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Sufficient conditions for starlike functions associated with the lemniscate of Bernoulli

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Abstract

Let $-1 \leq B < A \leq 1$. The condition on β is determined so that $1 + \beta zp'(z)/p^k(z) \prec (1 + Az)/(1 + Bz)$ $(-1 < k \leq 3)$ implies $p(z) \prec \sqrt{1+z}$. Similarly, the condition on β is determined so that $1 + \beta zp'(z)/p^n(z)$ or $p(z) + \beta zp'(z)/p^n(z) \prec \sqrt{1+z}$ $(n = 0, 1, 2)$ implies $p(z) \prec (1 + Az)/(1 + Bz)$ or $\sqrt{1+z}$. In addition to that, the condition on β is derived so that $p(z) \prec (1 + Az)/(1 + Bz)$ when $p(z) + \beta zp'(z)/p(z) \prec \sqrt{1+z}$. A few more problems of the similar flavor are also considered.

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1 Introduction

Let $\mathcal{A}$ be the class of analytic functions defined on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ normalized by the condition $f(0) = 0 = f'(0) - 1$. For two analytic functions f and g, we say that f is *subordinate* to g or g is *superordinate* to f, denoted by $f \prec g$, if there is a Schwarz function w with $|w(z)| \leq |z|$ such that $f(z) = g(w(z))$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. For an analytic function φ whose range is starlike with respect to $\varphi(0) = 1$ and is symmetric with respect to the real axis, let $\mathcal{S}^*(\varphi)$ denote the class of *Ma-Minda starlike functions* consisting of all $f \in \mathcal{A}$ satisfying $zf'(z)/f(z) \prec \varphi(z)$. For special choices of φ, $\mathcal{S}^*(\varphi)$ reduces to well-known subclasses of starlike functions. For example, when $-1 \leq B < A \leq 1$, $\mathcal{S}^*[A,B] := \mathcal{S}^*((1 + Az)/(1 + Bz))$ is the class of Janowski starlike functions [1] (see [2]) and $\mathcal{S}^*[1 - 2\alpha, -1]$ is the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α and $\mathcal{S}^* := \mathcal{S}^*(0)$ is the class of starlike functions. For $\varphi(z) := \sqrt{1+z}$, the class $\mathcal{S}^*(\varphi)$ reduces to the class $\mathcal{SL}$ introduced by Sokół and Stankiewicz [3] and studied recently by Ali *et al.* [4, 5]. A function $f \in \mathcal{A}$ is in the class $\mathcal{SL}$ if $zf'(z)/f(z)$ lies in the region bounded by the right half-plane of the lemniscate of Bernoulli given by $|w^2 - 1| < 1$. Analytically, $\mathcal{SL} := \{f \in \mathcal{A} : |(zf'(z)/f(z))^2 - 1| < 1\}$. For $b \geq 1/2$ and $a \geq 1$, a more general class $S^*[a,b]$ of the functions f satisfying $|(zf'(z)/f(z))^a - b| < b$ was considered by Paprocki and Sokół [6]. Clearly, $S^*[2,1] =: \mathcal{SL}$. For some radius problems related with the lemniscate of Bernoulli, see [3, 5, 7, 8]. Estimates for the initial coefficients of functions in the class $\mathcal{SL}$ are available in [8].

Let p be an analytic function defined on $\mathbb{D}$ with $p(0) = 1$. Recently Ali *et al.* [4] determined conditions for $p(z) \prec \sqrt{1+z}$ when $1 + \beta zp'(z)/p^k(z)$ with $k = 0, 1, 2$ or $(1 - \beta)p(z) + \beta p^2(z) + \beta zp'(z)$ is subordinated to $\sqrt{1+z}$. Motivated by the works in [4–8], in Section 2 the condition on β is determined so that $p(z) \prec \sqrt{1+z}$ when $1 + \beta zp'(z)/p^k(z) \prec (1 + Az)/(1 + Bz)$ $(-1 < k \leq 3)$. Similarly, the condition on β is determined so that $p(z) \prec (1 + Az)/(1 + Bz)$

when $1 + \beta zp'(z)/p^n(z) \prec \sqrt{1+z}$, $n = 0, 1, 2$. Further, the condition on β is obtained in each case so that $p(z) \prec \sqrt{1+z}$ when $p(z) + \beta zp'(z)/p^n(z) \prec \sqrt{1+z}$, $n = 0, 1, 2$. At the end of this section, the problem $p(z) + \beta zp'(z)/p(z) \prec \sqrt{1+z}$ implies $p(z) \prec (1 + Az)/(1 + Bz)$ is also considered.

Silverman [9] introduced the class $\mathcal{G}_b$ by

$$\mathcal{G}_b := \left\{ f \in \mathcal{A} : \left| \frac{zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < b \right\}$$

and proved $\mathcal{G}_b \subset \mathcal{S}^*(2/(1 + \sqrt{1+8b}))$, $0 < b \leq 1$. Further, this result was improved by Obradovič and Tuneski [10] by showing $\mathcal{G}_b \subset S^*[0, b] \subset \mathcal{S}^*(2/(1 + \sqrt{1+8b}))$, $0 < b \leq 1$. Tuneski [11] further obtained the condition for $\mathcal{G}_b \subset S^*[A, B]$. Inspired by the work of Silverman [9], Nunokawa *et al.* [12] obtained the sufficient conditions for a function in the class $\mathcal{G}_b$ to be strongly starlike, strongly convex, or starlike in $\mathbb{D}$. By setting $p(z) = zf'(z)/f(z)$, the inclusion $\mathcal{G}_b \subset S^*[A, B]$ can be written as

$$1 + \frac{zp'(z)}{p^2(z)} \prec 1 + bz \quad \implies \quad p(z) \prec \frac{1 + Az}{1 + Bz}.$$

Recently Ali *et al.* [13], obtained the condition on the constants $A, B, D, E \in [-1, 1]$ and β so that $p(z) \prec (1 + Az)/(1 + Bz)$ when $1 + \beta zp'(z)/p^n(z) \prec (1 + Dz)/(1 + Ez)$, $n = 0, 1$. In Section 3, alternate and easy proofs of results [13, Lemmas 2.1, 2.10] are discussed. Further, this section is concluded with the condition on $A, B, D, E \in [-1, 1]$ and β such that $1 + \beta zp'(z)/p^2(z) \prec (1 + Dz)/(1 + Ez)$ implies $p(z) \prec (1 + Az)/(1 + Bz)$.

The following results are required in order to prove our main results.

Lemma 1.1 [14, Corollary 3.4h, p.135] *Let q be univalent in $\mathbb{D}$, and let φ be analytic in a domain D containing $q(\mathbb{D})$. Let $zq'(z)\varphi(q(z))$ be starlike. If p is analytic in $\mathbb{D}$, $p(0) = q(0)$ and satisfies*

$$zp'(z)\varphi\big(p(z)\big) \prec zq'(z)\varphi\big(q(z)\big),$$

then $p \prec q$ and q is the best dominant.

The following is a more general form of the above lemma.

Lemma 1.2 [14, Corollary 3.4i, p.134] *Let q be univalent in $\mathbb{D}$, and let φ and ν be analytic in a domain D containing $q(\mathbb{D})$ with $\varphi(w) \neq 0$ when $w \in q(\mathbb{D})$. Set*

$$Q(z) := zq'(z)\varphi\big(q(z)\big), \qquad h(z) := \nu\big(q(z)\big) + Q(z).$$

Suppose that

(1) *h is convex or $Q(z)$ is starlike univalent in $\mathbb{D}$ and*
(2) $\operatorname{Re}(\frac{zh'(z)}{Q(z)}) > 0$ *for $z \in \mathbb{D}$.*

If

$$\nu\big(p(z)\big) + zp'(z)\varphi\big(p(z)\big) \prec \nu\big(q(z)\big) + zq'(z)\varphi\big(q(z)\big), \tag{1.1}$$

then $p \prec q$ and q is the best dominant.

Lemma 1.3 [14, Corollary 3.4a, p.120] *Let q be analytic in $\mathbb{D}$, let ϕ be analytic in a domain D containing $q(\mathbb{D})$ and suppose*

(1) $\operatorname{Re}\phi[q(z)] > 0$ *and either*
(2) q *is convex, or*
(3) $Q(z) = zq'(z) \cdot \phi[q(z)]$ *is starlike.*

If p is analytic in $\mathbb{D}$, with $p(0) = q(0)$, $p(\mathbb{D}) \subset D$ and

$$p(z) + zp'(z)\phi\bigl[p(z)\bigr] \prec q(z),$$

then $p(z) \prec q(z)$.

2 Results associated with the lemniscate of Bernoulli

In the first result, condition on β is obtained so that the subordination

$$1 + \beta\frac{zp'(z)}{p^k(z)} \prec \frac{1 + Az}{1 + Bz} \quad (-1 < B < A \leq 1)$$

implies $p(z) \prec \sqrt{1 + z}$.

Lemma 2.1 *Let $|\beta| \geq 2^{(k+3)/2}(A - B) + |B\beta|$, $-1 < k \leq 3$. Let p be an analytic function defined on $\mathbb{D}$ with $p(0) = 1$ satisfying*

$$1 + \beta\frac{zp'(z)}{p^k(z)} \prec \frac{1 + Az}{1 + Bz} \quad (-1 < B < A \leq 1),$$

then $p(z) \prec \sqrt{1 + z}$.

Proof Let $q(z) = \sqrt{1 + z}$. A computation shows that the function

$$Q(z) := \beta\frac{zq'(z)}{q^k(z)} = \frac{\beta z}{2(1 + z)^{(k+1)/2}} \quad (-1 < k \leq 3)$$

is starlike in the unit disk $\mathbb{D}$. Consider the subordination

$$1 + \beta\frac{zp'(z)}{p^k(z)} \prec 1 + \beta\frac{zq'(z)}{q^k(z)}.$$

Thus in view of Lemma 1.1, it follows that $p(z) \prec q(z)$. In order to prove our result, we need to prove

$$\frac{1 + Az}{1 + Bz} \prec 1 + \frac{\beta zq'(z)}{q^k(z)} = 1 + \frac{\beta z}{2(1 + z)^{(k+1)/2}} := h(z).$$

Let $w = \Phi(z) = \frac{1+Az}{1+Bz}$. Then $\Phi^{-1}(w) = \frac{w-1}{A-Bw}$. The subordination $\Phi(z) \prec h(z)$ is equivalent to $z \prec \Phi^{-1}(h(z))$. Thus in order to prove the result, we need only to show $|\Phi^{-1}(h(e^{it}))| \geq 1$. For $z = e^{it}$, $-\pi \leq t \leq \pi$, we have

$$\bigl|\Phi^{-1}\bigl(h\bigl(e^{it}\bigr)\bigr)\bigr| \geq \frac{|\beta|}{2(A - B)(2\cos(t/2))^{(k+1)/2} + |B\beta|} =: g(t).$$

A calculation shows that $g(t)$ attains its minimum at $t = 0$. Further, the value of $g(t)$ at π or $-\pi$ comes out to be $1/|B|$ which is naturally greater than the value at the extreme point $t = 0$ because if $g(0) \geq g(\pi)$, then $(A - B)|\beta| \leq 0$ which is absurd. Thus

$$g(0) = \frac{|\beta|}{2^{(k+3)/2}(A - B) + |B\beta|} \geq 1$$

for $|\beta| \geq 2^{(k+3)/2}(A - B) + |B\beta|$. Hence $\Phi(z) \prec h(z)$, and the proof is complete now. □

Next result depicts the condition on β such that $1 + \beta zp'(z) \prec \sqrt{1+z}$ implies $p(z) \prec (1 + Az)/(1 + Bz)$ $(-1 \leq B < A \leq 1)$. On subsequent lemmas, similar results are obtained by considering the expressions $1 + \beta zp'(z)/p(z)$ and $1 + \beta zp'(z)/p^2(z)$.

Lemma 2.2 *Let* $(A - B)\beta \geq \sqrt{2}(1 + |B|)^2 + (1 - B)^2$ *and* $-1 \leq B < A \leq 1$. *Let p be an analytic function defined on* $\mathbb{D}$ *with* $p(0) = 1$ *satisfying*

$$1 + \beta zp'(z) \prec \sqrt{1+z},$$

then $p(z) \prec \frac{1+Az}{1+Bz}$.

Proof Define the function $q : \mathbb{D} \to \mathbb{C}$ by

$$q(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1)$$

with $q(0) = 1$. A computation shows that

$$Q(z) = \beta zq'(z) = \frac{\beta(A - B)z}{(1 + Bz)^2}$$

and

$$\frac{zQ'(z)}{Q(z)} = \frac{1 - Bz}{1 + Bz}.$$

Let $z = re^{it}$, $r \in (0,1)$, $-\pi \leq t \leq \pi$. Then

$$\operatorname{Re}\left(\frac{1 - Bz}{1 + Bz}\right) = \operatorname{Re}\left(\frac{1 - Bre^{it}}{1 + Bre^{it}}\right) = \frac{1 - B^2r^2}{|1 + Bre^{it}|^2}.$$

Since $1 - B^2r^2 > 0$ $(|B| \leq 1, 0 < r < 1)$ and so $\operatorname{Re}(zQ'(z)/Q(z)) > 0$, this shows that Q is starlike in $\mathbb{D}$. It follows from Lemma 1.1 that the subordination

$$1 + \beta zp'(z) \prec 1 + \beta zq'(z)$$

implies $p(z) \prec q(z)$. Now we need to prove the following in order to prove the lemma:

$$\sqrt{1+z} \prec 1 + \beta zq'(z) = 1 + \beta\frac{(A - B)z}{(1 + Bz)^2} =: h(z).$$

Let $w = \Phi(z) = \sqrt{1+z}$. Then $\Phi^{-1}(w) = w^2 - 1$. The subordination $\Phi(z) \prec h(z)$ is equivalent to the subordination $z \prec \Phi^{-1}(h(z))$. Now in order to prove the result, it is enough to show $|\Phi^{-1}(h(e^{it}))| \geq 1$, $z = e^{it}$, $-\pi \leq t \leq \pi$. Now

$$\left|\Phi^{-1}\left(h\left(e^{it}\right)\right)\right| = \left|\left(1 + \beta\frac{(A-B)e^{it}}{(1+Be^{it})^2}\right)^2 - 1\right| \geq 1 \quad \text{implies that} \quad \left|1 + \beta\frac{(A-B)e^{it}}{(1+Be^{it})^2}\right| \geq \sqrt{2}.$$

Further,

$$\begin{aligned}\left|1 + \beta\frac{(A-B)e^{it}}{(1+Be^{it})^2}\right| &= \frac{|1 + (2B + \beta(A-B))e^{it} + B^2 e^{2it}|}{|1 + 2Be^{it} + B^2 e^{2it}|} \\ &\geq \frac{\operatorname{Re}(2B + \beta(A-B) + B^2 e^{it} + e^{-it})}{1 + 2|B| + B^2} \\ &= \frac{2B + \beta(A-B) + (1+B^2)\cos t}{(1+|B|)^2} \\ &\geq \frac{2B + \beta(A-B) - (1+B^2)}{(1+|B|)^2} \geq \sqrt{2}\end{aligned}$$

for $(A-B)\beta \geq \sqrt{2}(1+|B|)^2 + (1-B)^2$. Therefore $\Phi(z) \prec h(z)$ and this completes the proof. □

Lemma 2.3 *Let* $(A-B)\beta \geq (\sqrt{2}-1)(1+|A|)(1+|B|)$ *and* $-1 \leq B < A \leq 1$. *Let* p *be an analytic function defined on* $\mathbb{D}$ *with* $p(0) = 1$ *satisfying*

$$1 + \beta\frac{zp'(z)}{p(z)} \prec \sqrt{1+z},$$

then $p(z) \prec \frac{1+Az}{1+Bz}$.

Proof Let the function $q : \mathbb{D} \to \mathbb{C}$ be defined by

$$q(z) = \frac{1+Az}{1+Bz} \quad (-1 \leq B < A \leq 1).$$

A computation shows that

$$Q(z) := \frac{\beta zq'(z)}{q(z)} = \frac{\beta(A-B)z}{(1+Az)(1+Bz)}$$

and

$$\frac{zQ'(z)}{Q(z)} = \frac{1 - ABz^2}{(1+Az)(1+Bz)}.$$

Let $z = re^{it}$, $r \in (0,1)$, $-\pi \leq t \leq \pi$. Then

$$\begin{aligned}\operatorname{Re}\left(\frac{1 - ABz^2}{(1+Az)(1+Bz)}\right) &= \operatorname{Re}\left(\frac{1 - ABr^2 e^{2it}}{(1+Are^{it})(1+Bre^{it})}\right) \\ &= \frac{(1-ABr^2)(1 + (A+B)r\cos t + ABr^2)}{|1+Are^{it}|^2|1+Bre^{it}|^2}.\end{aligned}$$

Since $1 + ABr^2 + (A + B)r\cos t \geq (1 - Ar)(1 - Br) > 0$ for $A + B \geq 0$ and, similarly, $1 + ABr^2 + (A + B)r\cos t \geq (1 + Ar)(1 + Br) > 0$ for $A + B \leq 0$, it follows that Q is starlike in $\mathbb{D}$. Lemma 1.1 suggests that the subordination

$$1 + \beta\frac{zp'(z)}{p(z)} \prec 1 + \beta\frac{zq'(z)}{q(z)}$$

implies $p(z) \prec q(z)$. Now we have to prove

$$\sqrt{1+z} \prec 1 + \beta\frac{zq'(z)}{q(z)} = 1 + \frac{\beta(A-B)z}{(1+Az)(1+Bz)} =: h(z).$$

Let $w = \Phi(z) = \sqrt{1+z}$. Then $\Phi^{-1}(w) = w^2 - 1$. The subordination $\Phi(z) \prec h(z)$ is equivalent to the subordination $z \prec \Phi^{-1}(h(z))$. Now in order to prove the result, it is enough to show $|\Phi^{-1}(h(e^{it}))| \geq 1$, $-\pi \leq t \leq \pi$. Now

$$\left|\Phi^{-1}\left(h\left(e^{it}\right)\right)\right| = \left|\left(1 + \frac{\beta(A-B)e^{it}}{(1+Ae^{it})(1+Be^{it})}\right)^2 - 1\right| \geq 1 \quad \text{implies that}$$

$$\left|1 + \frac{\beta(A-B)e^{it}}{(1+Ae^{it})(1+Be^{it})}\right| \geq \sqrt{2}.$$

Further,

$$\begin{aligned}\left|1 + \frac{\beta(A-B)e^{it}}{(1+Ae^{it})(1+Be^{it})}\right| &\geq \operatorname{Re}\left(1 + \frac{\beta(A-B)e^{it}}{(1+Ae^{it})(1+Be^{it})}\right) \\ &\geq 1 + \frac{(A-B)\beta}{(1+|A|)(1+|B|)} \geq \sqrt{2}\end{aligned}$$

for $(A - B)\beta \geq (\sqrt{2} - 1)(1 + |A|)(1 + |B|)$. Therefore $\Phi(z) \prec h(z)$ and this completes the proof. □

Lemma 2.4 *Let* $(A - B)\beta \geq (\sqrt{2} - 1)(1 + |A|)^2 + (1 - A)^2$ *and* $-1 \leq B < A \leq 1$. *Let* p *be an analytic function defined on* $\mathbb{D}$ *with* $p(0) = 1$ *satisfying*

$$1 + \beta\frac{zp'(z)}{p^2(z)} \prec \sqrt{1+z},$$

then $p(z) \prec \frac{1+Az}{1+Bz}$.

Proof Let the function $q : \mathbb{D} \to \mathbb{C}$ be defined by

$$q(z) = \frac{1+Az}{1+Bz} \quad (-1 \leq B < A \leq 1)$$

with $q(0) = 1$. Then

$$Q(z) = \frac{\beta zq'(z)}{q^2(z)} = \frac{\beta(A-B)z}{(1+Az)^2}$$

and

$$\frac{zQ'(z)}{Q(z)} = \frac{1-Az}{1+Az}.$$

Let $z = re^{it}$, $-\pi \le t \le \pi$, $0 < r < 1$. Then

$$\operatorname{Re}\left(\frac{1-Az}{1+Az}\right) = \frac{1-A^2r^2}{|1+Are^{it}|^2}.$$

Since $1 - A^2r^2 > 0$ ($|A| \le 1$, $0 < r < 1$). Hence $\operatorname{Re}(zQ'(z))/Q(z) > 0$, this shows that Q is starlike in $\mathbb{D}$. An application of Lemma 1.1 reveals that the subordination

$$1 + \beta\frac{zp'(z)}{p^2(z)} \prec 1 + \beta\frac{zq'(z)}{q^2(z)}$$

implies $p(z) \prec q(z)$. Now our result is established if we prove

$$\sqrt{1+z} \prec 1 + \beta\frac{zq'(z)}{q^2(z)} = 1 + \beta\frac{(A-B)z}{(1+Az)^2} =: h(z).$$

The rest of the proof is similar to that of Lemma 2.2, and therefore it is skipped here. □

In the next result, the condition on β is obtained so that $p(z) + \beta zp'(z) \prec \sqrt{1+z}$ implies $p(z) \prec \sqrt{1+z}$. On subsequent lemmas, similar results are discussed by considering the expressions $p(z) + \beta zp'(z)/p(z)$ and $p(z) + \beta zp'(z)/p^2(z)$.

Lemma 2.5 *Let p be an analytic function defined on $\mathbb{D}$ with $p(0) = 1$ satisfying $p(z) + \beta zp'(z) \prec \sqrt{1+z}$, $\beta > 0$. Then $p(z) \prec \sqrt{1+z}$.*

Proof Define the function $q : \mathbb{D} \to \mathbb{C}$ by $q(z) = \sqrt{1+z}$ with $q(0) = 1$. Since $q(\mathbb{D}) = \{w : |w^2 - 1| < 1\}$ is the right half of the lemniscate of Bernoulli, $q(\mathbb{D})$ is a convex set, and hence q is a convex function. Let us define $\phi(w) = \beta$, then

$$\operatorname{Re}\phi\big[q(z)\big] = \beta > 0.$$

Consider the function Q defined by

$$Q(z) := zq'(z)\phi\big(q(z)\big) = \beta\frac{z}{2\sqrt{1+z}}.$$

Further,

$$\begin{aligned}\operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) &= 1 - \operatorname{Re}\left(\frac{z}{2(1+z)}\right)\\ &\ge \frac{3}{4} > 0.\end{aligned}$$

Thus the function Q is starlike, and the result now follows by an application of Lemma 1.3. □

Lemma 2.6 *Let p be an analytic function defined on $\mathbb{D}$ with $p(0) = 1$ satisfying*

$$p(z) + \beta\frac{zp'(z)}{p(z)} \prec \sqrt{1+z}, \quad \beta > 0.$$

Then $p(z) \prec \sqrt{1+z}$.

Proof As before, let q be given by $q(z) = \sqrt{1+z}$ with $q(0) = 1$. Then q is a convex function. Let us define $\phi(w) = \beta/w$. Since $q(\mathbb{D}) = \{w : |w^2 - 1| < 1\}$ is the right half of the lemniscate of Bernoulli, so

$$\operatorname{Re}\phi\bigl[q(z)\bigr] = \frac{\beta}{|\sqrt{1+z}|^2}\operatorname{Re}(\sqrt{1+z}) > 0.$$

Consider the function Q defined by

$$Q(z) := \beta\frac{zq'(z)}{q(z)} = \beta\frac{z}{2(1+z)}.$$

Further,

$$\operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) = 1 - \operatorname{Re}\left(\frac{z}{1+z}\right) \geq \frac{1}{2} > 0.$$

Thus the function Q is starlike, and the result now follows by an application of Lemma 1.3. □

Lemma 2.7 *Let p be an analytic function defined on $\mathbb{D}$ with $p(0) = 1$ satisfying*

$$p(z) + \beta\frac{zp'(z)}{p^2(z)} \prec \sqrt{1+z}, \quad \beta > 0.$$

Then $p(z) \prec \sqrt{1+z}$.

Proof Let q be given by $q(z) = \sqrt{1+z}$ with $q(0) = 1$. Then q is a convex function. Let us define $\phi(w) = \beta/w^2$ and

$$\operatorname{Re}\phi\bigl[q(z)\bigr] = \operatorname{Re}\left(\frac{\beta}{1+z}\right) > \frac{\beta}{2} > 0.$$

Consider the function Q defined by

$$Q(z) := \beta\frac{zq'(z)}{q^2(z)} = \beta\frac{z}{2(1+z)^{\frac{3}{2}}}.$$

Further,

$$\operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) = 1 - \frac{3}{2}\operatorname{Re}\left(\frac{z}{1+z}\right) \geq \frac{1}{4} > 0.$$

Thus the function Q is starlike, and the result now follows by an application of Lemma 1.3. □

In the next result, the condition on β is obtained such that $p(z) + \beta zp'(z)/p(z) \prec \sqrt{1+z}$ implies that $p(z) \prec (1+Az)/(1+Bz)$.

Lemma 2.8 *Let* $-1 \leq B < A \leq 1$, $(A-B)\beta \geq \sqrt{2}(1+|A|)(1+|B|) + |A|^2 - 1$ *and*

$$\frac{1}{\beta} \geq \max\left\{0, \frac{A-B}{(1+|A|)(1+|B|)} - \frac{1-|B|}{1+|B|}\right\}.$$

Let p *be an analytic function defined on* $\mathbb{D}$ *with* $p(0) = 1$ *satisfying*

$$p(z) + \beta\frac{zp'(z)}{p(z)} \prec \sqrt{1+z}.$$

Then $p(z) \prec \frac{1+Az}{1+Bz}$.

Proof Define the function $q : \mathbb{D} \to \mathbb{C}$ by $q(z) = (1+Az)/(1+Bz)$, $-1 \leq B < A \leq 1$. Consider the subordination

$$p(z) + \beta\frac{zp'(z)}{p(z)} \prec q(z) + \beta\frac{zq'(z)}{q(z)}.$$

Thus, in view of Lemma 1.2, the above subordination can be written as (1.1) by defining the functions ν and φ as $\nu(w) := w$ and $\varphi(w) := \beta/w$ ($\beta \neq 0$). Clearly, the functions ν and φ are analytic in $\mathbb{C}$ and $\varphi(w) \neq 0$. Let the functions $Q(z)$ and $h(z)$ be defined by

$$Q(z) := zq'(z)\varphi\bigl(q(z)\bigr) = \beta\frac{zq'(z)}{q(z)}$$

and

$$h(z) := \nu\bigl(q(z)\bigr) + Q(z) = q(z) + \beta\frac{zq'(z)}{q(z)}.$$

A computation shows that $Q(z)$ is starlike univalent in $\mathbb{D}$. Further,

$$\frac{zh'(z)}{Q(z)} = \frac{1}{\beta} + 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}.$$

Let $z = e^{it}$, $-\pi \leq t \leq \pi$. Then

$$\begin{aligned}\operatorname{Re}\left(\frac{e^{it}h'(e^{it})}{Q(e^{it})}\right) &= \frac{1}{\beta} + \operatorname{Re}\left(\frac{1-Be^{it}}{1+Be^{it}} - \frac{(A-B)e^{it}}{(1+Ae^{it})(1+Be^{it})}\right) \\ &\geq \frac{1}{\beta} + \frac{1-|B|}{1+|B|} - \frac{A-B}{(1+|A|)(1+|B|)} > 0.\end{aligned}$$

Thus by Lemma 1.2, it follows that $p(z) \prec q(z)$. In order to prove our result, we need to prove that

$$\Phi(z) := \sqrt{1+z} \prec q(z) + \beta\frac{zq'(z)}{q(z)} = \frac{1+Az}{1+Bz} + \frac{\beta(A-B)z}{(1+Az)(1+Bz)} := h(z).$$

The subordination $\Phi(z) \prec h(z)$ is equivalent to the subordination $z \prec \Phi^{-1}(h(z))$. Now in order to prove the result, it is enough to show $|\Phi^{-1}(h(e^{it}))| \geq 1$, $-\pi \leq t \leq \pi$. Now

$$\left|\Phi^{-1}\big(h\big(e^{it}\big)\big)\right| = \left|\left(\frac{1+Ae^{it}}{1+Be^{it}} + \frac{\beta(A-B)e^{it}}{(1+Ae^{it})(1+Be^{it})}\right)^2 - 1\right| \geq 1$$

implies

$$\left|\frac{1+Ae^{it}}{1+Be^{it}} + \frac{\beta(A-B)e^{it}}{(1+Ae^{it})(1+Be^{it})}\right| \geq \sqrt{2}.$$

Further,

$$\begin{aligned}\left|\frac{1+Ae^{it}}{1+Be^{it}} + \frac{\beta(A-B)e^{it}}{(1+Ae^{it})(1+Be^{it})}\right| &\geq \operatorname{Re}\left(\frac{1+Ae^{it}}{1+Be^{it}} + \frac{\beta(A-B)e^{it}}{(1+Ae^{it})(1+Be^{it})}\right) \\ &\geq \frac{1-|A|}{1+|B|} + \frac{(A-B)\beta}{(1+|A|)(1+|B|)} \geq \sqrt{2}\end{aligned}$$

for $(A-B)\beta \geq \sqrt{2}(1+|A|)(1+|B|) + |A|^2 - 1$. This completes the proof. □

3 Sufficient condition for Janowski starlikeness

The following first two results (Lemmas 3.1, 3.2) are essentially due to Ali *et al.* [13, Lemmas 2.1, 2.10]. However, an alternate proof of the same result, which is much easier than that given by Ali *et al.* [13], is presented below.

Lemma 3.1 *Assume that* $-1 \leq B < A \leq 1$, $-1 \leq E < D \leq 1$ *and* $\beta(A-B) \geq (D-E)(1+B^2) + |2B(D-E) - E\beta(A-B)|$. *Let* p *be an analytic function defined on* $\mathbb{D}$ *with* $p(0) = 1$ *satisfying*

$$1 + \beta z p'(z) \prec \frac{1+Dz}{1+Ez}, \quad \beta \neq 0.$$

Then $p(z) \prec \frac{1+Az}{1+Bz}$.

Proof Define the function $q : \mathbb{D} \to \mathbb{C}$ by

$$q(z) = \frac{1+Az}{1+Bz}, \quad -1 \leq B < A \leq 1.$$

Then q is convex in $\mathbb{D}$ with $q(0) = 1$. Further computation shows that

$$Q(z) = \beta z q'(z) = \frac{\beta(A-B)z}{(1+Bz)^2}$$

and Q is starlike in $\mathbb{D}$. It follows from Lemma 1.1 that the subordination

$$1 + \beta z p'(z) \prec 1 + \beta z q'(z)$$

implies $p(z) \prec q(z)$. In view of the above result, it is sufficient to prove

$$\frac{1+Dz}{1+Ez} \prec 1 + \beta z q'(z) = 1 + \beta\frac{(A-B)z}{(1+Bz)^2} = h(z).$$

Let $w = \Phi(z) = \frac{1+Dz}{1+Ez}$. Then $\Phi^{-1}(w) = \frac{w-1}{D-Ew}$ and

$$\begin{aligned}\Phi^{-1}(h(z)) &= \frac{\beta(A-B)z}{D(1+Bz)^2 - E(1+Bz)^2 - \beta E(A-B)z} \\ &= \frac{\beta(A-B)z}{(D-E)(1+B^2z^2) + (2B(D-E) - \beta E(A-B))z}.\end{aligned}$$

Let $z = e^{it}$, $\pi \le t \le \pi$. Thus

$$\left|\Phi^{-1}(h(e^{it}))\right| \ge \frac{|\beta|(A-B)}{(D-E)(1+B^2) + |(2B(D-E) - \beta E(A-B))|} \ge 1$$

for $|\beta|(A-B) \ge (D-E)(1+B^2) + |(2B(D-E) - E\beta(A-B))|$. Hence $q(\mathbb{D}) \subset h(\mathbb{D})$, that is, $q(z) \prec h(z)$, this completes the proof. □

It should be noted that Ali *et al.* [13] made the assumption $AB > 0$ in order to prove the result [13, Lemma 2.10], whereas in the following lemma this condition has been dropped.

Lemma 3.2 *Assume that* $-1 \le B < A \le 1$, $-1 \le E < D \le 1$ *and* $\beta(A-B) \ge (D-E)(1+|AB|) + |(A+B)(D-E) - E\beta(A-B)|$. *Let p be an analytic function defined on* $\mathbb{D}$ *with* $p(0) = 1$ *satisfying*

$$1 + \beta\frac{zp'(z)}{p(z)} \prec \frac{1+Dz}{1+Ez}, \quad \beta \neq 0.$$

Then $p(z) \prec \frac{1+Az}{1+Bz}$.

Proof As above, define the function $q : \mathbb{D} \to \mathbb{C}$ by

$$q(z) = \frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1.$$

Then q is convex in $\mathbb{D}$ with $q(0) = 1$. A computation shows that

$$Q(z) = \frac{\beta zq'(z)}{q(z)} = \frac{\beta(A-B)z}{(1+Az)(1+Bz)}$$

and Q is starlike in $\mathbb{D}$. It follows from Lemma 1.1 that the subordination

$$1 + \beta\frac{zp'(z)}{p(z)} \prec 1 + \beta\frac{zq'(z)}{q(z)}$$

implies $p(z) \prec q(z)$. Now we need to prove

$$\frac{1+Dz}{1+Ez} \prec 1 + \beta\frac{zq'(z)}{q(z)} = 1 + \beta\frac{(A-B)z}{(1+Bz)^2} = h(z).$$

Let $w = \Phi(z) = \frac{1+Dz}{1+Ez}$. Then $\Phi^{-1}(w) = \frac{w-1}{D-Ew}$ and

$$\begin{aligned}\Phi^{-1}(h(z)) &= \frac{\beta(A-B)z}{(D-E)(1+Az)(1+Bz) - \beta E(A-B)z} \\ &= \frac{\beta(A-B)z}{(D-E)(1+ABz^2) + ((A+B)(D-E) - \beta E(A-B))z}.\end{aligned}$$

Let $z = e^{it}$, $\pi \le t \le \pi$. Thus

$$\left|\Phi^{-1}\left(h\left(e^{it}\right)\right)\right| \ge \frac{|\beta|(A-B)}{(D-E)(1+|AB|)+|(A+B)(D-E)-\beta E(A-B)|} \ge 1$$

for $|\beta|(A-B) \ge (D-E)(1+|AB|)+|(A+B)(D-E)-E\beta(A-B)|$. Hence $q(\mathbb{D}) \subset h(\mathbb{D})$, that is, $q(z) \prec h(z)$, this completes the proof. □

Lemma 3.3 *Assume that* $-1 \le B < A \le 1$, $-1 \le E < D \le 1$ *and* $|\beta|(A-B) \ge (D-E)(1+A^2)+|2A(D-E)-E\beta(A-B)|$. *Let p be an analytic function defined on* $\mathbb{D}$ *with* $p(0)=1$ *satisfying*

$$1+\beta\frac{zp'(z)}{p^2(z)} \prec \frac{1+Dz}{1+Ez}.$$

Then $p(z) \prec \frac{1+Az}{1+Bz}$.

Proof Define the function $q:\mathbb{D}\to\mathbb{C}$ by

$$q(z)=\frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1.$$

Then q is convex in $\mathbb{D}$ with $q(0)=1$. A computation shows that

$$Q(z)=\frac{\beta zq'(z)}{q^2(z)}=\frac{\beta(A-B)z}{(1+Az)^2}$$

and

$$\frac{zQ'(z)}{Q(z)}=\frac{1-Az}{1+Az}.$$

As before, a computation shows Q is starlike in $\mathbb{D}$. It follows from Lemma 1.1 that the subordination

$$1+\beta\frac{zp'(z)}{p^2(z)} \prec 1+\beta\frac{zq'(z)}{q^2(z)}$$

implies $p(z) \prec q(z)$. To prove result, it is enough to show that

$$\frac{1+Dz}{1+Ez} \prec 1+\beta\frac{zq'(z)}{q^2(z)}=1+\beta\frac{(A-B)z}{(1+Az)^2}=h(z).$$

The remaining part of the proof is similar to that of Lemma 3.1, and therefore it is skipped here. □

Remark 3.4 When $\beta=1$, Lemma 3.3 reduces to [13, Lemma 2.6] due to Ali *et al.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results and they read and approved the final manuscript.

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