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Global existence of solutions and energy decay for a Kirchhoff-type equation with nonlinear dissipation

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Abstract

This paper deals with the initial boundary value problem for a class of nonlinear Kirchhoff-type equation with dissipative term

$$u_{tt} - \varphi(\|\nabla u\|_2^2)\Delta u + a|u_t|^{\alpha-2}u_t = b|u|^{\beta-2}u, \quad x \in \Omega, t > 0$$

in a bounded domain, where $a, b > 0$ and $\alpha, \beta > 2$ are constants. We obtain the global existence of solutions by constructing a stable set in $H_0^1(\Omega)$ and show the energy decay estimate by applying a lemma of Komornik.

MSC: 35B40; 35L70

Keywords: nonlinear Kirchhoff-type equation; initial boundary value problem; stable set; energy decay estimate

1 Introduction

In this paper, we investigate the existence and asymptotic stability of global solutions for the initial boundary value problem of the following Kirchhoff-type equation with nonlinear dissipative term in a bounded domain

$$u_{tt} - \varphi(\|\nabla u\|_2^2)\Delta u + a|u_t|^{\alpha-2}u_t = b|u|^{\beta-2}u, \quad x \in \Omega, t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, t \geq 0, \tag{1.3}$$

where Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$, $a, b > 0$ and $\alpha, \beta > 2$ are constants, $\varphi(s)$ is a C^1-class function on $[0, +\infty)$ satisfying

$$\varphi(s) \geq m_0, \qquad s\varphi(s) \geq \int_0^s \varphi(\theta)\,d\theta, \quad \forall s \in [0, +\infty) \tag{1.4}$$

with $m_0 \geq 1$ is a constant.

If $\Omega = [0, L]$ is an interval of the real line, equation (1.1) describes a small amplitude vibration of an elastic string with fixed endpoints. The original equation is

$$\rho h u_{tt} + \delta u_t + f = \left(\gamma_0 + \frac{Eh}{2L}\int_0^L |u_x|^2\,ds\right)u_{xx},$$

where L is the rest length, E is the Young modulus, ρ is the mass density, h is the cross-section area, γ_0 is the initial axial tension, δ is the resistance modulus and f is a nonlinear perturbation effect.

When $a = b = 0$, $\varphi(s) = s^r$, $r \geq 1$ and $u_0 \neq 0$ (the mildly degenerate case), the local existence of solutions in Sobolev space was investigated by many author [1–6]. Concerning a global existence of solutions for mildly degenerate Kirchhoff equations, it is natural to add a dissipative term u_t or Δu_t.

For $a = 1$, $b = 0$, $\alpha = 2$, $\varphi(s) = s^r$, $r \geq 1$, the problem (1.1)-(1.3) was treated by Nishihara and Yamada [7]. They proved the existence and uniqueness of a global solution $u(t)$ for small data $(u_0, u_1) \in (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$ with $u_0 \neq 0$ and the polynomial decay of the solution. Aassila and Benaissa [8] extended the global existence part of [7] to the case where $\varphi(s) \geq 0$ with $\varphi(\|\nabla u_0\|^2) \neq 0$ and the case of nonlinear dissipative term case ($a \neq 0$).

In the case $a = 0$, for large β and $\varphi(s) \geq r > 0$, D'Ancona and Spagnolo [9] proved that if $u_0, u_1 \in C_0^\infty(R^n)$ are small, then problem (1.1)-(1.3) has a global solution. The nondegenerate case with $\alpha = 2$, $a > 0$ and $b = 0$ was considered by De Brito, Yamada and Nishihara [10–13], they proved that for small initial data $(u_0, u_1) \in (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$ there exists a unique global solution of (1.1)-(1.3) that decays exponentially as $t \to +\infty$.

When $\varphi(s) \geq 0$, Ghisi and Gobbino [14] proved the existence and uniqueness of a global solution $u(t)$ of the problem (1.1)-(1.3) for small initial data $(u_0, u_1) \in (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$ with $m(\|\nabla u_0\|^2) \neq 0$ and the asymptotic behavior $(u(t), u_t(t), u_{tt}(t)) \to (u_\infty, 0, 0)$ in $(H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega) \times L^2(\Omega)$ as $t \to +\infty$, where either $u_\infty = 0$ or $\varphi(\|\nabla u_\infty\|^2) = 0$.

The case $\varphi(s) \geq r > 0$ has been considered by Hosoya and Yamada [15] under the following condition:

$$0 \leq \beta < \frac{2}{n-4}, \quad n \geq 5; \qquad 0 \leq \beta < +\infty, \quad n \leq 4.$$

They proved that, if the initial datas are small enough, the problem (1.1)-(1.3) has a global solution which decays exponentially as $t \to +\infty$.

In this paper, we prove the global existence for the problem (1.1)-(1.3) by applying the potential well theory introduced by Sattinger [16] and Payne and Sattinger [17]. Meanwhile, we obtain the asymptotic stability of global solutions by use of the lemma of Komornik [18].

We adopt the usual notation and convention. Let $H^m(\Omega)$ denote the Sobolev space with the norm $\|u\|_{H^m(\Omega)} = (\sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^2(\Omega)}^2)^{\frac{1}{2}}$, $H_0^m(\Omega)$ denotes the closure in $H^m(\Omega)$ of $C_0^\infty(\Omega)$. For simplicity of notation, hereafter we denote by $\|\cdot\|_p$ the Lebesgue space $L^p(\Omega)$ norm, $\|\cdot\|$ denotes $L^2(\Omega)$ norm and we write equivalent norm $\|\nabla \cdot\|$ instead of $H_0^1(\Omega)$ norm $\|\cdot\|_{H_0^1(\Omega)}$. Moreover, M denotes various positive constants depending on the known constants and it may be difference in each appearance.

This paper is organized as follows: In the next section, we will give some preliminaries. Then in Section 3, we state the main results and give their proof.

2 Preliminaries

In order to state and prove our main results, we first define the following functionals:

$$K(u) = m_0\|\nabla u\|^2 - b\|u\|_\beta^\beta, \qquad J(u) = \frac{m_0}{2}\|\nabla u\|^2 - \frac{b}{\beta}\|u\|_\beta^\beta,$$

for $u \in H_0^1(\Omega)$. Then we define the stable set S by

$$S = \left\{u \in H_0^1(\Omega), K(u) > 0\right\} \cup \{0\}.$$

We denote the total energy functional associated with (1.1)-(1.3) by

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\int_0^{\|\nabla u\|^2} \varphi(s)\,ds - \frac{b}{\beta}\|u\|_\beta^\beta \tag{2.1}$$

for $u \in H_0^1(\Omega)$, $t \geq 0$, and $E(0) = \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\int_0^{\|\nabla u_0\|^2} \varphi(s)\,ds - \frac{b}{\beta}\|u_0\|_\beta^\beta$ is the total energy of the initial data.

Lemma 2.1 *Let q be a number with $2 \leq q < +\infty$, $n \leq 2$ and $2 \leq q \leq \frac{2n}{n-2}$, $n > 2$. Then there is a constant C depending on Ω and q such that*

$$\|u\|_q \leq C\|u\|_{H_0^1(\Omega)}, \quad \forall u \in H_0^1(\Omega).$$

Lemma 2.2 [18] *Let $y(t): R^+ \to R^+$ be a nonincreasing function and assume that there are two constants $\mu \geq 1$ and $A > 0$ such that*

$$\int_s^{+\infty} y(t)^{\frac{\mu+1}{2}}\,dt \leq Ay(s), \quad 0 \leq s < +\infty,$$

then $y(t) \leq Cy(0)(1+t)^{-\frac{2}{\mu-1}}$, $\forall t \geq 0$, if $\mu > 1$, where C is positive constants independent of $y(0)$.

Lemma 2.3 *Let $u(t,x)$ be a solutions to the problem* (1.1)-(1.3). *Then $E(t)$ is a nonincreasing function for $t > 0$ and*

$$\frac{d}{dt}E(t) = -a\left\|u_t(t)\right\|_\alpha^\alpha. \tag{2.2}$$

Proof By multiplying equation (1.1) by u_t and integrating over Ω, we get

$$\frac{d}{dt}E\bigl(u(t)\bigr) = -a\left\|u_t(t)\right\|_\alpha^\alpha \leq 0.$$

Therefore, $E(t)$ is a nonincreasing function on t. □

We state a local existence result, which is known as a standard one (see [6, 19]).

Theorem 2.1 *Suppose that $\alpha, \beta > 2$ satisfy*

$$2 < \beta < +\infty, \quad n \leq 2; \qquad 2 < \beta \leq \frac{2(n-1)}{n-2}, \quad n > 2, \tag{2.3}$$

$$2 < \alpha < +\infty, \quad n \leq 2; \qquad 2 < \alpha \leq \frac{2n}{n-2}, \quad n > 2, \tag{2.4}$$

and let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. *Then there exists* $T > 0$ *such that the problem* (1.1)-(1.3) *has a unique local solution* $u(t)$ *in the class*

$$u \in C([0,T); H_0^1(\Omega)), \qquad u_t \in C([0,T); L^2(\Omega)) \cap L^\alpha(\Omega \times [0,T)). \tag{2.5}$$

In order to prove the existence of global solutions of the problem (1.1)-(1.3), we need the following lemma.

Lemma 2.4 *Supposed that* (2.3) *holds, If* $u_0 \in S$, $u_1 \in L^2(\Omega)$ *such that*

$$\delta = bC^\beta \left(\frac{2\beta}{(\beta-2)m_0} E(0) \right)^{\frac{\beta-2}{2}} < 1, \tag{2.6}$$

then $u \in S$, *for each* $t \in [0,T)$.

Proof Assume that there exists a number $t^* \in [0,T)$ such that $u(t) \in S$ on $[0,t^*)$ and $u(t^*) \notin S$. Then we have

$$K(u(t^*)) = 0, \quad u(t^*) \neq 0. \tag{2.7}$$

Since $u(t) \in S$ on $[0,t^*)$, it holds that

$$\begin{aligned} J(u(t)) &= \frac{m_0}{2} \|\nabla u(t)\|^2 - \frac{b}{\beta} \|u(t)\|_\beta^\beta \\ &\geq \frac{m_0}{2} \|\nabla u(t)\|^2 - \frac{m_0}{\beta} \|\nabla u(t)\|^2 = \frac{(\beta-2)m_0}{2\beta} \|\nabla u(t)\|^2, \end{aligned} \tag{2.8}$$

we have from $K(u(t^*)) = 0$ that

$$\begin{aligned} J(u(t^*)) &= \frac{m_0}{2} \|\nabla u(t^*)\|^2 - \frac{b}{\beta} \|u(t^*)\|_\beta^\beta \\ &= \frac{m_0}{2} \|\nabla u(t^*)\|^2 - \frac{m_0}{\beta} \|\nabla u(t^*)\|^2 = \frac{(\beta-2)m_0}{2\beta} \|\nabla u(t^*)\|^2, \end{aligned} \tag{2.9}$$

we conclude from (1.4) and (2.1) that

$$\begin{aligned} E(t) &\geq \frac{1}{2} \|u_t(t)\|^2 + \frac{m_0}{2} \|\nabla u(t)\|^2 - \frac{b}{\beta} \|u(t)\|_\beta^\beta \\ &= \frac{1}{2} \|u_t(t)\|^2 + J(u(t)). \end{aligned} \tag{2.10}$$

Therefore, we obtain from (2.8), (2.9) and (2.10) that

$$\|\nabla u(t)\|^2 \leq \frac{2\beta}{(\beta-2)m_0} J(u(t)) \leq \frac{2\beta}{(\beta-2)m_0} E(t) \leq \frac{2\beta}{(\beta-2)m_0} E(0), \tag{2.11}$$

for $\forall t \in [0,t^*]$.

By exploiting Lemma 2.1, (2.6) and (2.11), we easily arrive at

$$b\|u(t)\|_\beta^\beta \le bC^\beta \|\nabla u(t)\|^\beta = bC^\beta \|\nabla u(t)\|^{\beta-2} \|\nabla u(t)\|^2 \le bC^\beta \left(\frac{2\beta}{(\beta-2)m_0}E(0)\right)^{\frac{\beta-2}{2}} \|\nabla u(t)\|^2 < \|\nabla u(t)\|^2, \tag{2.12}$$

for all $t \in [0, t^*]$.

Therefore, we obtain

$$K(u(t^*)) = m_0 \|\nabla u(t^*)\|^2 - b\|u(t^*)\|_\beta^\beta > 0, \tag{2.13}$$

which contradicts (2.7). Thus, we conclude that $u(t) \in S$ on $[0, T)$. □

3 The global existence and nonexistence

Theorem 3.1 *Suppose that* (2.3) *and* (2.4) *hold, and* $u(t)$ *is a local solution of problem* (1.1)-(1.3) *on* $[0, T)$. *If* $u_0 \in S$ *and* $u_1 \in L^2(\Omega)$ *satisfy* (2.6), *then* $u(x, t)$ *is a global solution of the problem* (1.1)-(1.3).

Proof It suffices to show that $\|\nabla u(t)\|^2 + \|u_t(t)\|^2$ is bounded independently of t.

Under the hypotheses in Theorem 3.1, we get from Lemma 2.4 that $u(t) \in S$ on $[0, T)$. So the formula (2.8) holds on $[0, T)$.

Therefore, we have from (2.8) that

$$\frac{1}{2}\|u_t\|^2 + \frac{(\beta-2)m_0}{2\beta}\|\nabla u(t)\|^2 \le \frac{1}{2}\|u_t(t)\|^2 + J(u(t)) = E(t) \le E(0). \tag{3.1}$$

Hence, we get

$$\|u_t(t)\|^2 + \|\nabla u(t)\|^2 \le \max\left(2, \frac{2\beta}{(\beta-2)m_0}\right)E(0) < +\infty.$$

The above inequality and the continuation principle lead to the global existence of the solution, that is, $T = +\infty$. Thus, the solution $u(t)$ is a global solution of the problem (1.1)-(1.3). □

Now we employ the analysis method to discuss the solution of the problem (1.1)-(1.3) occurs blow-up in finite time. Our result reads as follows.

Theorem 3.2 *Assume that* (i) $2 < \beta < \frac{2n}{n-2}$, *if* $n > 2$; (ii) $0 < \beta < +\infty$, *if* $n \le 2$. *If* $u_0 \in S$ *and* $u_1 \in L^2(\Omega)$ *such that*

$$E(0) < Q_0, \qquad \|u_0\|_\beta > S_0,$$

where

$$Q_0 = \frac{(\beta-2)b}{2\beta}\left(\frac{m_0}{bC^2}\right)^{\frac{\beta}{\beta-2}}, \qquad S_0 = \left(\frac{m_0}{bC^2}\right)^{\frac{1}{\beta-2}}$$

with $C > 0$ *is a positive Sobolev constant. Then the solution of the problem* (1.1)-(1.3) *does not exist globally in time.*

Proof On the contrary, under the conditions in Theorem 3.2, suppose that $u(x,t)$ is a global solution of the problem (1.1)-(1.3); then by Lemma 2.1, it is well known that there exists a constant $C > 0$ depending only n, β such that $\|u\|_\beta \le C\|\nabla u\|$ for all $u \in H_0^1(\Omega)$.

From the above inequality, we conclude that

$$\|\nabla u\|^2 \ge C^{-2}\|u\|_\beta^2. \tag{3.2}$$

It follows from (1.4), (2.1) and (3.2) that

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\int_0^{\|\nabla u\|^2} \varphi(s)\,ds - \frac{b}{\beta}\|u\|_\beta^\beta \\ &\ge \frac{m_0}{2}\|\nabla u\|^2 - \frac{b}{\beta}\|u\|_\beta^\beta \ge \frac{m_0}{2C^2}\|u\|_\beta^2 - \frac{b}{\beta}\|u\|_\beta^\beta. \end{aligned} \tag{3.3}$$

Setting

$$s = s(t) = \|u(t)\|_\beta = \left\{\int_\Omega |u(x,t)|^\beta\,dx\right\}^{\frac{1}{\beta}}.$$

We denote the right side of (3.3) by $Q(s) = Q(\|u(t)\|_\beta)$, then

$$Q(s) = \frac{m_0}{2C^2}s^2 - \frac{b}{\beta}s^\beta, \quad s \ge 0. \tag{3.4}$$

By (3.4), we obtain

$$Q'(s) = \frac{m_0}{C^2}s - bs^{\beta-1}.$$

Let $Q'(s) = 0$, then we obtain $S_0 = (\frac{m_0}{bC^2})^{\frac{1}{\beta-2}}$.

As $s = S_0$, we have

$$Q''(s)|_{s=S_0} = \left(\frac{m_0}{C^2} - b(\beta-1)s^{\beta-2}\right)\bigg|_{s=S_0} = -\frac{m_0(\beta-2)}{C^2} < 0.$$

Consequently, the function $Q(s)$ has a single maximum value Q_0 at S_0, where

$$Q_0 = Q(S_0) = \frac{(\beta-2)b}{2\beta}\left(\frac{m_0}{bC^2}\right)^{\frac{\beta}{\beta-2}}.$$

Since the initial data is such that $E(0)$, $s(0)$ satisfies $E(0) < Q_0$, $\|u_0\|_\beta > S_0$.

Therefore, we have from Lemma 2.3 that

$$E(t) \le E(0) < Q_0, \quad \forall t > 0.$$

At the same time, by (3.3) and (3.4) it is evident that there can be no time $t > 0$ for which

$$E(t) < Q_0, \qquad s(t) = S_0.$$

Hence, we have also $s(t) > S_0$ for all $t > 0$ from the continuity of $E(t)$ and $s(t)$.

According to the above contradiction we know that the global solution of the problem (1.1)-(1.3) does not exist, *i.e.*, the solution blows up in some finite time.

This completes the proof of Theorem 3.2. □

4 Energy decay estimate

The following theorem shows the asymptotic behavior of global solutions of the problem (1.1)-(1.3).

Theorem 4.1 *If the hypotheses in Theorem* 3.2 *are valid, then the global solutions of the problem* (1.1)-(1.3) *has the following asymptotic property*:

$$E(t) \le M(1+t)^{-\frac{2}{\alpha-2}},$$

where $M > 0$ *is a constant depending on initial energy* $E(0)$.

Proof Multiplying by $E(t)^{\frac{\alpha-2}{2}}u$ on both sides of the equation (1.1) and integrating over $\Omega \times [S,T]$, we obtain that

$$0 = \int_S^T \int_\Omega E(t)^{\frac{\alpha-2}{2}} u\bigl[u_{tt} - \varphi\bigl(\|\nabla u\|^2\bigr)\Delta u + a|u_t|^{\alpha-2}u_t - bu|u|^{\beta-2}\bigr]\,dx\,dt, \tag{4.1}$$

where $0 \le S < T < +\infty$.

Since

$$\begin{aligned}\int_S^T \int_\Omega E(t)^{\frac{\alpha-2}{2}} uu_{tt}\,dx\,dt ={}& \int_\Omega E(t)^{\frac{\alpha-2}{2}} uu_t\,dx\Big|_S^T - \int_S^T \int_\Omega E(t)^{\frac{\alpha-2}{2}} |u_t|^2\,dx\,dt \\ &- \frac{\alpha-2}{2}\int_S^T \int_\Omega E(t)^{\frac{\alpha-4}{2}} E'(t)uu_t\,dx\,dt.\end{aligned} \tag{4.2}$$

So, substituting the formula (4.2) into the right-hand side of (4.1), we get that

$$\begin{aligned}0 ={}& \int_S^T E(t)^{\frac{\alpha-2}{2}} \left(\|u_t\|^2 + \varphi\bigl(\|\nabla u\|^2\bigr)\|\nabla u\|^2 - \frac{2b}{\beta}\|u\|_\beta^\beta\right) dt \\ &- \int_S^T \int_\Omega E(t)^{\frac{\alpha-2}{2}} \bigl[2|u_t|^2 - a|u_t|^{\alpha-2}u_t u\bigr]\,dx\,dt \\ &- \frac{\alpha-2}{2}\int_S^T \int_\Omega E(t)^{\frac{\alpha-4}{2}} E'(t)uu_t\,dx\,dt + \int_\Omega E(t)^{\frac{\alpha-2}{2}} uu_t\,dx\Big|_S^T \\ &+ \left(\frac{2}{\beta}-1\right) b\int_S^T E(t)^{\frac{\alpha-2}{2}} \|u\|_\beta^\beta\,dt.\end{aligned} \tag{4.3}$$

We obtain from (2.12) and (2.11) that

$$b\left(1-\frac{2}{\beta}\right)\|u\|_\beta^\beta \le \delta\frac{\beta-2}{\beta}\|\nabla u\|^2 \le \delta\frac{\beta-2}{\beta}\cdot\frac{2\beta}{(\beta-2)m_0}E(t) = \frac{2\delta}{m_0}E(t). \tag{4.4}$$

We derive from (1.4) that

$$\int_0^{\|\nabla u\|^2} \varphi(s)\,ds \le \varphi\big(\|\nabla u\|^2\big)\|\nabla u\|^2. \tag{4.5}$$

It follows from (4.3), (4.4) and (4.5) that

$$\begin{aligned} &2\left(1-\frac{\delta}{m_0}\right)\int_S^T E(t)^{\frac{\alpha}{2}}\,dt \\ &\quad\le \int_S^T\int_\Omega E(t)^{\frac{\alpha-2}{2}}\big[2|u_t|^2 - a|u_t|^{\alpha-2}u_t u\big]\,dx\,dt \\ &\qquad + \frac{\alpha-2}{2}\int_S^T\int_\Omega E(t)^{\frac{\alpha-4}{2}}E'(t)uu_t\,dx\,dt - \int_\Omega E(t)^{\frac{\alpha-2}{2}}uu_t\,dx\bigg|_S^T. \end{aligned} \tag{4.6}$$

We have from Lemma 2.1 and (3.1) that

$$\begin{aligned} &\left|\frac{\alpha-2}{2}\int_S^T\int_\Omega E(t)^{\frac{\alpha-4}{2}}E'(t)uu_t\,dx\,dt\right| \\ &\quad\le \frac{\alpha-2}{2}\int_S^T E(t)^{\frac{\alpha-4}{2}}\big(-E'(t)\big)\left(\frac{1}{2}\|u\|^2+\frac{1}{2}\|u_t\|^2\right)dt \\ &\quad\le -\frac{\alpha-2}{2}\int_S^T E(t)^{\frac{\alpha-4}{2}}E'(t)\left(\frac{\beta C^2}{(\beta-2)m_0}\cdot\frac{(\beta-2)m_0}{2\beta}\|\nabla u\|^2+\frac{1}{2}\|u_t\|^2\right)dt \\ &\quad\le -\frac{\alpha-2}{2}\max\left(\frac{\beta C^2}{(\beta-2)m_0},1\right)\int_S^T E(t)^{\frac{\alpha-2}{2}}E'(t)\,dt \\ &\quad= -\frac{\alpha-2}{\alpha}\max\left(\frac{\beta C^2}{(\beta-2)m_0},1\right)E(t)^{\frac{\alpha}{2}}\bigg|_S^T \le ME(S)^{\frac{\alpha}{2}}, \end{aligned} \tag{4.7}$$

similarly, we have

$$\begin{aligned} \left|-\int_\Omega E(t)^{\frac{\alpha-2}{2}}uu_t\,dx\bigg|_S^T\right| &\le \max\left(\frac{\beta C^2}{(\beta-2)m_0},1\right)E(t)^{\frac{\alpha}{2}}\bigg|_S^T \\ &\le ME(S)^{\frac{\alpha}{2}}. \end{aligned} \tag{4.8}$$

Substituting the estimates (4.7) and (4.8) into (4.6), we conclude that

$$\begin{aligned} &2\left(1-\frac{\delta}{m_0}\right)\int_S^T E(t)^{\frac{\alpha}{2}}\,dt \\ &\quad\le \int_S^T\int_\Omega E(t)^{\frac{\alpha-2}{2}}\big[2|u_t|^2 - a|u_t|^{\alpha-2}u_t u\big]\,dx\,dt + ME(S)^{\frac{\alpha}{2}}. \end{aligned} \tag{4.9}$$

We get from Young inequality and Lemma 2.3 that

$$\begin{aligned} 2\int_S^T\int_\Omega E(t)^{\frac{\alpha-2}{2}}|u_t|^2\,dx\,dt &\le \int_S^T\int_\Omega\big(\varepsilon_1 E(t)^{\frac{\alpha}{2}} + M(\varepsilon_1)|u_t|^\alpha\big)\,dx\,dt \\ &\le M\varepsilon_1\int_S^T E(t)^{\frac{\alpha}{2}}\,dt + M(\varepsilon_1)\int_S^T\|u_t\|_\alpha^\alpha\,dt \end{aligned}$$

$$\begin{aligned} &= M\varepsilon_1 \int_S^T E(t)^{\frac{\alpha}{2}}\,dt - \frac{M(\varepsilon_1)}{a}\big(E(T)-E(S)\big) \\ &\le M\varepsilon_1 \int_S^T E(t)^{\frac{\alpha}{2}}\,dt + ME(S). \qquad (4.10)\end{aligned}$$

From Young inequality, Lemma 2.1, Lemma 2.3 and (2.11), We receive that

$$\begin{aligned} &-a\int_S^T \int_\Omega E(t)^{\frac{\alpha-2}{2}} uu_t|u_t|^{\alpha-2}\,dx\,dt \\ &\le a\int_S^T E(t)^{\frac{\alpha-2}{2}}\big(\varepsilon_2\|u\|_\alpha^\alpha + M(\varepsilon_2)\|u_t\|_\alpha^\alpha\big)\,dt \\ &\le aC^\alpha\varepsilon_2 E(0)^{\frac{\alpha-2}{2}}\int_S^T \|\nabla u\|^\alpha\,dt + aM(\varepsilon_2)E(S)^{\frac{\alpha-2}{2}}\int_S^T \|u_t\|_\alpha^\alpha\,dt \\ &= aC^\alpha\varepsilon_2 E(0)^{\frac{\alpha-2}{2}}\int_S^T \left(\frac{2\beta}{(\beta-2)m_0}E(t)\right)^{\frac{\alpha}{2}}dt + M(\varepsilon_2)E(S)^{\frac{\alpha-2}{2}}\big(E(S)-E(T)\big) \\ &\le C^\alpha\varepsilon_2 E(0)^{\frac{\alpha-2}{2}}\left(\frac{2\beta}{(\beta-2)m_0}\right)^{\frac{\alpha}{2}}\int_S^T E(t)^{\frac{\alpha}{2}}\,dt + ME(S)^{\frac{\alpha}{2}}. \qquad (4.11)\end{aligned}$$

Choosing small enough ε_1 and ε_2, such that

$$\frac{1}{2}\left[M\varepsilon_1 + E(0)^{\frac{\alpha-2}{2}}\left(\frac{2\beta C^2}{(\beta-2)m_0}\right)^{\frac{\alpha}{2}}\varepsilon_2\right] + \frac{\delta}{m_0} < 1,$$

then, substituting (4.10) and (4.11) into (4.9), we get

$$\int_S^T E(t)^{\frac{\alpha}{2}}\,dt \le ME(S) + ME(S)^{\frac{\alpha}{2}} \le M\big(1+E(0)\big)^{\frac{\alpha-2}{2}}E(S).$$

Therefore, we have from Lemma 2.2 that

$$E(t) \le M(1+t)^{-\frac{\alpha-2}{2}}, \quad t\in[0,+\infty).$$

The proof of Theorem 4.1 is thus finished. □

Competing interests
The author declares that he has no competing interests.

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