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Hermite-Hadamard-type inequalities for (g, φ_h)-convex dominated functions

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Abstract

In this paper, we introduce the notion of (g, φ_h)-convex dominated function and present some properties of them. Finally, we present a version of Hermite-Hadamard-type inequalities for (g, φ_h)-convex dominated functions. Our results generalize the Hermite-Hadamard-type inequalities in Dragomir *et al.* (Tamsui Oxford Univ. J. Math. Sci. 18(2):161-173, 2002), Kavurmacı *et al.* (New Definitions and Theorems via Different Kinds of Convex Dominated Functions, 2012) and Özdemir *et al.* (Two new different kinds of convex dominated functions and inequalities via Hermite-Hadamard type, 2012).

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1 Introduction

The inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

which holds for all convex functions $f : [a, b] \to \mathbb{R}$, is known in the literature as Hermite-Hadamard's inequality.

In [1], Dragomir and Ionescu introduced the following class of functions.

Definition 1 Let $g : I \to \mathbb{R}$ be a convex function on the interval I. The function $f : I \to \mathbb{R}$ is called g-convex dominated on I if the following condition is satisfied:

$$\begin{aligned} &\left|\lambda f(x) + (1-\lambda)f(y) - f\big(\lambda x + (1-\lambda)y\big)\right| \\ &\quad \le \lambda g(x) + (1-\lambda)g(y) - g\big(\lambda x + (1-\lambda)y\big) \end{aligned}$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

In [2], Dragomir *et al.* proved the following theorem for g-convex dominated functions related to (1.1).

Let $g : I \to \mathbb{R}$ be a convex function and $f : I \to \mathbb{R}$ be a g-convex dominated mapping. Then, for all $a, b \in I$ with $a < b$,

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{1}{b-a}\int_a^b g(x)\,dx - g\left(\frac{a+b}{2}\right)$$

and

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(x)\,dx.$$

In [1] and [2], the authors connect together some disparate threads through a Hermite-Hadamard motif. The first of these threads is the unifying concept of g-convex-dominated function. In [3], Hwang *et al.* established some inequalities of Fejér type for g-convex-dominated functions. Finally, in [4, 5] and [6], authors introduced several new different kinds of convex-dominated functions and then gave Hermite-Hadamard-type inequalities for these classes of functions.

In [7], Varošanec introduced the following class of functions.

I and J are intervals in $\mathbb{R}$, $(0,1) \subseteq J$ and functions h and f are real non-negative functions defined on J and I, respectively.

Definition 2 Let $h : J \to \mathbb{R}$ be a non-negative function, $h \not\equiv 0$. We say that $f : I \to \mathbb{R}$ is an h-convex function, or that f belongs to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I$, $\alpha \in (0,1]$, we have

$$f\big(\alpha x + (1-\alpha)y\big) \leq h(\alpha)f(x) + h(1-\alpha)f(y). \tag{1.2}$$

If the inequality (1.2) is reversed, then f is said to be h-concave, *i.e.* $f \in SV(h, I)$.

Youness have defined the φ-convex functions in [8]. A function $\varphi : [a,b] \to [c,d]$ where $[a,b] \subset \mathbb{R}$:

Definition 3 A function $f : [a,b] \to \mathbb{R}$ is said to be φ-convex on $[a,b]$ if for every two points $x \in [a,b]$, $y \in [a,b]$ and $t \in [0,1]$ the following inequality holds:

$$f\big(t\varphi(x) + (1-t)\varphi(y)\big) \leq tf\big(\varphi(x)\big) + (1-t)f\big(\varphi(y)\big).$$

In [9], Sarıkaya defined a new kind of φ-convexity using h-convexity as following:

Definition 4 Let I be an interval in $\mathbb{R}$ and $h : (0,1) \to (0,\infty)$ be a given function. We say that a function $f : I \to [0,\infty)$ is φ_h-convex if

$$f\big(t\varphi(x) + (1-t)\varphi(y)\big) \leq h(t)f\big(\varphi(x)\big) + h(1-t)f\big(\varphi(y)\big) \tag{1.3}$$

for all $x, y \in I$ and $t \in (0,1)$.

If inequality (1.3) is reversed, then f is said to be φ_h-concave. In particular, if f satisfies (1.3) with $h(t) = t$, $h(t) = t^s$ $(s \in (0,1))$, $h(t) = \frac{1}{t}$, and $h(t) = 1$, then f is said to be φ-convex, φ_s-convex, φ-Godunova-Levin function and φ-P-function, respectively.

In the following sections, our main results are given: we introduce the notion of (g,φ_h)-convex dominated function and present some properties of them. Finally, we present a version of Hermite-Hadamard-type inequalities for (g,φ_h)-convex dominated functions. Our results generalize the Hermite-Hadamard-type inequalities in [2, 4] and [6].

2 (g,φ_h)-convex dominated functions

Definition 5 Let $h:(0,1)\to(0,\infty)$ be a given function, $g:I\to[0,\infty)$ be a given φ_h-convex function. The real function $f:I\to[0,\infty)$ is called (g,φ_h)-convex dominated on I if the following condition is satisfied:

$$\begin{aligned}&\bigl|h(t)f\bigl(\varphi(x)\bigr)+h(1-t)f\bigl(\varphi(y)\bigr)-f\bigl(t\varphi(x)+(1-t)\varphi(y)\bigr)\bigr|\\&\quad\le h(t)g\bigl(\varphi(x)\bigr)+h(1-t)g\bigl(\varphi(y)\bigr)-g\bigl(t\varphi(x)+(1-t)\varphi(y)\bigr)\end{aligned}\tag{2.1}$$

for all $x,y\in I$ and $t\in(0,1)$.

In particular, if f satisfies (2.1) with $h(t)=t$, $h(t)=t^s$ ($s\in(0,1)$), $h(t)=\frac{1}{t}$ and $h(t)=1$, then f is said to be (g,φ)-convex-dominated, (g,φ_s)-convex-dominated, $(g,\varphi_{Q(I)})$-convex-dominated and $(g,\varphi_{P(I)})$-convex-dominated functions, respectively.

The next simple characterization of (g,φ_h)-convex dominated functions holds.

Lemma 1 *Let $h:(0,1)\to(0,\infty)$ be a given function, $g:I\to[0,\infty)$ be a given φ_h-convex function and $f:I\to[0,\infty)$ be a real function. The following statements are equivalent*:

1. *f is (g,φ_h)-convex dominated on I.*
2. *The mappings $g-f$ and $g+f$ are φ_h-convex on I.*
3. *There exist two φ_h-convex mappings l, k defined on I such that*

$$f=\frac{1}{2}(l-k)\quad\textit{and}\quad g=\frac{1}{2}(l+k).$$

Proof $1\iff2$ The condition (2.1) is equivalent to

$$\begin{aligned}&g\bigl(t\varphi(x)+(1-t)\varphi(y)\bigr)-h(t)g\bigl(\varphi(x)\bigr)-h(1-t)g\bigl(\varphi(y)\bigr)\\&\quad\le h(t)f\bigl(\varphi(x)\bigr)+h(1-t)f\bigl(\varphi(y)\bigr)-f\bigl(t\varphi(x)+(1-t)\varphi(y)\bigr)\\&\quad\le h(t)g\bigl(\varphi(x)\bigr)+h(1-t)g\bigl(\varphi(y)\bigr)-g\bigl(t\varphi(x)+(1-t)\varphi(y)\bigr)\end{aligned}$$

for all $x,y\in I$ and $t\in[0,1]$. The two inequalities may be rearranged as

$$(g+f)\bigl(t\varphi(x)+(1-t)\varphi(y)\bigr)\le h(t)(g+f)\bigl(\varphi(x)\bigr)+h(1-t)(g+f)\bigl(\varphi(y)\bigr)$$

and

$$(g-f)\bigl(t\varphi(x)+(1-t)\varphi(y)\bigr)\le h(t)(g-f)\bigl(\varphi(x)\bigr)+h(1-t)(g-f)\bigl(\varphi(y)\bigr)$$

which are equivalent to the φ_h-convexity of $g+f$ and $g-f$, respectively.

$2\iff3$ Let we define the mappings f, g as $f=\frac{1}{2}(l-k)$ and $g=\frac{1}{2}(l+k)$. Then if we sum and subtract f and g, respectively, we have $g+f=l$ and $g-f=k$. By the condition 2 in

Lemma 1, the mappings $g-f$ and $g+f$ are φ_h-convex on I, so l, k are φ_h-convex mappings on I, also. □

Theorem 1 *Let $h:(0,1)\to(0,\infty)$ be a given function, $g:I\to[0,\infty)$ be a given φ_h-convex function. If $f:I\to[0,\infty)$ is Lebesgue integrable and (g,φ_h)-convex dominated on I for linear continuous function $\varphi:[a,b]\to[a,b]$, then the following inequalities hold*:

$$\begin{aligned}&\left|\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}f(x)\,dx-\frac{1}{2h(\frac{1}{2})}f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right|\\&\quad\le\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}g(x)\,dx-\frac{1}{2h(\frac{1}{2})}g\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\end{aligned}\tag{2.2}$$

and

$$\begin{aligned}&\left|\left[f\big(\varphi(a)\big)+f\big(\varphi(b)\big)\right]\int_0^1 h(t)\,dt-\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}f(x)\,dx\right|\\&\quad\le\left[g\big(\varphi(a)\big)+g\big(\varphi(b)\big)\right]\int_0^1 h(t)\,dt-\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}g(x)\,dx\end{aligned}\tag{2.3}$$

for all $x,y\in I$ and $t\in[0,1]$.

Proof By the Definition 5 with $t=\frac{1}{2}$, $x=\lambda a+(1-\lambda)b$, $y=(1-\lambda)a+\lambda b$ and $\lambda\in[0,1]$, as the mapping f is (g,φ_h)-convex dominated function, we have that

$$\begin{aligned}&\left|h\left(\frac{1}{2}\right)\left[f\big(\varphi\big(\lambda a+(1-\lambda)b\big)\big)+f\big(\varphi\big((1-\lambda)a+\lambda b\big)\big)\right]\right.\\&\qquad\left.-f\left(\frac{\varphi(\lambda a+(1-\lambda)b)+\varphi((1-\lambda)a+\lambda b)}{2}\right)\right|\\&\quad\le h\left(\frac{1}{2}\right)\left[g\big(\varphi\big(\lambda a+(1-\lambda)b\big)\big)+g\big(\varphi\big((1-\lambda)a+\lambda b\big)\big)\right]\\&\qquad-g\left(\frac{\varphi(\lambda a+(1-\lambda)b)+\varphi((1-\lambda)a+\lambda b)}{2}\right).\end{aligned}$$

Then using the linearity of φ-function, we have

$$\begin{aligned}&\left|h\left(\frac{1}{2}\right)\left[f\big(\lambda\varphi(a)+(1-\lambda)\varphi(b)\big)+f\big((1-\lambda)\varphi(a)+\lambda\varphi(b)\big)\right]-f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right|\\&\quad\le h\left(\frac{1}{2}\right)\left[g\big(\lambda\varphi(a)+(1-\lambda)\varphi(b)\big)+g\big((1-\lambda)\varphi(a)+\lambda\varphi(b)\big)\right]-g\left(\frac{\varphi(a)+\varphi(b)}{2}\right).\end{aligned}$$

If we integrate the above inequality with respect to λ over $[0,1]$, the inequality in (2.2) is proved.

To prove the inequality in (2.3), firstly we use the Definition 5 for $x=a$ and $y=b$, we have

$$\begin{aligned}&\left|h(t)f\big(\varphi(a)\big)+h(1-t)f\big(\varphi(b)\big)-f\big(t\varphi(a)+(1-t)\varphi(b)\big)\right|\\&\quad\le h(t)g\big(\varphi(a)\big)+h(1-t)g\big(\varphi(b)\big)-g\big(t\varphi(a)+(1-t)\varphi(b)\big).\end{aligned}$$

Then we integrate the above inequality with respect to t over $[0,1]$, we get

$$\left|f(\varphi(a))\int_0^1 h(t)\,dt+f(\varphi(b))\int_0^1 h(1-t)\,dt-\int_0^1 f(t\varphi(a)+(1-t)\varphi(b))\,dt\right|$$
$$\leq g(\varphi(a))\int_0^1 h(t)\,dt+g(\varphi(b))\int_0^1 h(1-t)\,dt-\int_0^1 g(t\varphi(a)+(1-t)\varphi(b))\,dt.$$

If we substitute $x=t\varphi(a)+(1-t)\varphi(b)$ and use the fact that $\int_0^1 h(t)\,dt=\int_0^1 h(1-t)\,dt$, we get

$$\left|[f(\varphi(a))+f(\varphi(b))]\int_0^1 h(t)\,dt-\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)\,dx\right|$$
$$\leq [g(\varphi(a))+g(\varphi(b))]\int_0^1 h(t)\,dt-\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} g(x)\,dx.$$

So, the proof is completed. □

Corollary 1 *Under the assumptions of Theorem* 1 *with* $h(t)=t$, $t\in(0,1)$, *we have*

$$\left|\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)\,dx-f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right|$$
$$\leq \frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} g(x)\,dx-g\left(\frac{\varphi(a)+\varphi(b)}{2}\right) \tag{2.4}$$

and

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)\,dx\right|$$
$$\leq \frac{g(\varphi(a))+g(\varphi(b))}{2}-\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} g(x)\,dx. \tag{2.5}$$

Remark 1 If function φ is the identity in (2.4) and (2.5), then they reduce to Hermite-Hadamard type inequalities for convex dominated functions proved by Dragomir, Pearce and Pečarić in [2].

Corollary 2 *Under the assumptions of Theorem* 1 *with* $h(t)=t^s$, $t,s\in(0,1)$, *we have*

$$\left|\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)\,dx-2^{s-1}f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right|$$
$$\leq \frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} g(x)\,dx-2^{s-1}g\left(\frac{\varphi(a)+\varphi(b)}{2}\right) \tag{2.6}$$

and

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{s+1}-\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)\,dx\right|$$
$$\leq \frac{g(\varphi(a))+g(\varphi(b))}{s+1}-\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} g(x)\,dx. \tag{2.7}$$

Remark 2 If function φ is the identity in (2.6) and (2.7), then they reduce to Hermite-Hadamard type inequalities for (g,s)-convex dominated functions proved by Kavurmacı, Özdemir and Sarıkaya in [4].

Corollary 3 *Under the assumptions of Theorem* 1 *with* $h(t)=\frac{1}{t}$, $t\in(0,1)$, *we have*

$$\begin{aligned}&\left|\frac{4}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}f(x)\,dx-f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right|\\&\quad\le\frac{4}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}g(x)\,dx-g\left(\frac{\varphi(a)+\varphi(b)}{2}\right).\end{aligned}\tag{2.8}$$

Remark 3 If function φ is the identity in (2.8), then it reduces to Hermite-Hadamard type inequality for $(g,Q(I))$-convex dominated functions proved by Özdemir, Tunç and Kavurmacı in [6].

Corollary 4 *Under the assumptions of Theorem* 1 *with* $h(t)=1$, $t\in(0,1)$, *we have*

$$\begin{aligned}&\left|\frac{2}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}f(x)\,dx-f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right|\\&\quad\le\frac{2}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}g(x)\,dx-g\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\end{aligned}\tag{2.9}$$

and

$$\begin{aligned}&\left|\left[f\left(\varphi(a)\right)+f\left(\varphi(b)\right)\right]-\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}f(x)\,dx\right|\\&\quad\le\left[g\left(\varphi(a)\right)+g\left(\varphi(b)\right)\right]-\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}g(x)\,dx.\end{aligned}\tag{2.10}$$

Remark 4 If function φ is the identity in (2.9) and (2.10), then they reduce to Hermite-Hadamard type inequalities for $(g,P(I))$-convex dominated functions proved by Özdemir, Tunç and Kavurmacı in [6].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
MG and HK carried out the design of the study and performed the analysis. MEÖ participated in its design and coordination. All authors read and approved the final manuscript.

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