0^d = \overbrace{\mathbb{N}_0 \times \mathbb{N}_0 \times \cdots \times \mathbb{N}_0}^{d}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $|x| = \sum_{i=1}^{d} x_i$, $\mathrm{x}^{\mathrm{k}} = x_1^{k_1} x_2^{k_2} \cdots \mathrm{x}_d^{k_d}$, $|\mathrm{k}| = \sum_{i=1}^{d} k_i$, $\mathrm{k}! = k_1! k_2! \cdots k_d!$.

For $\mathrm{x} \in \mathrm{T}$, we denote

$$\varphi_{ij}^2(\mathrm{x}) = \begin{cases} x_i(1-|\mathrm{x}|) & \text{for } i=j=1,2,\dots,d, \\ x_i x_j & \text{for } 1 \le i < j \le d. \end{cases}$$

Let $D_i = D_{ii} = \frac{\partial}{\partial x_i}$, $1 \le i \le d$; $D_{ij} = D_i - D_j$, $1 \le i < j \le d$; $D^k = D_1^{k_1} D_2^{k_2} \cdots D_d^{k_d}$, $\mathrm{k} \in N_0^d$, and

$$D_{ij}^2(\mathrm{x}) = \begin{cases} \frac{\partial^2}{\partial x_i^2} & \text{for } i=j=1,2,\dots,d, \\ (\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j})^2 & \text{for } 1 \le i < j \le d. \end{cases}$$

Definition 1.1 Let $L_p := L_p(\mathrm{T})$, $1 \le p < \infty$, and weighted Sobolev spaces are given by

$$W_{\Phi,p}^2 = \{g \mid g \in L_p, D^k g, |k| \le 2 \text{ are in } L_{\mathrm{loc}}(\overset{\circ}{\mathrm{T}}), \text{ and } \varphi_{ij}^2 D_{ij}^2 f \in L_p, 1 \le i \le j \le d\},$$

where the derivatives are in the sense of distributions, and $\overset{\circ}{\mathrm{T}}$ is the interior of T.

The K-functional of Ditzian-Totik type is given by

$$K_\varphi^2(f;t_l^2)_p = \inf_{g \in W_{\Phi,p}^2} \{\|f-g\|_p + t_l^2 \Phi(g)_p\}, \quad t_l > 0, l = 1,2,\dots,d,$$

where $\mathrm{t} = (t_1, t_2, \dots, t_d)$, $\Phi(g)_p := \|g\|_p + \sum_{1 \le i \le j \le d} \|\varphi_{ij}^2 D_{ij}^2 g\|_p$.

The anisotropic Besov spaces [2] are given by

$$B_{p,q}^{\frac{\theta}{2}} := (L_p, W_{\Phi,p}^2)_{\frac{\theta}{2},q},$$

where $\theta = (\theta_1, \theta_2, \dots, \theta_d)$, $1 \le p, q < \infty$, $n \in \mathbb{N}$, $n > 2 > \theta_l > 0$.

By [3] and the definition of anisotropic Besov spaces, it is not difficult to get the following.

Theorem 1.2 *Suppose* $1 \le p, q < \infty$, $n \in \mathbb{N}$, $n > 2 > \theta_l > 0$, $l = 1,2,\dots,d$. *Then*

$$f \in B_{p,q}^{\frac{\theta}{2}} \quad \Leftrightarrow \quad \int_0^\infty [t_l^{-\frac{\theta_l}{2}} K_\varphi^2(f;t_l^2)_p]^q \frac{dt_l}{t_l} < \infty, \tag{1.2}$$

and

$$\int_0^\infty [t_l^{-\frac{\theta_l}{2}} K_\varphi^2(f;t_l^2)_p]^q \frac{dt_l}{t_l} < \infty \quad \Leftrightarrow \quad \int_0^1 [t_l^{-\frac{\theta_l}{2}} K_\varphi^2(f;t_l^2)_p]^q \frac{dt_l}{t_l} < \infty. \tag{1.3}$$

In this paper, we use the multivariate Bernstein-Durrmeyer operators defined on the simplex to characterize anisotropic Besov spaces. We will show, for $1 \le p, q < \infty$, $n \in \mathbb{N}$, $n > 2 > \theta_l > 0$, that

$$f \in B_{p,q}^{\frac{\theta}{2}} \quad \Leftrightarrow \quad \left\{\sum_{n=1}^\infty [n^{\frac{\theta_l}{2}} \|L_n(f) - f\|_p]^q \frac{1}{n}\right\}^{\frac{1}{q}} < \infty.$$

For convenience, throughout this paper, M denotes a positive constant independent of x, n and f which may be different in different places.

2 Auxiliary lemmas

To prove the theorems, we need the following lemmas. The following two lemmas were proved in [4].

Lemma 2.1 *If* $1 \le p < \infty$, $f \in L_p$, $n \in \mathbb{N}$, *then*

$$\left\| M_{n,d}(f) \right\|_p \le M \|f\|_p, \tag{2.1}$$

$$\left\| \varphi_{ij}^2 D_{ij}^2 M_{n,d}(f) \right\|_p \le Mn \|f\|_p, \quad 1 \le i \le j \le d. \tag{2.2}$$

Lemma 2.2 *If* $1 \le p < \infty$, $f \in W_{\Phi,p}^2$, $n \in \mathbb{N}$, $n > 2$, *then*

$$\left\| \varphi_{ij}^2 D_{ij}^2 M_{n,d}(f) \right\|_p \le M \left\| \varphi_{ij}^2 D_{ij}^2 f \right\|_p, \quad i = 1, 2, \ldots, d. \tag{2.3}$$

Lemma 2.3 *Suppose* $1 \le p < \infty$, $f \in L_p$, $n \in \mathbb{N}$, $n > 2$. *Then*

$$\left\| M_{n,d}(f) - f \right\|_p \le M K_\varphi^2\left(f; n^{-1}\right)_p. \tag{2.4}$$

Proof Let $f \in L_p$, It is shown in [5] that there exists a constant $M > 0$ such that

$$M^{-1} \omega_\varphi^2(f; t_l)_p \le K_\varphi^{*,2}\left(f; t_l^2\right)_p \le M \omega_\varphi^2(f; t_l)_p,$$

where $\omega_\varphi^2(f; t_l)_p$ is the modulus of smoothness of Ditzian-Totik type defined by

$$\omega_\varphi^2(f; t_l)_p := \sup_{0 \le h \le t_l} \sum_{1 \le i \le j \le d} \left\| \Delta_{h\varphi_{ij} e_{ij}}^2 f \right\|_p, \quad t_l > 0, l = 1, 2, \ldots, d,$$

$$\left\| \Delta_{h\mathrm{e}}^2 f(\mathrm{x}) \right\|_p = \begin{cases} f(\mathrm{x} + \frac{he}{2}) - 2f(\mathrm{x} + \frac{he}{2}) + f(\mathrm{x} - \frac{he}{2}), & \mathrm{x} \pm \frac{he}{2} \in \mathrm{T}, \\ 0, & \text{otherwise}, \end{cases}$$

$h > 0$, $\mathrm{e_i} = (0, 0, \ldots, \overset{\text{ith}}{1}, 0, \ldots, 0)$ is the unit vector in $\mathbb{R}^d$, $\mathrm{e_{ij}} = \mathrm{e_i} - \mathrm{e_j}$, $\mathrm{e} \in \mathbb{R}^n$. $K_\varphi^{*,2}(f; t_l^2)_p$ is another K-functional of Ditzian-Totik type defined by

$$K_\varphi^{*,2}\left(f; t_l^2\right)_p = \inf_{g \in W_{\Phi,p}^2} \left\{ \left\| f - g \right\|_p + t_l^2 \sum_{1 \le i \le j \le d} \left\| \varphi_{ij}^2 D_{ij}^2 g \right\|_p \right\}, \quad t_l > 0, l = 1, 2, \ldots, d.$$

We notice that [6] for $f \in L_p$, we have

$$\left\| M_{n,d}(f) - f \right\|_p \le M\left(\omega_\varphi^2(f; \sqrt{n})_p + n^{-1} \|f\|_p\right).$$

Thus, for $g \in W_{\Phi,p}^2$, by the definition of K-functional $K_\varphi^{*,2}(f; t_l^2)_p$, we have

$$\begin{aligned} \left\| M_{n,d}(f) - f \right\|_p &\le M\left(\omega_\varphi^2(f; \sqrt{n})_p + n^{-1} \|f\|_p\right) \\ &\le M\left(K_\varphi^{*,2}\left(f; n^{-\frac{1}{2}}\right) + n^{-1} \left\| f - g \right\|_p + n^{-1} \|g\|_p\right) \\ &\le M\left(2\|f - g\|_p + n^{-1} \|g\|_p + n^{-1} \sum_{1 \le i \le j \le d} \left\| \varphi_{ij}^2 D_{ij}^2 g \right\|_p\right). \end{aligned}$$

According to the definition of K-functional $K_\varphi^2(f; t_l^2)_p$, Lemma 2.3 has been proved. □

Lemma 2.4 *Suppose* $1 \le p < \infty, f \in L_p$, $n \in \mathbb{N}$, $n > 2$. *Then*

$$\Phi\big(M_{n,d}(f)\big)_p \le MnK_\varphi^2\big(f; n^{-1}\big)_p. \tag{2.5}$$

Proof For $f \in L_p$, $g \in W_{\Phi,p}^2$, by Lemma 2.1 and Lemma 2.2, we get

$$\begin{aligned}\Phi\big(M_{n,d}(f)\big)_p &= \big\|M_{n,d}(f)\big\|_p + \sum_{1\le i\le j\le d} \big\|\varphi_{ij}^2 D_{ij}^2 M_{n,d}(f)\big\|_p \\ &\le \big\|M_{n,d}(f-g)\big\|_p + \big\|M_{n,d}(g)\big\|_p \\ &\quad + \sum_{1\le i\le j\le d} \big\|\varphi_{ij}^2 D_{ij}^2 M_{n,d}(f-g)\big\|_p + \sum_{1\le i\le j\le d} \big\|\varphi_{ij}^2 D_{ij}^2 M_{n,d}(g)\big\|_p \\ &\le M\Bigg(n\|f-g\|_p + \|g\|_p + \sum_{1\le i\le j\le d} \big\|\varphi_{ij}^2 D_{ij}^2 g\big\|_p\Bigg) \\ &\le Mn\Bigg(\|f-g\|_p + n^{-1}\Bigg(\|g\|_p + \sum_{1\le i\le j\le d} \big\|\varphi_{ij}^2 D_{ij}^2 g\big\|_p\Bigg)\Bigg).\end{aligned}$$

According to the definition of K-functional $K_\varphi^2(f; t_l^2)_p$, Lemma 2.4 has been proved. □

3 Main results

In this section we will prove our main results.

Theorem 3.1 *Let* $1 \le p, q < \infty$, $n \in \mathbb{N}$, $n > 2 > \theta_l > 0$, $l = 1, 2, \ldots, d$. *Then*

$$\begin{aligned} f \in B_{p,q}^{\frac{\theta}{2}} \quad &\Leftrightarrow \quad \Bigg\{\sum_{n=1}^{\infty}\big(n^{\frac{\theta_l}{2}}\big\|M_{n,d}(f)-f\big\|_p\big)^q \frac{1}{n}\Bigg\}^{\frac{1}{q}} < \infty \\ &\Leftrightarrow \quad n^{-\frac{1}{q}} n^{\frac{\theta_l}{2}}\big(M_{n,d}(f;\mathrm{x}) - f(\mathrm{x})\big) \in l^q(L_p). \end{aligned} \tag{3.1}$$

Proof First we prove the direct result of (3.1). By applying Lemma 2.3, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\big[n^{\frac{\theta_l}{2}}\big\|M_{n,d}(f)-f\big\|_p\big]^q\frac{1}{n} &\le \sum_{r=0}^{\infty}\sum_{n=2^r}^{2^{r+1}-1}\big[n^{\frac{\theta_l}{2}} MK_\varphi^2\big(f;n^{-1}\big)_p\big]^q n^{-1} \\ &\le M\sum_{r=0}^{\infty}\big[n^{(r+1)\frac{\theta_l}{2}} K_\varphi^2\big(f;2^{-r}\big)_p\big]^q \\ &\le M\frac{1}{\ln 2}\big(2^{1+\frac{\theta_l}{2}}\big)^q \sum_{r=0}^{\infty}\int_{2^{-(r+1)}}^{2^{-r}}\big[t^{-\frac{\theta_l}{2}} K_\varphi^2(f;t)_p\big]^q\frac{dt}{t} \\ &\le M\frac{1}{\ln 2}\big(2^{1+\frac{\theta_l}{2}}\big)^q \int_0^1\big[t^{-\frac{\theta_l}{2}} K_\varphi^2(f;t)_p\big]^q\frac{dt}{t}.\end{aligned}$$

In virtue of $f \in B_{p,q}^{\frac{\theta}{2}}$ and by Theorem 1.2, we have

$$\sum_{n=1}^{\infty}\big[n^{\frac{\theta_l}{2}}\big\|M_{n,d}(f)-f\big\|_p\big]^q\frac{1}{n} < \infty. \tag{3.2}$$

The necessity has been proved.

Next, we prove the inverse result of (3.1). We take a constant $A \in \mathbb{N}$, which will be determined later. For $r \in \mathbb{N}$, we take $n_r \in \mathbb{N}$, which satisfies the following conditions:

$$(1)\ A^{r-1} \le n_r < A^r; \qquad (2)\ \|M_{n_r,d}(f) - f\|_p = \min_{A^{r-1} \le m < A^r} \|M_{m,d}(f) - f\|_p.$$

By using the definition of K-functional and Lemma 2.4, we derive by induction

$$\begin{aligned}
A^{\frac{\theta}{2}} K_\varphi^2(f; A^{-r})_p &\le A^{\frac{\theta_l}{2}} \|f - M_{n_r,d}(f)\|_p + M A^{(\frac{\theta_l}{2}-r)} n_r K_\varphi^2(f; n_r^{-1})_p \\
&\le A^{\frac{\theta_l}{2}} \|f - M_{n_r,d}(f)\|_p + A^{r(\frac{\theta_l}{2}-1)} \big[M n_r \|f - M_{n_{r-1},d}(f)\|_p \\
&\quad + M^2 n_{r-1} K_\varphi^2(f; n_{r-1}^{-1})_p \big] \\
&\le \cdots \\
&\le A^{\frac{\theta_l}{2}} \|f - M_{n_r,d}(f)\|_p + A^{r(\frac{\theta_l}{2}-1)} \left[\sum_{\nu=1}^{r-1} M^l n_{r-\nu+1} \|f - M_{n_{r-\nu},d}(f)\|_p \right. \\
&\quad \left. + M^r n_1 K_\varphi^2(f; n_1^{-1})_p \right] \\
&\le A^{1+\frac{\theta_l}{2}} \sum_{\nu=1}^{r-1} \big(M A^{\nu(\frac{\theta_l}{2}-1)} \big)^\nu \big[n_{r-\nu}^{\frac{\theta_l}{2}} \|f - M_{n_{r-\nu},d}(f)\|_p \big] \\
&\quad + A \big(M A^{\frac{\theta_l}{2}-1} \big)^r \|f\|_p.
\end{aligned}$$

We now choose $A \in \mathbb{N}$, $A \ge 2$, such that $\alpha := M A^{\frac{\theta_l}{2}-1} < \frac{1}{2}$. For $1 < q < \infty$, we have

$$\begin{aligned}
&\int_0^{A^{-1}} \big[t_l^{-\frac{\theta_l}{2}} K_\varphi^2(f; t_l)_p \big]^q \frac{dt_l}{t_l} \qquad (3.3)\\
&\le A^{\frac{\theta_l q}{2}} \ln A \sum_{r=0}^{\infty} \big[A^{\frac{k\theta_l}{2}} K_\varphi^2(f; A^{-r})_p \big]^q \\
&\le 2^q A^{\frac{\theta_l q}{2}} (\ln A) A^{(1+\frac{\theta_l}{2})q} \sum_{r=1}^{\infty} \left\{ \left[\sum_{\nu=0}^{r-1} \alpha^\nu n_{r-1}^{\frac{\theta_l}{2}} \|f - M_{n_{r-\nu},d}(f)\|_p \right]^q + A^q \big(\alpha^\nu \|f\|_p \big)^q \right\} \\
&\le A_1 A^q \frac{\alpha}{1-\alpha} \|f\|_p^q + 2^{q-1} A_1 \sum_{\nu=1}^{\infty} \sum_{r=\nu+1}^{\infty} \alpha^{r-\nu} \big[n_\nu^{\frac{\theta_l}{2}} \|f - M_{n_\nu,d}(f)\|_p \big]^q \\
&\le 2 A_1 A^q \|f\|_p^q + 2^{q-1} A_1 \sum_{\nu=1}^{\infty} \big[n_\nu^{\frac{\theta_l}{2}} \|f - M_{n_\nu,d}(f)\|_p \big]^q \\
&\le M \left\{ \|f\|_p^q + \sum_{\nu=1}^{\infty} \sum_{A^{\nu-1} \le m < A^\nu} \big[m^{\frac{\theta_l}{2}} \|f - M_{m,d}(f)\|_p \big]^q \right\} < \infty. \qquad (3.4)
\end{aligned}$$

The proof for $q = 1$ is easy and we shall omit it. Thus, we have

$$\int_0^1 \big[t_l^{-\frac{\theta_l}{2}} K_\varphi^2(f; t_l)_p \big]^q \frac{dt_l}{t_l} < \infty.$$

By Theorem 1.2, the sufficiency has also been proved. The proof is completed. $\square$

Remark 1 For other integral-type operators, the method and the results are similar.

Competing interests
The authors did not provide this information.

Authors' contributions
The authors did not provide this information.

Author details
[1]School of Mathematics and Information Engineering, Taizhou University, Taizhou, Zhejiang 317000, China. [2]School of Mathematical Sciences, Beijing Normal University, Beijing, 100875, China. [3]School of Science, China University of Mining and Technology, Beijing, 100083, China.

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