Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH   **Open Access**

Existence of four solutions for the elliptic problem with nonlinearity crossing one eigenvalue

Tacksun Jung[1]* and Q-Heung Choi[2]

*Correspondence:
tsjung@kunsan.ac.kr
[1] Department of Mathematics,
Kunsan National University, Kunsan,
573-701, Korea
Full list of author information is
available at the end of the article

Abstract

We investigate the multiplicity of the weak solutions for the nonlinear elliptic boundary value problem. We get a theorem which shows the existence of at least four weak solutions for the asymptotically linear elliptic problem with Dirichlet boundary condition. We obtain this result by using the Leray-Schauder degree theory, the variational reduction method and critical point theory.
MSC: 35J15; 35J25

Keywords: elliptic boundary value problem; jumping nonlinearity; inverse compact operator; variational reduction method; Leray-Schauder degree theory

1 Introduction

Let Ω be a bounded open subset of R^n with smooth boundary $\partial\Omega$. Let L be the self-adjoint strongly elliptic partial differential operator

$$L = \sum_{1 \le i,j \le n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j},$$

where $a_{ij}(x) \in L^\infty(\Omega)$. Let $f : R \to R$ be a C^1 function. Let λ_k be the eigenvalues and ϕ_k, $k = 1, 2, \ldots$, be the associated normalized eigenfunctions of the eigenvalue problem $Lu + \lambda u = 0$ in Ω, $u = 0$ on $\partial\Omega$. We note that $0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_k \to \infty$ and $\phi_1 > 0$.

In this paper, we consider the multiplicity of solutions of the following elliptic problem with Dirichlet boundary condition and jumping nonlinearity:

$$Lu + bu^+ - au^- = s\phi_1 \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega.$$

The physical model for this kind of the jumping nonlinearity problem can be furnished by traveling waves in suspension bridges. The nonlinear equations with jumping nonlinearity have been extensively studied by Lazer and McKenna [1], McKenna and Walter [2], Tarantello [3], Micheletti and Pistoia [4, 5], Jung and Choi [6–8] and many other authors. McKenna and Walter [2] proved that if $-\infty < f'(-\infty) < \lambda_1$ and $\lambda_n < f'(+\infty)$, then there exists s_0 such that for all $s \ge s_0$, (1.1) has at least two solutions if n is odd, and three solutions if n is even.

Our main result is as follows.

Theorem 1.1 *Assume that $-\infty < a < \lambda_1 < b < \lambda_2$, $s > 0$. Then (1.1) has at least four solutions.*

For the proof of Theorem 1.1 we use the Leray-Schauder degree theory, the variational reduction method and the critical point theory. The outline of this paper is as follows: In Section 2, we show the existence of the third weak solution for (1.1) by the variational reduction method and critical point theory. In Section 3, we show the existence of the fourth weak solution for (1.1) by the Leray-Schauder degree theory.

2 Existence of the third weak solution

We note that (1.1) has a positive solution $v_0 = \frac{s\phi_1}{b-\lambda_1}$ and a negative solution $v_1 = \frac{s\phi_1}{a-\lambda_1}$. To show the existence of the third solution of (1.1), we shall use the variational reduction method and the critical point theory. We assume that $-\infty < a < \lambda_1 < b < \lambda_2$ and $s > 0$. Let $g(\xi) = b\xi^+ - a\xi^-$ and $H_0^1(\Omega)$ be the Sobolev space with the norm

$$\|u\|^2 = \int_\Omega (-Lu) \cdot u \, dx.$$

We consider the following functional associated with (1.1):

$$I_{a,b}(u,s) = \int_\Omega \left[-\frac{1}{2} Lu \cdot u - G(u) + s\phi_1 u \right] dx, \tag{2.1}$$

where

$$G(u) = \frac{b|u^+|^2}{2} + \frac{a|u^-|^2}{2}.$$

For simplicity, we shall write $I = I_{a,b}$ when a and b are fixed. By the condition $-\infty < a < \lambda_1 < b < \lambda_2$ and $s > 0$, I is well defined. By the following Lemma 2.1, $I(u,s) \in C(H_0^1(\Omega), R)$ and *Fréchet* differentiable in $H_0^1(\Omega)$, so the solutions of (1.1) coincide with the critical points of $I(u,s)$.

Lemma 2.1 *Assume that $-\infty < a < \lambda_1 < b < \lambda_2$ and $s > 0$. Then $I(u,s)$ is continuous and Fréchet differentiable in $H_0^1(\Omega)$ and*

$$D_u I(u,s)(h) = \int_\Omega \left[-Lu \cdot h - \left(bu^+ - au^- \right) h + s\phi_1 h \right] dx$$

for $h \in H_0^1(\Omega)$. Moreover, if we set

$$F(u,s) = \int_\Omega \left[\frac{b|u^+|^2}{2} + \frac{a|u^-|^2}{2} - s\phi_1 u \right] dx,$$

then $D_u F(u,s)$ is continuous with respect to weak convergence, $D_u F(u,s)$ is compact, and

$$D_u F(u,s)h = \int_\Omega \left[\left(bu^+ - au^- \right) h - s\phi_1 h \right] dx \quad \text{for all } h \in H_0^1(\Omega).$$

This implies that $I \in C^1(H_0^1(\Omega), R)$ and $F(u,s)$ is weakly continuous.

Proof Let $u \in H_0^1(\Omega)$. First, we will prove that $I(u,s)$ is continuous with respect to the variable u. We consider

$$\left| I(u+v,s) - I(u,s) \right| = \int_\Omega \left[u \cdot (-Lv) + \frac{1}{2} v \cdot (-Lv) - G(u+v) + G(u) \right] dx.$$

Let $u = \sum h_m \phi_m$, $v = \sum \tilde{h}_m \phi_m$. Then we have

$$\left| \int_\Omega u \cdot (-Lv) \, dx \right| = \left| \sum \int_\Omega \lambda_m h_m \tilde{h}_m \right| \le \|u\| \|v\|,$$

$$\left| \int_\Omega v \cdot (-Lv) \, dx \right| = \left| \sum \lambda_m \tilde{h}_m^2 \right| \le \|v\|^2.$$

On the other hand,

$$\left| \left| (u+v)^+ \right|^2 - \left| u^+ \right|^2 \right| \le 2u^+ |v| + |v|^2,$$

$$\left| \left| (u+v)^- \right|^2 - \left| u^- \right|^2 \right| \le 2u^- |v| + |v|^2,$$

and hence

$$\left| \int_\Omega \left(\left| (u+v)^+ \right|^2 - \left| u^+ \right|^2 \right) dx \right|$$

$$\le \int_\Omega \left[2 \left| u^+ \right| |v| + |v|^2 \right] dx \le 2\|u\| \|v\| + \|v\|^2,$$

$$\left| \int_\Omega \left(\left| (u+v)^- \right|^2 - \left| u^- \right|^2 \right) dx \right|$$

$$\le \int_\Omega \left[2 \left| u^- \right| |v| + |v|^2 \right] dx \le 2\|u\| \|v\| + \|v\|^2.$$

With the above results, we see that $I(u,s)$ is continuous at u. To prove $I(u,s)$ is *Fréchet* differentiable at $u \in H_0^1(\Omega)$, it is enough to compute the following:

$$\left| I(u+v,s) - I(u,s) - D_u I(u,s)v \right| = \left| \int_\Omega \frac{1}{2} v(-Lv) - G(u+v) + G(u) - g(u)v \right|$$

$$\le \frac{1}{2} \|v\|^2 + C\gamma \|v\| \left(\|u\| + \|v\| \right) + M\|v\|$$

$$\le C' \|v\| \left(\|v\| + \|u\| + \|v\| + 1 \right). \qquad \square$$

Let V be the one-dimensional space spanned by the eigenfunction ϕ_1 whose eigenvalue is λ_1. Let W be the orthogonal complement of V in $H_0^1(\Omega)$. Let $P : H_0^1(\Omega) \to V$ be the orthogonal projection of $H_0^1(\Omega)$ onto V and $I - P : H_0^1(\Omega) \to W$ denote that of $H_0^1(\Omega)$ onto W. Then every element $u \in H_0^1(\Omega)$ is expressed by $u = v + z$, $v \in Pu$, $z = (I-P)u$. Then (2.1) is equivalent to the two systems in the two unknowns v and z:

$$Lv + P\left(b(v+w)^+ - a(v+w)^- \right) = s\phi_1 \quad \text{in } \Omega, \tag{2.2}$$

$$Lz + (I-P)\left(b(v+w)^+ - a(v+w)^- \right) = 0 \quad \text{in } \Omega, \tag{2.3}$$

$$v = z = 0 \quad \text{on } \partial\Omega.$$

The subspace W is spanned by eigenfunctions corresponding to the eigenvalues λ_k, $k \geq 2$. Let $v \in V$ be fixed and consider the function $h : W \rightarrow R$ defined by

$$h(w) = I(v + w, s).$$

The function h has continuous *Fréchet* derivatives Dh given by

$$Dh(w)(y) = D_u I(v + w, s)(y) \tag{2.4}$$

for $y \in W$. By Lemma 2.1, I is a function of class C^1.

By the following Lemma 2.2, we can get the critical points of the functional $I(u, s)$ on the infinite dimensional space $H_0^1(\Omega)$ from that of the reduced functional $\tilde{I}(v, s)$ on the finite dimensional subspace V.

Lemma 2.2 (Reduction Method) *Assume that $-\infty < a < \lambda_1 < b < \lambda_2$ and $s > 0$. Then*

(i) *there exists a unique solution $z \in W$ of the equation*

$$Lz + (I - P)\big(b(v + w)^+ - a(v + w)^-\big) = 0 \quad in \ \Omega,$$

$$z = 0 \quad on \ \Omega.$$

If we put $z = \theta(v, s)$, then θ is continuous on V and satisfies a uniform Lipschitz condition in v with respect to L^2 norm (also norm $\| \cdot \|$). Moreover,

$$D_u I\big(v + \theta(v, s), s\big)(w) = 0 \quad for \ all \ w \in W.$$

(ii) *There exists $m > 0$ such that if w and y are in W, then*

$$\big(Dh(w) - Dh(y)\big)(w - y) \geq m\|w - y\|^2.$$

(iii) *If $\tilde{I} : V \rightarrow R$ is defined by $\tilde{I}(v, s) = I(v + \theta(v, s), s)$, then $\tilde{I}$ has a continuous Fréchet derivative $D_v \tilde{I}$ with respect to v, and*

$$D_v \tilde{I}(v, s)(h) = D_u I\big(v + \theta(v, s), s\big)(h) \quad for \ all \ v, h \in V. \tag{2.5}$$

(iv) *If $v_0 \in V$ is a critical point of $\tilde{I}$ if and only if $v_0 + \theta(v_0, s)$ is a critical point of I.*

(v) *Let $S \subset V$ and $\Sigma \subset H_0^1(\Omega)$ be open bounded regions such that*

$$\big\{v + \theta(v, s); v \in S\big\} = \Sigma \cap \big\{v + \theta(v, s); v \in V\big\}.$$

If $D\tilde{I}(v, s) \neq 0$ for $v \in \partial S$, then

$$\deg(D\tilde{I}, S, 0) = \deg(DI, \Sigma, 0),$$

where d denote the Leray-Schauder degree.

(vi) *If $u_0 = v_0 + \theta(v_0)$ is a critical point of mountain pass type of I, then v_0 is a critical point of mountain pass type of $\tilde{I}$.*

Proof (i) Let $\delta = \frac{a+b}{2}$. If $g_1(\xi) = g(\xi) - \delta\xi$, then equation (2.3) is equivalent to

$$z = (-L - \delta)^{-1}(I - P)\big(g_1(v + z)\big). \tag{2.6}$$

The operator $(-L - \delta)^{-1}(I - P)$ is self adjoint, compact and linear map from $(I - P)L^2(\Omega)$ into itself and its L_2 norm is $|\lambda_2 - \delta|^{-1}$. Since $|g_1(\xi_2) - g_1(\xi_1)| \le \max\{|a - \delta|, |b - \delta|\}|\xi_2 - \xi_1| = \delta|\xi_2 - \xi_1|$, it follows that the right-hand side of (2.6) defines, for fixed $v \in V$, a Lipschitz mapping of $(I - P)L^2(\Omega)$ into itself with Lipschitz constant $r = \frac{\delta}{|\lambda_2 - \delta|} < 1$. Therefore, by the contraction mapping principle, for given $v \in V$, there exists a unique $z = (I - P)L^2(\Omega)$ which satisfies (2.6). If $\theta(v, s)$ denote the unique $z \in (I - P)L^2(\Omega)$ which solves (2.3), then θ is continuous and satisfies a uniform Lipschitz condition in v with respect to L^2 norm (also norm $\| \cdot \|$). In fact, if $z_1 = \theta(v_1, s)$ and $z_2 = \theta(v_2, s)$, then

$$
\begin{aligned}
\|z_1 - z_2\|_{L^2(\Omega)} &= \big\|(-L - \delta)^{-1}(I - P)\big(g_1(v_1 + z_1) - g_1(v_2 + z_2)\big)\big\|_{L^2(\Omega)} \\
&\le r\big\|(v_1 + z_1) - (v_2 + z_2)\big\|_{L^2(\Omega)} \\
&\le r\big(\|v_1 - v_2\|_{L^2(\Omega)} + \|z_1 - z_2\|_{L^2(\Omega)}\big) \le r\|v_1 - v_2\| + r\|z_1 - z_2\|.
\end{aligned}
$$

Hence,

$$\|z_1 - z_2\| \le C\|v_1 - v_2\|, \quad C = \frac{r}{1 - r}. \tag{2.7}$$

Let $u = v + z$, $v \in V$ and $z = \theta(v, s)$. If $w \in (I - P)L^2(\Omega) \cap H_0^1(\Omega)$,

$$
\begin{aligned}
D_u I\big(v + \theta(v, s), s\big)(w) = \int_\Omega \big[&-L(v + z) \cdot w - Pg(v + z)w \\
&- (I - P)g(v + w)w + s\phi_1 w\big]\,dx.
\end{aligned}
$$

From (2.3), we see that

$$\int_\Omega \big[(-Lz) \cdot w - (I - P)\big(g(v + z)w\big)\big]\,dx = 0.$$

Since

$$\int_\Omega Lv \cdot w = 0 \quad \text{and} \quad \int_\Omega \big[Pg(v + z)w - s\phi_1 w\big]\,dx = 0,$$

we have

$$D_u I\big(v + \theta(v, s), s\big)(w) = 0. \tag{2.8}$$

(ii) If w and y are in W, then

$$\big(Dh(w) - Dh(y)\big)(w - y) = \int_\Omega \big[-L(w - y)(w - y) - \big(g(v + w) - g(v + y)\big)(w - y)\big]\,dx.$$

Since $\int_\Omega [-L(w-y)(w-y)]\,dx = \|w-y\|^2$ and $(g(\xi_2) - g(\xi_1))(\xi_2 - \xi_1) < b(\xi_2 - \xi_1)^2$, we see that if w and y are in W, then $\|w-y\|_{L^2(\Omega)} \leq \frac{1}{\lambda_2}\|w-y\|^2$ and

$$\int_\Omega \left[\left(g(v+w) - g(v+y) \right)(w-y) \right] dx \leq \frac{b}{\lambda_2}\|w-y\|^2$$

and

$$\left(Dh(w) - Dh(y) \right)(w-y) \geq \left(1 + b\frac{b}{\lambda_2} \right)\|w-y\|^2,$$

where $(1 + b\frac{b}{\lambda_2}) > 0$.

(iii) Since the functional I has a continuous *Fréchet* derivative $D_u I$, $\tilde{I}$ has a continuous *Fréchet* derivative $D_v \tilde{I}$ with respect to v.

(iv) Suppose that there exists $v_0 \in V$ such that $D_v \tilde{I}(v_0, s) = 0$. From $D_v \tilde{I}(v, s)(h) = D_u I(v + \theta(v,s), s)(h)$ for all $v, h \in V$, $D_u I(v_0 + \theta(v_0, s), s)(h) = D_v \tilde{I}(v_0, s)(h) = 0$ for all $h \in V$. Since $D_u I(v + \theta(v,s), s)(w) = 0$ for all $w \in W$ and E is the direct sum of V and W, it follows that $D_u I(v_0 + \theta(v_0, s), s) = 0$. Thus, $v_0 + \theta(v_0, s)$ is a solution of (1.1). Conversely, if u is a solution of (1.1) and $v = Pu$, then $D_v \tilde{I}(v) = 0$.

(v) The proof of part (v) follows by arguing as in Lemma 2.6 of [1].

(vi) Suppose v_0 is not of mountain pass type of $\tilde{I}$. Let S be an open neighborhood of v_0 in V such that $\tilde{I}^{-1}(-\infty, \tilde{I}(v_0)) \cap S$ is empty or path connected. If $\tilde{I}^{-1}(-\infty, \tilde{I}(v_0)) \cap S$ is empty, by part (i) we see that $\{v + w : v \in V, w \in W\} \cap I^{-1}(-\infty, I(u_0))$ is also empty. Thus u_0 is not of mountain pass type for I. If $\tilde{I}^{-1}(-\infty, \tilde{I}(v_0)) \cap S$ is path connected, Letting $T = \{v + w : v \in V, \|\|w - \theta(v)\|\| < 1\}$ and using again (i) it is seen that $T \cap I^{-1}(-\infty, I(u_0))$ is also path connected. □

We define the functional on $H_0^1(\Omega)$

$$I_{a,b}^*(u) = I_{a,b}(u, 0) = \int_\Omega \left[\frac{1}{2}(-Lu)\cdot u - \frac{b}{2}\left|u^+\right|^2 - \frac{a}{2}\left|u^-\right|^2 \right] dx.$$

The critical points of $I_{a,b}^*(u)$ coincide with the solutions of the equation

$$Lu + bu^+ - au^- = 0 \quad \text{in } H_0^1(\Omega). \tag{2.9}$$

Under the assumption $-\infty < a < \lambda_1 < b < \lambda_2$, (2.9) has only the trivial solution and hence $I_{a,b}^*(u)$ has only one critical point $u = 0$. In fact, from (2.9), we obtain

$$\int_\Omega \left[(L + \lambda_1)u \cdot \phi_1(x) + (b - \lambda_1)u^+ \phi_1(x) - \left(a - \lambda_1(x) \right)u^- \cdot \phi_1(x) \right] dx = 0 \tag{2.10}$$

under the assumption $-\infty < a < \lambda_1 < b < \lambda_2$ the left-hand side of (2.10) is nonnegative, so the only possibility to hold (2.10) is that $u = 0$. Given $v \in V$, let $\theta^*(v) = \theta(v, 0)$ be the unique solution of the equation

$$Lz + (I - P)\left[b(v+z)^+ - a(v+z)^- \right] = 0 \quad \text{in } W.$$

Let us define $\tilde{I}_{a,b}^*(v) = I_{a,b}^*(v + \theta^*(v))$. We note that $\tilde{I}_{a,b}^*(v)$ has only the trivial critical point $v = 0$.

Lemma 2.3 *Let* $-\infty < a < \lambda_1 < b < \lambda_2$. *Then we have*

$$\tilde{I}_{a,b}^*(v) > 0 \quad \text{for all } v \in V \text{ with } v \neq 0.$$

Proof To prove the conclusion, it suffices to show that $\tilde{I}_{a,b}^*(v)$ does not satisfies the following cases:

(i) $\tilde{I}_{a,b}^*(v) \leq 0$ and $\tilde{I}_{a,b}^*(v_0) = 0$ for some $v_0 \in V$ with $v_0 \neq 0$.

(ii) $\tilde{I}_{a,b}^*(v) \geq 0$ and $\tilde{I}_{a,b}^*(v_1) = 0$ for some $v_1 \in V$ with $v_1 \neq 0$.

(iii) $\tilde{I}_{a,b}^*(v) < 0$ for all $v \in V$ with $v \neq 0$.

(iv) There exist v_1 and v_2 in V such that $\tilde{I}_{a,b}^*(v_1) < 0$ and $\tilde{I}_{a,b}^*(v_2) > 0$.

Suppose that (i) holds. It follows that $\tilde{I}_{a,b}^*(v)$ has an absolute maximum at v_0, and hence $D\tilde{I}_{a,b}^*(v_0) = 0$. Therefore, by Lemma 2.2, $u_0 = v_0 + \theta^*(v_0)$ is a nontrivial solution of (2.9), which is a contradiction. A similar argument show that it is impossible that (ii) holds. Suppose that (iii) holds. Then there exists $t_0 \in (0,1)$ such that for all t with $0 < t \leq t_0$,

$$t\tilde{I}_{a,b}^*(v) + (1-t)\tilde{I}_{0,b}^*(v) \leq 0 \quad \text{for all } v \neq 0.$$

We note that there exist $v_0 \neq 0$ and t with $0 < t \leq t_0$ such that $t\tilde{I}_{a,b}^*(v_0) + (1-t)\tilde{I}_{0,b}^*(v_0) = 0$ for some $v_0 \neq 0$ and t. Let t_1 be the greatest number such that

$$t\tilde{I}_{a,b}^*(v_0) + (1-t)\tilde{I}_{0,b}^*(v_0) = 0.$$

Then $0 < t_1 \leq t_0$. Since $t_1\tilde{I}_{a,b}^*(v) + (1-t_1)\tilde{I}_{0,b}^*(v) \leq 0$ for all $v \neq 0$, and hence v_0 is a point of maximum of $t_1\tilde{I}_{a,b}^*(v) + (1-t_1)\tilde{I}_{0,b}^*(v)$, we have

$$D\big[t_1\tilde{I}_{a,b}^*(v) + (1-t_1)\tilde{I}_{0,b}^*(v)\big] = 0.$$

Let $v \in V$ be given and $0 < t_1 < 1$. Let $\theta_{t_1}^*(v)$ be the unique solution of the equation

$$Lz + (I-P)\big(bu^+ - tau^-\big) = 0 \quad \text{in } W.$$

By Lemma 2.2, $v_{0+\theta_{t_1}^*(v_0)}$ is a nontrivial solution of the equation

$$t_1\big(Lu + bu^+ - au^-\big) + (1-t_1)\big(Lu + bu^+ - au^-\big) = 0 \quad \text{in } H_0^1(\Omega),$$

that is,

$$Lu + bu^+ - t_1au^- = 0 \quad \text{in } H_0^1(\Omega),$$

which contradicts the fact that the above equation has only the trivial solution because $-\infty < t_1a < \lambda_1 < b < \lambda_2$. Suppose that (iv) holds. Since the equation (2.9) has only the trivial solution, $I_{a,b}^*(u)$ has only one critical point $u = 0$. By the assumption (iv), the trivial solution $u = 0$ is a point of inflexion, and hence by Lemma 2.2, $v = 0$ is the critical point of $\tilde{I}_{a,b}^*(v)$, which is a point of inflexion. By the assumption (iv), there exist v_3, v_4 in V and $t_0 \in (0,1)$ such that for all t with $0 < t \leq t_0$, $t\tilde{I}_{a,b}^*(v_3) + (1-t)\tilde{I}_{0,b}^*(v_3) < 0$ and $t\tilde{I}_{a,b}^*(v_4) + (1-t)\tilde{I}_{0,b}^*(v_4) > 0$.

We note that there exists t with $0 < t \leq t_0$ such that $t\tilde{I}^*_{a,b}(v_3) + (1-t)\tilde{I}^*_{0,b}(v_3) < 0$ and $t\tilde{I}^*_{a,b}(v_4) + (1-t)\tilde{I}^*_{0,b}(v_4) > 0$. Let t_1 be the greatest number such that

$$t\tilde{I}^*_{a,b}(v_3) + (1-t)\tilde{I}^*_{0,b}(v_3) < 0 \quad \text{and} \quad t\tilde{I}^*_{a,b}(v_4) + (1-t)\tilde{I}^*_{0,b}(v_4) > 0.$$

Then $0 < t_1 \leq t_0$. Since $t_1\tilde{I}^*_{a,b}(v_3) + (1-t_1)\tilde{I}^*_{0,b}(v_3) < 0$ and $t_1\tilde{I}^*_{a,b}(v_4) + (1-t_1)\tilde{I}^*_{0,b}(v_4) > 0$. Thus, there exists a point of inflexion $v_0 \neq 0$ of $t_1\tilde{I}^*_{a,b}(v) + (1-t_1)\tilde{I}^*_{0,b}(v)$ and we have

$$Dt_1\tilde{I}^*_{a,b}(v) + (1-t_1)\tilde{I}^*_{0,b}(v_0) = 0.$$

Let $v \in V$ be given and $0 < t_1 < 1$. Let $\theta^*_{t_1}(v)$ be the unique solution of the equation

$$Lz + (I-P)\big(bu^+ - tau^-\big) = 0 \quad \text{in } W.$$

By Lemma 2.2, $v_0 + \theta^*_{t_1}(v_0)$ is a nontrivial solution of the equation

$$t_1\big(Lu + bu^+ - au^-\big) + (1-t_1)\big(Lu + bu^+ - au^-\big) = 0 \quad \text{in } H^1_0(\Omega),$$

that is,

$$Lu + bu^+ - t_1au^- = 0 \quad \text{in } H^1_0(\Omega),$$

which contradicts the fact that the above equation has only the trivial solution because $-\infty < t_1a < \lambda_1 < b < \lambda_2$. □

From now on, we shall denote that $I_{a,b} = I$.

Lemma 2.4 *Let $-\infty < a < \lambda_1 < b < \lambda_2$. Then we have*

$$\tilde{I}_{a,b}(v,s) \to \infty \quad \text{as } \|v\| \to \infty.$$

Proof We shall prove the lemma by contradiction. We suppose that there exists a sequence $\{v_n\}^\infty_1$ in V and a number $M > 0$ such that $\|v_n\| \to \infty$ as $n \to \infty$ and $\tilde{I}(v_n,s) \leq M$. For given $v_n \in V$, let $w_n = \theta(v_n,s)$ be the unique solution of the equation

$$Lw + (I-P)\big(b(v_n + w)^+ - a(v_n + w)^- - s\phi_1 - h(x)\big) = 0 \quad \text{in } W.$$

By Lemma 2.2, we have that for some constant k,

$$\big\|\theta(v_n) - \theta(0)\big\| \leq k\|v_n\|.$$

From this the sequence $\{\frac{\|v_n + w_n\|}{\|v_n\|}\}$ is bounded in H. Let $u_n = v_n + w_n$, $v^*_n = \frac{v_n}{\|v_n\|}$, $w^*_n = \frac{w_n}{\|v_n\|}$ and $u^*_n = v^*_n + w^*_n$ for $n \geq 1$. For $w_n = \theta(v_n,s)$ and $w^*_n = \frac{w_n}{\|v_n\|}$, we have

$$w^*_n = L^{-1}\left((I-P)\left(-b\frac{(v_n + w_n)^+}{\|v_n\|} + a\frac{(v_n + w_n)^-}{\|v_n\|} - s\frac{\phi_1}{\|v_n\|} - \frac{h(x)}{\|v_n\|}\right)\right) \quad \text{in } W.$$

Since $\{\frac{v_n+w_n}{\|v_n\|}\}$ is bounded and $\frac{\phi_1}{\|v_n\|} \to 0$, $-b\frac{(v_n+w_n)^+}{\|v_n\|} + a\frac{(v_n+w_n)^-}{\|v_n\|} - s\frac{\phi_1}{\|v_n\|} - \frac{h(x)}{\|v_n\|}$ is bounded in H. Since L^{-1} is a compact operator, by passing to a subsequence, we get that $\{w_n^*\}_1^\infty$ converges to w^*. Since V is 1-dimensional subspace, we may assume that $\{v_n^*\}_1^\infty$ converges to $v^* \in V$ with $\|v^*\| = 1$. Therefore, $\{u_n^*\}_1^\infty$ converges to u^* in $H_0^1(\Omega)$. Since $\tilde{I}(v_n, s) \le M$ for all n, we have, for all n,

$$\int_\Omega \left[-\frac{1}{2} Lu_n \cdot u_n - \frac{b}{2}|u_n^+|^2 - \frac{a}{2}|u_n^-|^2 + s\phi_1 u_n + h(x)u_n \right] dx \le M.$$

Dividing the above inequality by $\|v_n\|^2$, we obtain

$$\int_\Omega \left[-\frac{1}{2} Lu_n^* \cdot u_n^* - \frac{b}{2}|(u_n^*)^+|^2 - \frac{a}{2}|(u_n^*)^-|^2 + s\phi_1 \frac{u_n^*}{\|v_n\|} + h(x)\frac{u_n^*}{\|v_n\|} \right] dx$$
$$\le \frac{M}{\|v_n\|^2}. \tag{2.11}$$

From the definition of $w_n = \theta(v_n, s)$, we have that for any $y \in W$, $n \ge 1$,

$$\int_\Omega \left[Lu_n \cdot y + bu_n^+ y - au_n^- y - s\phi_1 y - h(x)y \right] dx = 0. \tag{2.12}$$

Let us set $y = w_n$ in (2.12) and divide by $\|v_n\|^2$. Then we have

$$\int_\Omega \left[Lu_n^* \cdot w_n^* + b(u_n^*)^+ w_n^* - a(u_n^*)_n^- w_n^* - s\phi_1 \frac{w_n^*}{\|v_n\|} - h(x)\frac{w_n^*}{\|v_n\|} \right] dx = 0 \tag{2.13}$$

for all $n \ge 1$. Let $y \in W$. Dividing (2.12) by $\|v_n\|$ and letting $n \to \infty$, we obtain

$$\int_\Omega \left[Lu^* \cdot y + b(u^*)^+ y - a(u^*)^- y - s\phi_1 y - h(x)y \right] dx = 0. \tag{2.14}$$

(2.14) can be rewritten in the form

$$D\tilde{I}_{a,b}^*(v^* + w^*)(y) = 0 \quad \text{for all } y \in W.$$

By $w^* = \theta(v^*)$, letting $n \to \infty$ in (2.13), we have

$$\lim_{n\to\infty} \int_\Omega Lw_n^* \cdot w_n^* \, dx = -\int_\Omega \left[bu^* w^* - au^* w^* \right] dx$$
$$= \int_\Omega \left[Lu^* \cdot w^* \right] dx = \int_\Omega \left[Lw^* \cdot w^* \right] dx.$$

Hence,

$$\lim_{n\to\infty} \int_\Omega Lu_n^* \cdot u_n^* \, dx = \int_\Omega \left[Lu^* \cdot u^* \right] dx.$$

Letting $n \to \infty$ in (2.11), we obtain

$$\tilde{I}_{a,b}^*(v^*) = \int_\Omega \left[-\frac{1}{2} Lu^* \cdot u^* - \frac{b}{2}|(u^*)^+|^2 - \frac{a}{2}|(u^*)^-|^2 \right] dx \le 0.$$

Since $\|v^*\| = 1$, this contradicts the fact that $\tilde{I}^*(v) > 0$ for all $v \neq 0$. Thus, $\tilde{I}(v,s) \to \infty$ as $\|v\| \to \infty$. $\qquad\square$

Lemma 2.5 *Assume that* $-\infty < a < \lambda_1 < b < \lambda_2$ *and* $s > 0$. *Then there exists a small open neighborhood* B *of* v_0 *in* V *such that* v_0 *is a strict local point maximum of* $\tilde{I}(v,s)$ *and there exists a small open neighborhood* D *of* v_1 *in* V *such that* v_1 *is a strict point of minimum of* $\tilde{I}(v,s)$, *where* $v_0 = \frac{s\phi_1}{b-\lambda_1}$ *is a positive solution and* $v_1 = \frac{s\phi_1}{a-\lambda_1}$ *is a negative solution of* (1.1).

Proof Since the positive solution $v_0 = \frac{s\phi_1}{b-\lambda_1}$ is in V, $\theta(v_0,s) = 0$ and $I + \theta$, where I is an identity map on V, is continuous on V, it follows that there exists a small open neighborhood B of v_0 in V such that if $v_0 + v \in B$, then $v_0 + v + \theta(v_0 + v, s) > 0$. Here, $\theta(v_0 + v, s) = \theta(v_0, s) = 0$. Therefore, if $v_0 + v \in B$, then for $z = \theta v_0 + v, s = \theta(v_0) = 0$, we have

$$\tilde{I}(v_0 + v, s) = I(v_0 + v, s)$$

$$= \int_\Omega \left[-\frac{1}{2} L(v_0 + v) \cdot (v_0 + v) - \frac{b|v_0 + v|^2}{2} + s\phi_1(v_0 + v) \right] dx$$

$$= \int_\Omega \left[-\frac{1}{2} Lv \cdot v - \frac{b|v|^2}{2} \right] dx + C,$$

where

$$C = \int_\Omega \left[-\frac{1}{2} Lv_0 \cdot v_0 - \frac{b|v_0|^2}{2} + s\phi_1 v_0 \right] dx = I(v_0, s) = \tilde{I}(v_0, s).$$

Each $v \in V$ has the form $v = c\phi_1$ Therefore, we have, in B,

$$\tilde{I}(v_0 + v, s) - \tilde{I}(v_0 + v, s)$$

$$= \int_\Omega \left[-\frac{1}{2} Lv \cdot v - \frac{b|v|^2}{2} \right] dx$$

$$= \frac{1}{2}(\lambda_1 - b) \int_\Omega v^2 < 0.$$

Since $\lambda_1 < b < \lambda_2$, v_0 is a strict local point maximum of $\tilde{I}(v,s)$. Since the negative solution $v_1 = \frac{s\phi_1}{a-\lambda_1}$ is in V, $\theta(v_1,s) = 0$ and $I + \theta$, where I is an identity map on V, is continuous on V, it follows that there exists a small open neighborhood D of v_1 in V such that if $v_1 + v \in B$, then $v_1 + v + \theta(v_1 + v, s) < 0$. Here, $\theta(v_1 + v, s) = \theta(v_1, s) = 0$. Therefore, if $v_1 + v \in B$, then for $z = \theta v_1 + v, s = \theta(v_1, s) = 0$, we have

$$\tilde{I}(v_1 + v, s) = I(v_1 + v, s) = \int_\Omega \left[-\frac{1}{2} L(v_1 + v) \cdot (v_1 + v) - \frac{a|v_1 + v|^2}{2} + s\phi_1(v_1 + v) \right] dx$$

$$= \int_\Omega \left[-\frac{1}{2} Lv \cdot v - \frac{a|v|^2}{2} \right] dx + C_1,$$

where

$$C_1 = \int_\Omega \left[-\frac{1}{2} Lv_1 \cdot v_1 - \frac{a|v_1|^2}{2} + s\phi_1 v_1 \right] dx$$

$$= I(v_1, s) = \tilde{I}(v_1, s).$$

Each $v \in V$ has the form $v = c\phi_1$ Therefore, we have, in B,

$$
\begin{aligned}
& \tilde{I}(v_1 + v, s) - \tilde{I}(v_1, s) \\
& = \int_\Omega \left[-\frac{1}{2} Lv \cdot v - \frac{a|v|^2}{2} \right] dx \\
& = \frac{1}{2} (\lambda_1 - a) \int_\Omega v^2 > 0
\end{aligned}
$$

since $-\infty < a < \lambda_1$. Thus, v_1 is a strict local point minimum of $\tilde{I}(v, s)$. $\qquad\square$

Lemma 2.6 (Existence of the Third Solution) *Assume that $-\infty < a < \lambda_1 < b < \lambda_2$ and $s > 0$. Then* (1.1) *has the third weak solution which is a strict point of minimum of $I_{a,b}(u, s)$.*

Proof We know that (1.1) has a positive solution $v_0 = \frac{s\phi_1}{b - \lambda_1}$ and a negative solution $v_1 = \frac{s\phi_1}{a - \lambda_1}$. By Lemma 2.2 (reduction method), we shall show the existence of the third weak solution of (1.1). By Lemma 2.1, $\tilde{I}(v, s)$ is continuous and *Fréchet* differentiable. By Lemma 2.4, $\tilde{I}(v, s) \to \infty$ as $\|v\| \to \infty$, so $\tilde{I}(v, s)$ is bounded from below, satisfies the (P.S.) condition. By Lemma 2.5, there exists a small open neighborhood B of v_0 in V such that v_0 is a strict local point of maximum of $\tilde{I}(v, s)$ and there exists a small open neighborhood D of v_1 in V such that v_1 is a strict local point of minimum of $\tilde{I}(v, s)$. By the shape of the graph of the functional $\tilde{I}$ on the 1-dimensional subspace V, there exists the third critical point $v_2 \neq 0$ which is a strict local point of minimum of $\tilde{I}(v, s)$. By Lemma 2.2, $v_2 + \theta(v_2, s)$ is a solution of (1.1) and a strict local point of minimum of $I_{a,b}(u, s)$. $\qquad\square$

3 Existence of the fourth weak solution

In this section, we shall consider the Leray-Schauder degree of the elliptic operator for the multiplicity of the solutions of (1.1).

Lemma 3.1 *Assume that $-\infty < a < \lambda_1 < b < \lambda_2$. Then there exists $s_1 > 0$, $\epsilon > 0$ such that the Leray-Schauder degree*

$$
\deg\left(u - L^{-1}\left(-bu^+ + au^- + s\phi_1\right), B_{\epsilon s}^*(v_0), 0\right) = -1 \tag{3.1}
$$

for $s \geq s_1$. Here, B_τ^* denotes a ball of radius τ in $H_0^1(\Omega)$ and $v_0 = \frac{s\phi_1}{b - \lambda_1}$ is a positive solution of (1.1).

The proof of this lemma has the similar process to that of Theorem 1 in [2].

Lemma 3.2 *If $-\infty < a < \lambda_1 < b < \lambda_2$, then there exist positive constants s_2, ϵ such that*

$$
\deg\left(u - L^{-1}\left(-bu^+ + au^- + s\phi_1\right), B_{\epsilon s}^*(v_1), 0\right) = 1
$$

for $s \geq s_2$, where $v_1 = \frac{s\phi_1}{a - \lambda_1}$ is a negative solution of (1.1).

The proof of this lemma has the same process as that of Lemma 3.1.

Lemma 3.3 *If* $-\infty < a < \lambda_1 < b < \lambda_2$, *then there exist positive constants* s_3, $\gamma > 0$ *such that*

$$\deg\left(u - L^{-1}\left(-bu^+ + au^- + s\phi_1\right), B_\gamma^*(v_2), 0\right) = 1$$

for $s \geq s_3$, *where* v_2 *is the third solution of* (1.1).

Proof We know that the third solution v_2 of (1.1) is an isolated local minimum of $I_{a,b}(u,s)$. By [9], there exist positive constants s_3, $\gamma > 0$ such that

$$\deg\left(D\tilde{I}, B_\gamma^*(v_2) \cap V, 0\right) = 1$$

for $s \geq s_3$, where $B_\gamma^*(v_2) \cap V$ is a ball centered at 0 containing no other critical point. By (v) of Lemma 2.2,

$$\deg\left(u - L^{-1}\left(-bu^+ + au^- + s\phi_1\right), B_\gamma^*(v_2), 0\right) = \deg\left(D\tilde{I}, B_\gamma^*(v_2) \cap V, 0\right) = 1.$$

So we prove the lemma. □

Lemma 3.4 *Assume that* $-\infty < a < \lambda_1 < b < \lambda_2$ *and s is bounded. Then there exist C and* $s^* > 0$ *such that any solution to* (1.1) *satisfies* $\|u\| \leq C$ *for any s with* $s \leq s^*$.

Proof If not, then there exist (u_n, s_n) with $\|u_n\| \to \infty$, $s_n \to s^*$, s_n is bounded, which satisfy equation (1.1). Now let $v_n = \frac{u_n}{\|u_n\|}$, and v_n satisfies

$$Lv_n + \left(bv_n^+ - av_n^-\right) - \frac{s_n\phi_1(x)}{\|u_n\|} = 0. \tag{3.2}$$

Taking the inner product of both sides of (3.2) with ϕ_1, we have

$$0 = \langle -Lv_n - \lambda_1 v_n, \phi_1 \rangle = \left\langle (b - \lambda_1)v_n^+ + (\lambda_1 - a)v_n^-, \phi_1 \right\rangle - \left\langle \frac{s_n\phi_1(x)}{\|u_n\|}, \phi_1 \right\rangle. \tag{3.3}$$

Thus,

$$\left|\left\langle \frac{s_n\phi_1(x)}{\|u_n\|}, \phi_1 \right\rangle\right| \geq \epsilon \int |v_n|\phi_1(x) \geq \epsilon \left|\int v_n\phi_1(x)\right|.$$

Thus, if v_n is a solution of (3.3), then

$$\left|\langle v_n, \phi_1(x) \rangle\right| \leq \frac{1}{\epsilon}\left|\left\langle \frac{s_n\phi_1(x)}{\|u_n\|}, \phi_1(x) \right\rangle\right|. \tag{3.4}$$

Since v_n's are precompact in $H_0^1(\Omega)$, there exists v with $\|v\| = 1$ such that $v_n \to v$. Taking the limit of both sides of (3.4), we have

$$\left|\langle v, \phi_1(x) \rangle\right| \leq 0.$$

Thus, we have

$$\left|\langle v, \phi_1(x) \rangle\right| = 0,$$

and hence $v = 0$. This is impossible, since $\|v\| = 1$. □

We have the following no solvability condition for (1.1).

Lemma 3.5 *Assume that* $-\infty < a < \lambda_1 < b < \lambda_2$. *Then there exists a constant* $s_0 < 0$ *so small enough that if* $s \le s_0$, *the problem*

$$-Lu = bu^+ - au^- - s\phi_1 \quad in\ H_0^1(\Omega)$$

has no solution.

Proof We can rewrite (1.1) as

$$(-L - \lambda_1)u = -\lambda_1 u^+ + bu^+ + (\lambda_1 - a)u^- - s\phi_1. \tag{3.5}$$

Taking the inner product of both sides of (3.5) with ϕ_1, we have

$$0 = \langle -(\lambda_1 - b)u^+ + (\lambda_1 - a)u^- - s\phi_1, \phi_1 \rangle \ge -s.$$

Thus, there is no solution for $s \le s_0$ if $s_0 < 0$. This completes the proof. □

Lemma 3.6 *Let* $-\infty < a < \lambda_1 < b < \lambda_2$. *Then there exists* $\beta > 0$, *depending on* C *and* s^* *such that*

$$\deg\big(u - L^{-1}\big(-bu^+ + au^- + s\phi_1\big), B_\beta^*(0), 0\big) = 0$$

for $s \le s^*$ *and* $\beta > C$.

Proof By Lemma 3.5, there exists a constant $s_0 < 0$ so small enough that if $s \le s_0$, (1.1) has no solution. By Lemma 3.4, there exist a constant C and $s^* > 0$ such that if u is a solution of (1.1) with $s \le s^*$, then $\|u\| \le C$. Let us choose β so large that $\beta > C$. We note that

$$u - L^{-1}\big(-bu^+ + au^- + s\phi_1\big) \ne 0 \quad \text{on } \partial B_\beta^*(0)$$

and

$$u - L^{-1}\big(-bu^+ + au^- + s_0\phi_1 + \lambda(s - s_0)\phi_1(x)\big) \ne 0 \quad \text{on } \partial B_\beta^*(0)$$

for $0 \le \lambda \le 1$. By the homotopy invariance property, we have that the Leray-Schauder degree

$$d_{LS}\big(u - L^{-1}\big(-bu^+ + au^- + s\phi_1\big), B_\beta^*(0), 0\big)$$
$$= d_{LS}\big(u - L^{-1}\big(-bu^+ + au^- + s_0\phi_1 + \lambda(s - s_0)\phi_1(x)\big), B_\beta^*(0), 0\big)$$
$$= d_{LS}\big(u - L^{-1}\big(-bu^+ + au^- + s_0\phi_1\big), B_\beta^*(0), 0\big) = 0,$$

where $s \le s^*$ and $0 \le \lambda \le 1$. Thus, we proved the lemma. □

Proof of Theorem 1.1 By Lemma 3.4 and Lemma 3.6, there exists a large number $\beta > 0$ (depending on C and s^*) such that all solutions of (1.1) are contained in the ball $B_\beta(0)$ and the Leray-Schauder degree

$$\deg\big(u - L^{-1}\big(-bu^+ + au^- + s\phi_1\big), B_\beta^*(0), 0\big) = 0$$

for $\beta > C$ and $0 < s < s^*$. Let us choose a positive number s^* such that $s_* = \max\{s_1, s_2, s_3\}$ and $s_* < s^*$. Then by Lemma 3.1, Lemma 3.2 and Lemma 3.3, for all s with $s_* \le s \le s^*$, there exist three disjoint balls $B_{\epsilon s}^*(v_0)$, $B_{\epsilon s}^*(v_1)$ and $B_\gamma^*(v_2)$ such that

$$\deg(u - L^{-1}\big(-bu^+ + au^- + s\phi_1\big), B_{\epsilon s}^*(v_0), 0) = -1,$$
$$\deg\big(u - L^{-1}\big(-bu^+ + au^- + s\phi_1\big), B_{\epsilon s}^*(v_1), 0\big) = 1,$$
$$\deg\big(u - L^{-1}\big(-bu^+ + au^- + s\phi_1\big), B_\gamma^*(v_2), 0\big) = 1.$$

By the excision property of Leray-Schauder degree,

$$\deg(u - L^{-1}\big(-bu^+ + au^- + s\phi_1\big), B_\beta^*(0) \backslash \big\{ B_{\epsilon s}^*\big(v_0 \cup B_{\epsilon s}^*(v_1) \cup B_\gamma^*(v_2)\big\}, 0\big) = -1.$$

Thus, (1.1) has at least four solutions. □

Author details
[1]Department of Mathematics, Kunsan National University, Kunsan, 573-701, Korea. [2]Department of Mathematics Education, Inha University, Incheon, 402-751, Korea.

References
1. Lazer, AC, McKenna, PJ: Multiplicity results for a class of semilinear elliptic and parabolic boundary value problems. J. Math. Anal. Appl. **107**(15), 371-395 (1985)
2. McKenna, PJ, Walter, W: On the multiplicity of the solution set of some nonlinear boundary value problems. Nonlinear Anal. TMA **8**(8), 893-907 (1984)
3. Tarantello, G: A note on a semilinear elliptic problem. Differ. Integral Equ. **5**, 561-565 (1992)
4. Micheletti, AM, Pistoia, A: Multiplicity results for a fourth-order semilinear elliptic problem. Nonlinear Anal. TMA **31**, 895-908 (1998)
5. Micheletti, AM, Pistoia, A: Nontrivial solutions for some fourth order semilinear elliptic problems. Nonlinear Anal. TMA **34**, 509-523 (1998)
6. Choi, QH, Jung, T: An application of a variational reduction method to a nonlinear wave equation. J. Differ. Equ. **117**, 390-410 (1995)
7. Jung, T, Choi, QH: Improved multiplicity results for fully nonlinear parabolic systems. Korean J. Math. **17**(3), 283-291 (2009)
8. Jung, T, Choi, QH: Topological approach for the multiple solutions of the nonlinear parabolic problem with variable coefficient jumping nonlinearity. Korean J. Math. **19**(1), 101-109 (2011)
9. Amman, H: A note on degree theory for gradient mappings. Proc. Am. Math. Soc. **85**, 591-595 (1982)