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Some generalized difference statistically convergent sequence spaces in 2-normed space

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Abstract
In this paper, we define a new generalized difference matrix $B^n_{(m)}$ and introduce some $B^n_{(m)}$-difference statistically convergent sequence spaces in a real linear 2-normed space. We also investigate some topological properties of these spaces.

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1 Introduction

We shall write w for the set of all real sequences $x = (x_k) = (x_k)_{k=0}^{\infty}$. Let c, c_0, $\overline{c}$, $\overline{c}_0$, l_∞, m and m_0 denote the sets of all convergent, null, statistically convergent, statistically null, bounded, bounded statistically convergent and bounded statistically null sequences, respectively. The difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ were first defined by Kızmaz in [1]. The idea of difference sequences is generalized by Et and Çolak [2] as

$$Z(\Delta^n) = \{x = (x_k) \in w : (\Delta^n x_k) \in Z\} \quad (n \in \mathbb{N})$$

for $Z = l_\infty, c, c_0$, where $\Delta^n x_k = \Delta^{n-1}x_k - \Delta^{n-1}x_{k+1}$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, the difference operator is equivalent to the following binomial representation:

$$\Delta^n x_k = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} x_{k+\nu}.$$

Et and Başarır [3] generalized these spaces to $E(\Delta^n)$, where $E = l_\infty(p), c(p), c_0(p)$ are Maddox's sequence spaces. Tripathy and Esi [4], who studied the spaces $l_\infty(\Delta_m)$, $c(\Delta_m)$ and $c_0(\Delta_m)$, gave a new type of generalization of the difference sequence spaces, where $\Delta_m x = (\Delta_m x_k) = (x_k - x_{k+m})$. Tripathy *et al.* [5] generalized this notion as follows:

$$Z(\Delta^n_m) = \{x = (x_k) \in w : (\Delta^n_m x_k) \in Z\} \quad (n, m \in \mathbb{N}),$$

where $\Delta^n_m x = (\Delta^n_m x_k) = (\Delta^{n-1}_m x_k - \Delta^{n-1}_m x_{k+m})$ and $\Delta^0_m x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation:

$$\Delta^n_m x_k = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} x_{k+m\nu}.$$

The difference sequence spaces have been studied by several authors, [3, 6–25].

The concept of 2-normed spaces has been initially introduced by Gähler in the 1960s [26], as an interesting non-linear generalization of a normed linear space, which has been subsequently studied by many authors [27–29]. Since then, a lot of activities have been started to study summability, sequence spaces and related topics on 2-normed spaces [30–33]. Recently, some difference sequence spaces have been introduced in 2-normed spaces by several authors [30, 31, 34].

Dutta [34] introduced the sequence spaces $\bar{c}(\|\cdot,\cdot\|,\Delta^n_{(m)},p)$, $\bar{c}_0(\|\cdot,\cdot\|,\Delta^n_{(m)},p)$, $l_\infty(\|\cdot,\cdot\|,\Delta^n_{(m)},p)$, $m(\|\cdot,\cdot\|,\Delta^n_{(m)},p)$ and $m_0(\|\cdot,\cdot\|,\Delta^n_{(m)},p)$, where $m,n\in\mathbb{N}$ and $\Delta^n_{(m)}x=(\Delta^n_{(m)}x_k)=(\Delta^{n-1}_{(m)}x_k-\Delta^{n-1}_{(m)}x_{k-m})$, and $\Delta^0_{(m)}x_k=x_k$ for all $k\in\mathbb{N}$, which is equivalent to the following binomial representation:

$$\Delta^n_{(m)}x_k=\sum_{v=0}^{n}(-1)^v\binom{n}{v}x_{k-mv}. \tag{1.1}$$

In [35], Başar and Altay introduced the generalized difference matrix $B(r,s)=(b_{nk}(r,s))$ which is a generalization of $\Delta^1_{(1)}$-difference operator as follows:

$$b_{nk}(r,s)=\begin{cases} r & (k=n),\\ s & (k=n-1),\\ 0 & (0\le k<n-1)\text{ or }(k>n)\end{cases}$$

for all $k,n\in\mathbb{N}$, $r,s\in\mathbb{R}-\{0\}$. Recently, Başarır and Kayıkçı [36] have defined the generalized difference matrix B^n of order n, which reduced the difference operator $\Delta^n_{(1)}$ in case $r=1$, $s=-1$ and the binomial representation of this operator is

$$B^nx_k=\sum_{v=0}^{n}\binom{n}{v}r^{n-v}s^vx_{k-v}, \tag{1.2}$$

where $r,s\in\mathbb{R}-\{0\}$ and $n\in\mathbb{N}$.

Thus, for any sequence space Z, the space $Z(B^n)$ is more general and more comprehensive than the corresponding consequences of the space $Z(\Delta^n_{(1)})$. For details, one may refer to [6, 15, 35–40].

The idea of statistical convergence was given by Zygmund [41] in 1935. The concept of statistical convergence was introduced by Fast [42] and Schoenberg [43], independently for the real sequences. Later on, it was further investigated from sequence point of view and linked with the summability theory by Fridy [44] and generalized to the concept of 2-normed space by Gürdal and Pehlivan [45]. The idea is based on the notion of natural density of subsets of $\mathbb{N}$, the set of positive integers, which is defined as follows: the natural density of a subset E of $\mathbb{N}$ is denoted by

$$\delta(E)=\lim_{n\to\infty}\frac{1}{n}\bigl|\{k\in E:k\le n\}\bigr|,$$

where the vertical bar denotes the cardinality of the enclosed set.

2 Definitions and preliminaries

A sequence space E is said to be solid (or normal) if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$.

A linear topological space X over the real field R is said to be a paranormed space if there is a sub-additive function $g : X \to R$ such that $g(\theta) = 0$, $g(x) = g(-x)$, $g(x + y) \leq g(x) + g(y)$ and scalar multiplication is continuous, *i.e.* $|\lambda_n - \lambda| \to 0$ and $g(x_n - x) \to 0$ imply that $g(\lambda_n x_n - \lambda x) \to 0$ for all λ's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X.

The following inequality will be used throughout the paper:

Let $p = (p_k)$ be a positive sequence of real numbers with $\inf_k p_k = h$, $\sup_k p_k = H$ and $D = \max\{1, 2^{H-1}\}$. Then for all $a_k, b_k \in \mathbb{C}$ for all $k \in \mathbb{N}$, we have

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\}$$

and $|\lambda|^{p_k} \leq \max\{|\lambda|^h, |\lambda|^H\}$ for $\lambda \in \mathbb{C}$.

A 2-norm on a vector space X of d dimension, where $d \geq 2$, is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$, which satisfies the following conditions:

(1) $\|x_1, x_2\| = 0$ if and only if x_1, x_2 are linearly dependent,
(2) $\|x_1, x_2\| = \|x_2, x_1\|$,
(3) $\|\alpha x_1, x_2\| = |\alpha| \|x_1, x_2\|$ for any $\alpha \in \mathbb{R}$,
(4) $\|x + x', x_1\| \leq \|x, x_1\| + \|x', x_1\|$.

The pair $(X, \|\cdot,\cdot\|)$ is then called a 2-normed space. For example, standard and Euclidean 2-norms on $\mathbb{R}^2$ are respectively given by

$$\|x_1, x_2\|_S = \begin{vmatrix} \langle x_1, x_1\rangle & \langle x_1, x_2\rangle \\ \langle x_2, x_1\rangle & \langle x_2, x_2\rangle \end{vmatrix}^{\frac{1}{2}}$$

and

$$\|x_1, x_2\|_E = \operatorname{abs}\left(\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}\right), \quad x_i = (x_{i1}, x_{i2}) \in \mathbb{R}^2 \ (i = 1, 2), \tag{2.1}$$

where $\langle\cdot,\cdot\rangle$ stands for the inner product on X [27].

Now we will give the following known example for 2-normed spaces.

Example 2.1 Consider the space Z for l_∞, c and c_0. Let us define:

$$\|x, y\| = \sup_{i\in\mathbb{N}} \sup_{j\in\mathbb{N}} |x_i y_j - x_j y_i|,$$

where $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots) \in Z$. Then $\|\cdot,\cdot\|$ is a 2-norm on Z.

A sequence (x_k) in a 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be convergent to some $L \in X$ in the 2-norm if

$$\lim_{k\to\infty} \|x_k - L, z\| = 0 \quad \text{for every } z \in X \text{ [45]}.$$

A sequence (x_k) in a 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be Cauchy sequence with respect to the 2-norm if

$$\lim_{k,l\to\infty} \|x_k - x_l, z\| = 0 \quad \text{for every } z \in X \text{ [45]}.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the 2-norm. Any complete 2-normed space is said to be a 2-Banach space [29].

Let recall that a sequence (x_k) is said to be statistically convergent to L if for every $\varepsilon > 0$ the set $\{k \in \mathbb{N} : \|x_k - L, z\| \geq \varepsilon\}$ has natural density zero for each nonzero z in X, in other words (x_k) statistically converges to L in 2-normed space $(X, \|\cdot,\cdot\|)$ if

$$\lim_{k\to\infty} \frac{1}{k} \left|\left\{k \in \mathbb{N} : \|x_k - L, z\| \geq \varepsilon\right\}\right| = 0,$$

for each nonzero z in X. For $L = 0$, we say this is statistically null [45].

Firstly, we give the following lemma, which we need to establish our main results.

Lemma 2.2 [34] *Every closed linear subspace F of an arbitrary linear normed space E, different from E, is a nowhere dense set in E.*

Throughout the paper $w(X)$, $c(X)$, $c_0(X)$, $\overline{c}(X)$, $\overline{c}_0(X)$, $l_\infty(X)$, $m(X)$ and $m_0(X)$ denote the spaces of all, convergent, null, statistically convergent, statistically null, bounded, bounded statistically convergent and bounded statistically null X valued sequence spaces, where $(X, \|\cdot,\cdot\|)$ is a real 2-normed space. By $\theta = (\theta, \theta, \theta, \ldots)$, we mean the zero element of X.

3 Main results

In this section, we define the generalized difference matrix $B^n_{(m)}$ and introduce difference sequence spaces $\overline{c}(B^n_{(m)}, p, \|\cdot,\cdot\|)$, $\overline{c}_0(B^n_{(m)}, p, \|\cdot,\cdot\|)$, $m(B^n_{(m)}, p, \|\cdot,\cdot\|)$, $m_0(B^n_{(m)}, p, \|\cdot,\cdot\|)$, $c(B^n_{(m)}, p, \|\cdot,\cdot\|)$, $c_0(B^n_{(m)}, p, \|\cdot,\cdot\|)$, $l_\infty(B^n_{(m)}, p, \|\cdot,\cdot\|)$, $W(B^n_{(m)}, p, \|\cdot,\cdot\|)$, which are defined on a real linear 2-normed space. We investigate some topological properties of the spaces $\overline{c}_0(B^n_{(m)}, p, \|\cdot,\cdot\|)$, $\overline{c}(B^n_{(m)}, p, \|\cdot,\cdot\|)$, $m(B^n_{(m)}, p, \|\cdot,\cdot\|)$ and $m_0(B^n_{(m)}, p, \|\cdot,\cdot\|)$ including linearity, existence of paranorm and solidity. Further, we show that the sequence spaces $m(B^n_{(m)}, p, \|\cdot,\cdot\|)$ and $m_0(B^n_{(m)}, p, \|\cdot,\cdot\|)$ are complete paranormed spaces when the base space is a 2-Banach space. Moreover, we give some inclusion relations.

By the notation $x_k \stackrel{\text{stat}}{\to} 0$, we will mean that x_k is statistically convergent to zero, throughout the paper. Let m, n be non-negative integers and $p = (p_k)$ be a sequence of strictly positive real numbers. Then we define new sequence spaces as follows:

$$\begin{aligned}
\overline{c}\big(B^n_{(m)}, p, \|\cdot,\cdot\|\big) = \big\{ & x = (x_k) \in w(X) : \big\|B^n_{(m)} x_k - L, z\big\|^{p_k} \stackrel{\text{stat}}{\to} 0, \\
& \text{for every nonzero } z \in X \text{ and some } L \in X\big\}, \\
\overline{c}_0\big(B^n_{(m)}, p, \|\cdot,\cdot\|\big) = \big\{ & x = (x_k) \in w(X) : \big\|B^n_{(m)} x_k, z\big\|^{p_k} \stackrel{\text{stat}}{\to} 0, \text{ for every nonzero } z \in X\big\}, \\
l_\infty\big(B^n_{(m)}, p, \|\cdot,\cdot\|\big) = \Big\{ & x = (x_k) \in w(X) : \sup_{k\geq 1}\big(\big\|B^n_{(m)} x_k, z\big\|^{p_k}\big) < \infty, \\
& \text{for every nonzero } z \in X\Big\},
\end{aligned}$$

$$c\left(B^n_{(m)},p,\|\cdot,\cdot\|\right)=\Bigl\{x=(x_k):\lim_{k\to\infty}\left\|B^n_{(m)}x_k-L,z\right\|^{p_k}=0,$$
$$\text{for every nonzero } z\in X \text{ and some } L\in X\Bigr\},$$
$$c_0\left(B^n_{(m)},p,\|\cdot,\cdot\|\right)=\Bigl\{x=(x_k):\lim_{k\to\infty}\left\|B^n_{(m)}x_k,z\right\|^{p_k}=0,\ \text{for every nonzero } z\in X\Bigr\},$$
$$W\left(B^n_{(m)},p,\|\cdot,\cdot\|\right)=\Biggl\{x=(x_k)\in w(X):\lim_{j\to\infty}\frac{1}{j}\sum_{k=1}^{j}\left\|B^n_{(m)}x_k-L,z\right\|^{p_k}=0,$$
$$\text{for every nonzero } z\in X \text{ and some } L\in X\Biggr\},$$
$$m\left(B^n_{(m)},p,\|\cdot,\cdot\|\right)=\bar{c}\left(B^n_{(m)},p,\|\cdot,\cdot\|\right)\cap l_\infty\left(B^n_{(m)},p,\|\cdot,\cdot\|\right)$$

and

$$m_0\left(B^n_{(m)},p,\|\cdot,\cdot\|\right)=\bar{c}_0\left(B^n_{(m)},p,\|\cdot,\cdot\|\right)\cap l_\infty\left(B^n_{(m)},p,\|\cdot,\cdot\|\right),$$

where $B^n_{(m)}x=B^n_{(m)}x_k=rB^{n-1}_{(m)}x_k+sB^{n-1}_{(m)}x_{k-m}$ and $B^0_{(m)}x_k=x_k$ for all $k\in\mathbb{N}$, which is equivalent to the binomial representation as follows:

$$B^n_{(m)}x_k=\sum_{\nu=0}^{n}\binom{n}{\nu}r^{n-\nu}s^{\nu}x_{k-m\nu}.$$

In this representation, we obtain the matrix $B^n_{(1)}$ defined in [36] for $n>1$ and in [35] for $n=1$.

(1) If we take $n=0$ then the above sequence spaces are reduced to $\bar{c}(p,\|\cdot,\cdot\|)$, $\bar{c}_0(p,\|\cdot,\cdot\|)$, $l_\infty(p,\|\cdot,\cdot\|)$, $c(p,\|\cdot,\cdot\|)$, $c_0(p,\|\cdot,\cdot\|)$, $W(p,\|\cdot,\cdot\|)$, $m(p,\|\cdot,\cdot\|)$ and $m_0(p,\|\cdot,\cdot\|)$, respectively.

(2) If we take $r=1$, $s=-1$, then the sequence spaces $\bar{c}(B^n_{(m)},p,\|\cdot,\cdot\|)$, $\bar{c}_0(B^n_{(m)},p,\|\cdot,\cdot\|)$, $l_\infty(B^n_{(m)},p,\|\cdot,\cdot\|)$, $W(B^n_{(m)},p,\|\cdot,\cdot\|)$, $m(B^n_{(m)},p,\|\cdot,\cdot\|)$, $m_0(B^n_{(m)},p,\|\cdot,\cdot\|)$ are reduced to $\bar{c}(\Delta^n_{(m)},p,\|\cdot,\cdot\|)$, $\bar{c}_0(\Delta^n_{(m)},p,\|\cdot,\cdot\|)$, $l_\infty(\Delta^n_{(m)},p,\|\cdot,\cdot\|)$, $W(\Delta^n_{(m)},p,\|\cdot,\cdot\|)$, $m(\Delta^n_{(m)},p,\|\cdot,\cdot\|)$ and $m_0(\Delta^n_{(m)},p,\|\cdot,\cdot\|)$, respectively, which are studied in [34].

(3) By taking $p_k=1$ for all $k\in\mathbb{N}$, then these sequence spaces are denoted by $\bar{c}(B^n_{(m)},\|\cdot,\cdot\|)$, $\bar{c}_0(B^n_{(m)},\|\cdot,\cdot\|)$, $l_\infty(B^n_{(m)},\|\cdot,\cdot\|)$, $c(B^n_{(m)},\|\cdot,\cdot\|)$, $c_0(B^n_{(m)},\|\cdot,\cdot\|)$, $W(B^n_{(m)},\|\cdot,\cdot\|)$, $m(B^n_{(m)},\|\cdot,\cdot\|)$ and $m_0(B^n_{(m)},\|\cdot,\cdot\|)$, respectively.

(4) If we replace the base space X, which is a real linear 2-normed space by $\mathbb{C}$, complete normed linear space, and take $m=1$ and take $r=1$, $s=-1$, then the above sequence spaces are denoted by $\bar{c}(\Delta^n_{(1)},p)$, $\bar{c}_0(\Delta^n_{(1)},p)$, $l_\infty(\Delta^n_{(1)},p)$, $c(\Delta^n_{(1)},p)$, $c_0(\Delta^n_{(1)},p)$, $W(\Delta^n_{(1)},p)$, $m(\Delta^n_{(1)},p)$ and $m_0(\Delta^n_{(1)},p)$, respectively.

(5) If we take $r=1$, $s=-1$, $p_k=1$ for all $k\in\mathbb{N}$, then these sequence spaces are denoted by $\bar{c}(\Delta^n_{(m)},\|\cdot,\cdot\|)$, $\bar{c}_0(\Delta^n_{(m)},\|\cdot,\cdot\|)$, $l_\infty(\Delta^n_{(m)},\|\cdot,\cdot\|)$, $c(\Delta^n_{(m)},\|\cdot,\cdot\|)$, $c_0(\Delta^n_{(m)},\|\cdot,\cdot\|)$, $W(\Delta^n_{(m)},\|\cdot,\cdot\|)$, $m(\Delta^n_{(m)},\|\cdot,\cdot\|)$ and $m_0(\Delta^n_{(m)},p,\|\cdot,\cdot\|)$, respectively.

(6) If we replace the base space X, which is a real linear 2-normed space by $\mathbb{C}$, we obtain the spaces $\bar{c}(B^n_{(m)},p)$, $\bar{c}_0(B^n_{(m)},p)$, $l_\infty(B^n_{(m)},p)$, $c(B^n_{(m)},p)$, $c_0(B^n_{(m)},p)$, $W(B^n_{(m)},p)$, $m(B^n_{(m)},p)$ and $m_0(B^n_{(m)},p)$, respectively.

(7) Moreover, if we take $X=\mathbb{C}$, $n=0$ and $p_k=1$ for all $k\in\mathbb{N}$, we get the spaces $\bar{c}$, $\bar{c}_0$, l_∞, c, c_0, W, m and m_0, respectively.

Theorem 3.1 *Let $p = (p_k)$ be a sequence of strictly positive real numbers. Then the sequence spaces $Z(B^n_{(m)}, p, \|\cdot,\cdot\|)$ are linear spaces where $Z = \overline{c}, \overline{c}_0, l_\infty, W, m, m_0$.*

Proof The proof of the theorem can be obtained by similar techniques in [34]. □

Theorem 3.2 *For any two sequences $p = (p_k)$ and $t = (t_k)$ of positive real numbers and for any two 2-norms $\|\cdot,\cdot\|_1$ and $\|\cdot,\cdot\|_2$ on X we have $Z(B^n_{(m)}, p, \|\cdot,\cdot\|_1) \cap Z(B^n_{(m)}, p, \|\cdot,\cdot\|_2) \neq \varnothing$, where $Z = \overline{c}, \overline{c}_0, m, m_0$.*

Proof The proof follows from the fact that the zero element belongs to each of the sequence spaces involved in the intersection. □

Theorem 3.3 *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space. Then the spaces $m(B^n_{(m)}, p, \|\cdot,\cdot\|)$, $m_0(B^n_{(m)}, p, \|\cdot,\cdot\|)$ are complete paranormed sequence spaces, paranormed by*

$$g(x) = \sup_{k\in\mathbb{N}, z\in X} \left(\left\|B^n_{(m)}x_k, z\right\|^{\frac{p_k}{M}}\right), \tag{3.1}$$

where $M = \max\{1, H\}$ and $H = \sup_k p_k$, $h = \inf_k p_k$.

Proof We will prove the theorem for the sequence space $m_0(B^n_{(m)}, p, \|\cdot,\cdot\|)$. It can be proved for the space $m(B^n_{(m)}, p, \|\cdot,\cdot\|)$ similarly.

Clearly $g(-x) = g(x)$ and $g(\theta) = 0$. From the following inequality, we have

$$\begin{aligned} g(x+y) &= \sup_{k\in\mathbb{N}, z\in X} \left(\left\|B^n_{(m)}(x_k + y_k), z\right\|^{\frac{p_k}{M}}\right) \\ &\le \sup_{k\in\mathbb{N}, z\in X} \left(\left\|B^n_{(m)}x_k, z\right\|^{\frac{p_k}{M}}\right) + \sup_{k\in\mathbb{N}, z\in X} \left(\left\|B^n_{(m)}y_k, z\right\|^{\frac{p_k}{M}}\right). \end{aligned}$$

This implies that $g(x+y) \le g(x) + g(y)$.

To prove the continuity of scalar multiplication, assume that (x^n) be any sequence of the points in $m_0(B^n_{(m)}, p, \|\cdot,\cdot\|)$ such that $g(x^n - x) \to 0$ and (λ_n) be any sequence of scalars such that $\lambda_n \to \lambda$. Since the inequality

$$g(x^n) \le g(x) + g(x^n - x)$$

holds by subadditivity of g, $(g(x^n))$ is bounded. Thus, we have

$$\begin{aligned} g(\lambda_n x^n - \lambda x) &= \sup_{k\in\mathbb{N}, z\in X} \left(\left\|B^n_{(m)}\lambda_n x^n_k - \lambda x_k, z\right\|^{\frac{p_k}{M}}\right) \\ &\le \left(\max\left\{|\lambda_n - \lambda|^h, |\lambda_n - \lambda|^H\right\}\right)^{\frac{1}{M}} \sup_{k\in\mathbb{N}, z\in X} \left(\left\|B^n_{(m)}x_k, z\right\|^{\frac{p_k}{M}}\right) \\ &\quad + \left(\max\left\{|\lambda|^h, |\lambda|^H\right\}\right)^{\frac{1}{M}} \sup_{k\in\mathbb{N}, z\in X} \left(\left\|B^n_{(m)}(x^n_k - x), z\right\|^{\frac{p_k}{M}}\right) \\ &= \left(\max\left\{|\lambda_n - \lambda|^h, |\lambda_n - \lambda|^H\right\}\right)^{\frac{1}{M}} g(x^n) \\ &\quad + \left(\max\left\{|\lambda|^h, |\lambda|^H\right\}\right)^{\frac{1}{M}} g(x^n - x) \end{aligned}$$

which tends to zero as $n \to \infty$. Hence, g is a paranorm on the sequence space $m_0(B^n_{(m)}, p, \|\cdot,\cdot\|)$.

To prove that $m_0(B^n_{(m)}, p, \|\cdot,\cdot\|)$ is complete, assume that (x^i) is a Cauchy sequence in $m_0(B^n_{(m)}, p, \|\cdot,\cdot\|)$. Then for a given ε $(0 < \varepsilon < 1)$, there exists a positive integer N_0 such that $g(x^i - x^j) < \varepsilon$, for all $i, j \geq N_0$. This implies that

$$\sup_{k\in\mathbb{N}, z\in X} \left(\left\|B^n_{(m)} x^i_k - B^n_{(m)} x^j_k, z\right\|^{\frac{p_k}{M}}\right) < \varepsilon,$$

for all $i, j \geq N_0$. It follows that for every nonzero $z \in X$,

$$\left\|B^n_{(m)} x^i_k - B^n_{(m)} x^j_k, z\right\| < \varepsilon,$$

for each $k \geq 1$ and for all $i, j \geq N_0$. Hence $(B^n_{(m)} x^i_k)$ is a Cauchy sequence in X for all $k \in \mathbb{N}$. Since X is a 2-Banach space, $(B^n_{(m)} x^i_k)$ is convergent in X for all $k \in \mathbb{N}$, so we write $(B^n_{(m)} x^i_k) \to (B^n_{(m)} x_k)$ as $i \to \infty$. Now we have for all $i, j \geq N_0$,

$$\begin{aligned}
&\sup_{k\in\mathbb{N}, z\in X} \left(\left\|B^n_{(m)}\left(x^i_k - x^j_k\right), z\right\|^{\frac{p_k}{M}}\right) < \varepsilon \\
&\quad\Rightarrow\quad \lim_{j\to\infty} \left\{\sup_{k\in\mathbb{N}, z\in X} \left(\left\|B^n_{(m)}\left(x^i_k - x^j_k\right), z\right\|^{\frac{p_k}{M}}\right)\right\} < \varepsilon \\
&\quad\Rightarrow\quad \sup_{k\in\mathbb{N}, z\in X} \left(\left\|B^n_{(m)}\left(x^i_k - x_k\right), z\right\|^{\frac{p_k}{M}}\right) < \varepsilon,
\end{aligned}$$

for all $i \geq N_0$. It follows that $(x^i - x) \in m_0(B^n_{(m)}, p, \|\cdot,\cdot\|)$. Since $(x^i) \in m_0(B^n_{(m)}, p, \|\cdot,\cdot\|)$ and $m_0(B^n_{(m)}, p, \|\cdot,\cdot\|)$ is a linear space, so we have $x = x^i - (x^i - x) \in m_0(B^n_{(m)}, p, \|\cdot,\cdot\|)$. This completes the proof. □

Theorem 3.4

(1) *If* $Z_1 \subset Z_2$, *then* $Z_1(B^n_{(m)}, p, \|\cdot,\cdot\|) \subset Z_2(B^n_{(m)}, p, \|\cdot,\cdot\|)$ *and the inclusion is strict, where* $Z_1, Z_2 = c, c_0, l_\infty$.

(2) *If* $n_1 < n_2$, *then* $Z(B^{n_1}_{(m)}, p, \|\cdot,\cdot\|) \subset Z(B^{n_2}_{(m)}, p, \|\cdot,\cdot\|)$ *and the inclusion is strict, where* $Z = c, c_0, l_\infty$.

Proof The parts of proof $Z_1(B^n_{(m)}, p, \|\cdot,\cdot\|) \subset Z_2(B^n_{(m)}, p, \|\cdot,\cdot\|)$ and $Z_1(B^{n_1}_{(m)}, p, \|\cdot,\cdot\|) \subset Z_2(B^{n_2}_{(m)}, p, \|\cdot,\cdot\|)$ are easy. To show the inclusions are strict, choose $Z_1 = c_0$, $Z_2 = c$, $x = (x_k) = (k^2, k^2)$ and consider the 2-norm as defined in (2.1), let $p_k = 1$ for all $k \in \mathbb{N}$, $m = 1$, $n = 2$, $r = 1$, $s = -1$, then $x \in c(B^2_{(1)}, \|\cdot,\cdot\|)$ but $x \notin c_0(B^2_{(1)}, \|\cdot,\cdot\|)$. If we choose $Z = c$, $x = (x_k) = (k^2, k^2)$ and $p_k = 1$ for all $k \in \mathbb{N}$, $m = 1$, $n = 2$, $r = 1$, $s = -1$, then $x \in c(B^2_{(1)}, \|\cdot,\cdot\|)$ but $x \notin c(B^1_{(1)}, \|\cdot,\cdot\|)$. These complete the proofs of parts (1) and (2) of the theorem, respectively. □

Theorem 3.5

(1) $c(B^n_{(m)}, \|\cdot,\cdot\|) \subset \overline{c}(B^n_{(m)}, \|\cdot,\cdot\|)$ *and the inclusion is strict.*

(2) $\overline{c}(\|\cdot,\cdot\|) \subset \overline{c}(B^n_{(m)}, \|\cdot,\cdot\|)$ *and the inclusion is strict.*

(3) $\overline{c}(B^n_{(m)}, \|\cdot,\cdot\|)$ *and* $l_\infty(B^n_{(m)}, \|\cdot,\cdot\|)$ *overlap but neither one contains the other.*

Proof

(1) It is clear that $c(B^n_{(m)}, \|\cdot,\cdot\|) \subset \overline{c}(B^n_{(m)}, \|\cdot,\cdot\|)$. To show that the inclusion is strict, choose the sequence $x = (x_k)$ such that,

$$B^n_{(m)} x_k = \begin{cases} (0, \sqrt{k}), & k = n^2, \\ (0, 0), & k \neq n^2, \end{cases} \tag{3.2}$$

where $n \in \mathbb{N} - \{0\}$, and consider the 2-norm as defined in (2.1). Then we obtain $B^n_{(m)} x_k \in \overline{c}(\|\cdot,\cdot\|)$, but $B^n_{(m)} x_k \notin c(\|\cdot,\cdot\|)$. That is, $x_k \in \overline{c}(B^n_{(m)}, p, \|\cdot,\cdot\|)$, but $x_k \notin c(B^n_{(m)}, p, \|\cdot,\cdot\|)$.

(2) It is easy to see that $\overline{c}(\|\cdot,\cdot\|) \subset \overline{c}(B^n_{(m)}, \|\cdot,\cdot\|)$. To show that the inclusion is strict, let us take $x = (x_k) = (k, k)$ and consider the 2-norm as defined in (2.1), $m = 1$, $n = 1$, $r = 1$, $s = -1$, then $x \in \overline{c}(B^1_{(1)}, \|\cdot,\cdot\|)$ but $x \notin \overline{c}(\|\cdot,\cdot\|)$.

(3) Since the sequence $x = \theta$ belongs to each of the sequence spaces, the overlapping part of the proof is obvious. For the other part of the proof, consider the sequence defined by (3.2) and the 2-norm as defined in (2.1). Then $x \in \overline{c}(B^n_{(m)}, \|\cdot,\cdot\|)$, but $x \notin l_\infty(B^n_{(m)}, \|\cdot,\cdot\|)$. Conversely if we choose $(B^n_{(m)} x_k) = (\overline{1}, \overline{0}, \overline{1}, \overline{0}, \ldots)$ where $\overline{k} = (k, k)$ for all $k = 0, 1$, then $B^n_{(m)} x_k \in l_\infty(\|\cdot,\cdot\|)$ but $B^n_{(m)} x_k \notin \overline{c}(\|\cdot,\cdot\|)$. That is, $x \in l_\infty(B^n_{(m)}, \|\cdot,\cdot\|)$ but $x \notin \overline{c}(B^n_{(m)}, \|\cdot,\cdot\|)$. □

Theorem 3.6 *The space $Z(B^n_{(m)}, p, \|\cdot,\cdot\|)$ is not solid in general, where $Z = \overline{c}, \overline{c}_0, m, m_0$.*

Proof To show that the space is not solid in general, consider the following examples. □

Example 3.7 Let $m = 3$, $n = 1$, $r = 1$, $s = -1$ and consider the 2-normed space as defined in Example 2.1. Let $p_k = 5$ for all $k \in \mathbb{N}$. Consider the sequence (x_k), where $x_k = (x^i_k)$ is defined by $(x^i_k) = (k, k, k, \ldots)$ for each fixed $k \in \mathbb{N}$. Then $x_k \in Z(B^1_{(3)}, p, \|\cdot,\cdot\|)$ for $Z = \overline{c}, m$. Let $\alpha_k = (-1)^k$, then $(\alpha_k x_k) \notin Z(B^1_{(3)}, p, \|\cdot,\cdot\|)$ for $Z = \overline{c}, m$. Thus $Z(B^1_{(3)}, p, \|\cdot,\cdot\|)$ for $Z = \overline{c}, m$ is not solid in general.

Example 3.8 Let $m = 3$, $n = 1$, $r = 1$, $s = -1$ and consider the 2-normed space as defined in Example 2.1. Let $p_k = 1$ for all odd k and $p_k = 2$ for all even k. Consider the sequence (x_k), where $x_k = (x^i_k)$ is defined by $(x^i_k) = (3, 3, 3, \ldots)$ for each fixed $k \in \mathbb{N}$. Then $x_k \in Z(B^1_{(3)}, p, \|\cdot,\cdot\|)$ for $Z = \overline{c}_0, \overline{m}_0$. Let $\alpha_k = (-1)^k$, then $(\alpha_k x_k) \notin Z(B^1_{(3)}, p, \|\cdot,\cdot\|)$ for $Z = \overline{c}_0, \overline{m}_0$. Thus $Z(B^1_{(3)}, p, \|\cdot,\cdot\|)$ for $Z = \overline{c}_0, \overline{m}_0$ is not solid in general.

Theorem 3.9 *The spaces $m_0(B^n_{(m)}, p, \|\cdot,\cdot\|)$ and $m(B^n_{(m)}, p, \|\cdot,\cdot\|)$ are nowhere dense subsets of $l_\infty(B^n_{(m)}, p, \|\cdot,\cdot\|)$.*

Proof From Theorem 3.3, it follows that $m_0(B^n_{(m)}, p, \|\cdot,\cdot\|)$ and $m(B^n_{(m)}, p, \|\cdot,\cdot\|)$ are closed subspaces of $l_\infty(B^n_{(m)}, p, \|\cdot,\cdot\|)$. Since the inclusion relations

$$m_0\big(B^n_{(m)}, p, \|\cdot,\cdot\|\big) \subset l_\infty\big(B^n_{(m)}, p, \|\cdot,\cdot\|\big), \qquad m\big(B^n_{(m)}, p, \|\cdot,\cdot\|\big) \subset l_\infty\big(B^n_{(m)}, p, \|\cdot,\cdot\|\big)$$

are strict, the spaces $m_0(B^n_{(m)}, p, \|\cdot,\cdot\|)$ and $m(B^n_{(m)}, p, \|\cdot,\cdot\|)$ are nowhere dense subsets of $l_\infty(B^n_{(m)}, p, \|\cdot,\cdot\|)$ by Lemma 2.2. □

Theorem 3.10 *Let $p = (p_k)$ be a sequence of non-negative bounded real numbers such that $\inf_k p_k > 0$. Then*

$$W\big(B^n_{(m)}, p, \|\cdot,\cdot\|\big) \cap l_\infty\big(B^n_{(m)}, p, \|\cdot,\cdot\|\big) \subset m\big(B^n_{(m)}, p, \|\cdot,\cdot\|\big).$$

Proof Let $(x_k) \in W(B^n_{(m)}, p, \|\cdot,\cdot\|) \cap l_\infty(B^n_{(m)}, p, \|\cdot,\cdot\|)$. Then for a given $\varepsilon > 0$, we have

$$\begin{aligned}\frac{1}{j}\sum_{k=1}^{j}\big\|B^n_{(m)}x_k - L, z\big\|^{p_k} &\geq \frac{1}{j}\sum_{\substack{k=1 \\ \|B^n_{(m)}x_k - L, z\|^{p_k} \geq \varepsilon}}^{j}\big\|B^n_{(m)}x_k - L, z\big\|^{p_k} \\ &\geq \varepsilon\frac{1}{j}\big|\big\{k \leq j : \big\|B^n_{(m)}x_k - L, z\big\|^{p_k} \geq \varepsilon\big\}\big|.\end{aligned}$$

If we take the limit for $j \to \infty$, it follows that $(x_k) \in c(B^n_{(m)}, p, \|\cdot,\cdot\|)$ from the inequality above. Since $(x_k) \in l_\infty(B^n_{(m)}, p, \|\cdot,\cdot\|)$, we have the result. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in the preparation of this article. All authors read and approved the final manuscript.

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