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Composition operators from Zygmund spaces into Q_K spaces

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Abstract
The boundedness and compactness of composition operators from Zygmund and little Zygmund spaces into Q_K and $Q_{K,0}$ spaces are characterized in this paper.
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1 Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the open unit disk of complex plane $\mathbb{C}$. Denote by $H(\mathbb{D})$ the class of functions analytic in $\mathbb{D}$. Let $g(z,a)$ denote the Green's function with pole at $a \in \mathbb{D}$, *i.e.*, $g(z,a) = \log \frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ is a Möbius transformation of $\mathbb{D}$. An $f \in H(\mathbb{D})$ is said to belong to the Zygmund space, denoted by $\mathcal{Z}$, if

$$\sup \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty,$$

where the supremum is taken over all $e^{i\theta} \in \partial\mathbb{D}$ and $h > 0$. By Theorem 5.3 in [1], we see that $f \in \mathcal{Z}$ if and only if

$$\|f\|_{\mathcal{Z}} = \left|f(0)\right| + \left|f'(0)\right| + \sup_{z\in\mathbb{D}}\left(1 - |z|^2\right)\left|f''(z)\right| < \infty. \tag{1}$$

It is easy to check that $\mathcal{Z}$ is a Banach space under the above norm. Let $\mathcal{Z}_0$ denote the subspace of $\mathcal{Z}$ consisting of those $f \in \mathcal{Z}$ for which

$$\lim_{|z|\to 1}\left(1 - |z|^2\right)\left|f''(z)\right| = 0.$$

The space $\mathcal{Z}_0$ is called the little Zygmund space. Throughout this paper, the closed unit ball in $\mathcal{Z}$ and $\mathcal{Z}_0$ will be denoted by $\mathbb{B}_{\mathcal{Z}}$ and $\mathbb{B}_{\mathcal{Z}_0}$, respectively.

Let $K : [0,\infty) \to [0,\infty)$ be a nondecreasing continuous function. We say that an $f \in H(\mathbb{D})$ belongs to the space Q_K if (see, *e.g.*, [2–4])

$$\|f\|_{Q_K}^2 = \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left|f'(z)\right|^2 K\left(g(z,a)\right)\,dA(z) < \infty. \tag{2}$$

Here, dA is the normalized Lebesgue area measure in $\mathbb{D}$. Modulo constants, Q_K is a Banach space under the norm $|f(0)| + \|f\|_{Q_K}$ and Q_K is Möbius invariant. When (see [2])

$$\int_0^1 K(-\log r)\big(1-r^2\big)^{-2} r\,dr < \infty,$$

$Q_K = \mathcal{B}$. Here, $\mathcal{B}$ is the Bloch space defined as follows:

$$\mathcal{B} = \Big\{f : f \in H(\mathbb{D}), \|f\|_{\mathcal{B}} = \big|f(0)\big| + \sup_{a\in\mathbb{D}}\big|f'(z)\big|\big(1-|z|^2\big) < \infty\Big\}.$$

If $K(t) = t^p$, then $Q_K = Q_p$ (see [5, 6]). If $f \in H(\mathbb{D})$ such that

$$\lim_{|a|\to 1}\int_{\mathbb{D}}\big|f'(z)\big|^2 K\big(g(z,a)\big)\,dA(z) = 0, \tag{3}$$

we say that f belongs to the space $Q_{K,0}$. If Q_K consists of just constant functions, we say that it is trivial. Q_K is nontrivial if and only if (see [2])

$$\int_0^{1/e} K(-\log r) r\,dr < \infty. \tag{4}$$

To avoid that Q_K is trivial, we assume from now that (4) is satisfied. See [2–4, 7–15] for the study of the space Q_K.

Let φ be an analytic self-map of $\mathbb{D}$. The composition operator C_φ is defined by

$$C_\varphi(f)(z) = f\big(\varphi(z)\big), \quad f \in H(\mathbb{D}).$$

It is interesting to provide a function theoretic characterization of when φ induces a bounded or compact composition operator on various spaces. For a study of the composition operators, see [16] and [17].

The composition operator from Bloch spaces to Q_K and $Q_{K,0}$ was studied in [9, 10, 18]. Some characterizations of the boundedness and compactness of the composition operator, as well as Volterra type operator, on the Zygmund space can be found in [19–23].

The purpose of this paper is to study the boundedness and compactness of the operator C_φ from the Zygmund space and little Zygmund space into Q_K and $Q_{K,0}$.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other.

2 Main results and proofs

In this section, we state and prove our main results. In order to formulate our main results, we need some auxiliary results which are incorporated in the following lemmas. The following lemma, can be proved in a standard way (see, *e.g.*, Theorem 3.11 in [16]).

Lemma 1 *Let K be a nonnegative nondecreasing function on $[0,\infty)$. Assume that φ is an analytic self-map of $\mathbb{D}$. Then $C_\varphi : \mathcal{Z} \to Q_K$ is compact if and only if $C_\varphi : \mathcal{Z} \to Q_K$ is bounded and for every bounded sequence $\{f_n\}$ in $\mathcal{Z}$ which converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$, $\lim_{n\to\infty}\|C_\varphi f_n\|_{Q_K} = 0$.*

By using the methods of [10] (see also [24]), we can obtain the following lemma. Since the proof is similar, we omit the details.

Lemma 2 *Let K be a nonnegative nondecreasing function on* $[0,\infty)$. *Assume that φ is an analytic self-map of $\mathbb{D}$. If $C_\varphi : \mathcal{Z}(\mathcal{Z}_0) \to Q_K$ is compact, then for any $\varepsilon > 0$ there exists a δ, $0 < \delta < 1$, such that for all f in $\mathcal{Z}(\mathcal{Z}_0)$,*

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \left|f'\big(\varphi(z)\big)\right|^2 \left|\varphi'(z)\right|^2 K\big(g(z,a)\big)\, dA(z) < \varepsilon \tag{5}$$

holds whenever $\delta < r < 1$.

By modifying the proof of Theorem 3.1 of [7] (or see [25]), we can prove the following lemma. We omit the details.

Lemma 3 *Let K be a nonnegative nondecreasing function on* $[0,\infty)$. *Assume that φ is an analytic self-map of $\mathbb{D}$. Then $C_\varphi : \mathcal{Z} \to Q_{K,0}$ is compact if and only if $C_\varphi : \mathcal{Z} \to Q_{K,0}$ is bounded and*

$$\lim_{|a|\to 1} \sup_{\|f\|_{\mathcal{Z}}\le 1} \int_{\mathbb{D}} \left|(C_\varphi f)'(z)\right|^2 K\big(g(z,a)\big)\, dA(z) = 0.$$

Lemma 4 [20] *Suppose that $f \in \mathcal{Z}_0$, then*

$$\lim_{|z|\to 1} \left|f'(z)\right| \Big/ \ln \frac{e}{1-|z|^2} = 0.$$

Lemma 5 [26] *Suppose that $\{n_k\}$ is an increasing sequence of positive integers satisfying $\frac{n_{k+1}}{n_k} \ge \lambda > 1$ for all $k \in \mathbb{N}$. Let $0 < p < \infty$. Then there are two positive constants C_1 and C_2, depending only on p and λ such that*

$$C_1 \left(\sum_{k=1}^{\infty} |a_k|^2\right)^{\frac{1}{2}} \le \left(\frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{k=1}^{\infty} a_k e^{in_k\theta}\right|^p d\theta\right)^{\frac{1}{p}} \le C_2 \left(\sum_{k=1}^{\infty} |a_k|^2\right)^{\frac{1}{2}}.$$

Now we are in a position to state and prove our main results in this paper.

Theorem 1 *Let K be a nonnegative nondecreasing function on* $[0,\infty)$. *Assume that φ is an analytic self-map of $\mathbb{D}$. Then the following statements hold*:

(i) *If*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|\varphi'(z)\right|^2 \left(\ln \frac{e}{1-|\varphi(z)|^2}\right)^2 K\big(g(z,a)\big)\, dA(z) < \infty, \tag{6}$$

then $C_\varphi : \mathcal{Z}(\mathcal{Z}_0) \to Q_K$ is bounded.

(ii) *If $C_\varphi : \mathcal{Z}(\mathcal{Z}_0) \to Q_K$ is bounded, then*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|\varphi'(z)\right|^2 \ln \frac{1}{1-|\varphi(z)|^2} K\big(g(z,a)\big)\, dA(z) < \infty. \tag{7}$$

Proof (i) Let $f \in \mathcal{Z}$. Then by the following result (see [20]):

$$\left|f'(z)\right| \le C\|f\|_{\mathcal{Z}} \ln \frac{e}{1-|z|^2}, \tag{8}$$

we have

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|(C_\varphi f)'(z)\right|^2 K\big(g(z,a)\big)\, dA(z) \\
&\quad = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f'\big(\varphi(z)\big)\right|^2 \left|\varphi'(z)\right|^2 K\big(g(z,a)\big)\, dA(z) \\
&\quad \le C\|f\|_{\mathcal{Z}} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|\varphi'(z)\right|^2 \left(\ln \frac{e}{1-|\varphi(z)|^2}\right)^2 K\big(g(z,a)\big)\, dA(z) < \infty.
\end{aligned}$$

In addition, by the well-known fact that $\|f\|_\infty \le C\|f\|_{\mathcal{Z}}$, we obtain

$$\left|f\big(\varphi(0)\big)\right| \le C\|f\|_{\mathcal{Z}}.$$

Therefore, $C_\varphi : \mathcal{Z} \to Q_K$ is bounded, and hence $C_\varphi : \mathcal{Z}_0 \to Q_K$ is bounded.

(ii) First, we suppose that $C_\varphi : \mathcal{Z} \to Q_K$ is bounded. Let $g(z) = z \in \mathcal{Z}$. By the boundedness of $C_\varphi : \mathcal{Z} \to Q_K$ we have that $\varphi = C_\varphi g \in Q_K$. Hence, we have

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|\le \frac{1}{\sqrt{e}}} \left|\varphi'(z)\right|^2 \ln \frac{1}{1-|\varphi(z)|^2} K\big(g(z,a)\big)\, dA(z) \\
&\quad \le \ln \frac{e}{e-1} \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|\le \frac{1}{\sqrt{e}}} \left|\varphi'(z)\right|^2 K\big(g(z,a)\big)\, dA(z) \\
&\quad \le \ln \frac{e}{e-1} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|\varphi'(z)\right|^2 K\big(g(z,a)\big)\, dA(z) < \infty.
\end{aligned} \tag{9}$$

For $z \in \mathbb{D}$, such that $|z| = r \ge \frac{1}{\sqrt{e}}$. Let

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{2^k+1} z^{2^k+1}.$$

Then by the fact that $p(z) = \sum_{k=0}^{\infty} z^{2^k}$ belongs to Bloch space (see [27, Theorem 1]) and the relationship of Bloch function and Zygmund function, we see that $f \in \mathcal{Z}$. Let

$$h_\theta(z) = f\big(e^{i\theta} z\big) = \sum_{k=0}^{\infty} \frac{1}{2^k+1} \big(e^{i\theta} z\big)^{2^k+1}.$$

Then $h_\theta \in \mathcal{Z}$ and $\|h_\theta\|_{\mathcal{Z}} = \|f\|_{\mathcal{Z}}$. We have

$$\begin{aligned}
\infty &> \|C_\varphi\|^2 \|h_\theta\|_{\mathcal{Z}}^2 \ge \|C_\varphi h_\theta\|_{Q_K}^2 \\
&\ge \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|(C_\varphi h_\theta)'(z)\right|^2 K\big(g(z,a)\big)\, dA(z) \\
&\ge \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|> \frac{1}{\sqrt{e}}} \left|\sum_{k=0}^{\infty} e^{i(2^k+1)\theta} \varphi^{2^k}(z)\right|^2 \left|\varphi'(z)\right|^2 K\big(g(z,a)\big)\, dA(z).
\end{aligned} \tag{10}$$

Since

$$\frac{1}{2\pi}\int_0^{2\pi}\|C_\varphi\|^2\|h_\theta\|_{\mathcal{Z}}^2\,d\theta=\frac{1}{2\pi}\int_0^{2\pi}\|C_\varphi\|^2\|f\|_{\mathcal{Z}}^2\,d\theta=\|C_\varphi\|^2\|f\|_{\mathcal{Z}}^2=\|C_\varphi\|^2\|h_\theta\|_{\mathcal{Z}}^2,$$

by (10), Lemma 5 and Fubini's theorem we have

$$\begin{aligned}\infty &> \frac{1}{2\pi}\int_0^{2\pi}\|C_\varphi\|^2\|h_\theta\|_{\mathcal{Z}}^2\,d\theta\\
&\ge \frac{1}{2\pi}\int_0^{2\pi}\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>\frac{1}{\sqrt{e}}}\left|\sum_{k=0}^\infty e^{i(2^k+1)\theta}\varphi^{2^k}(z)\right|^2\left|\varphi'(z)\right|^2K\big(g(z,a)\big)\,dA(z)\,d\theta\\
&= \sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>\frac{1}{\sqrt{e}}}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=0}^\infty e^{i(2^k+1)\theta}\varphi^{2^k}(z)\right|^2d\theta\right\}\left|\varphi'(z)\right|^2K\big(g(z,a)\big)\,dA(z)\\
&\ge \sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>\frac{1}{\sqrt{e}}}\sum_{k=0}^\infty\left|\varphi(z)\right|^{2^{k+1}}\left|\varphi'(z)\right|^2K\big(g(z,a)\big)\,dA(z).\end{aligned}$$

For any $r\in(0,1)$, a calculation shows that

$$\begin{aligned}\ln\frac{1}{1-r^2} &= -\ln(1+r)-\ln(1-r)=\int_0^r\left(-\sum_{n=0}^\infty(-1)^nt^n+\sum_{n=0}^\infty t^n\right)dt\\
&=\sum_{n=0}^\infty\big(1-(-1)^n\big)\frac{r^{n+1}}{n+1}=2\sum_{k=1}^\infty\frac{r^{2k}}{2k}\\
&=\sum_{k=1}^\infty\frac{r^{2k}}{k}=\sum_{k=0}^\infty\sum_{j=2^k}^{2^{k+1}-1}\frac{r^{2j}}{j}\\
&\le\sum_{k=0}^\infty\left(\frac{1}{2^k}+\cdots+\frac{1}{2^k}\right)r^{2\cdot 2^k}=\sum_{k=0}^\infty r^{2^{k+1}},\end{aligned}\tag{11}$$

since the number of terms in the sum from 2^k to $2^{k+1}-1$ is 2^k. Therefore,

$$\begin{aligned}\infty &> \frac{1}{2\pi}\int_0^{2\pi}\|C_\varphi\|^2\|h_\theta\|_{\mathcal{Z}}^2\,d\theta\\
&\ge \sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>\frac{1}{\sqrt{e}}}\left|\varphi'(z)\right|^2\ln\frac{1}{1-|\varphi(z)|^2}K\big(g(z,a)\big)\,dA(z),\end{aligned}\tag{12}$$

which together with (9) implies that (7) holds.

Now suppose that $C_\varphi:\mathcal{Z}_0\to Q_K$ is bounded. Take the function $f(z)$ given by the above. Set

$$f_r(z)=f(rz)=\sum_{k=1}^\infty\frac{1}{2^k+1}(rz)^{2^k+1},\quad r\in(0,1).$$

Then $f_r\in\mathcal{Z}_0$. Then, as argued the same with the case of $C_\varphi:\mathcal{Z}\to Q_K$ and let $r\to1$, we get the desired result. The proof of the theorem is finished. □

Theorem 2 *Let K be a nonnegative nondecreasing function on $[0,\infty)$. Assume that φ is an analytic self-map of $\mathbb{D}$. Then the following statements holds*:

(i) *If $\varphi \in Q_K$ and*

$$\lim_{r\to 1}\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}\left|\varphi'(z)\right|^2\left(\ln\frac{e}{1-|\varphi(z)|^2}\right)^2 K\big(g(z,a)\big)\,dA(z)=0, \tag{13}$$

then $C_\varphi:\mathcal{Z}(\mathcal{Z}_0)\to Q_K$ is compact.

(ii) *If $C_\varphi:\mathcal{Z}(\mathcal{Z}_0)\to Q_K$ is compact, then $\varphi\in Q_K$ and*

$$\lim_{r\to 1}\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}\left|\varphi'(z)\right|^2\ln\frac{1}{1-|\varphi(z)|^2}K\big(g(z,a)\big)\,dA(z)=0. \tag{14}$$

Proof (i) Assume that $\varphi\in Q_K$ and (13) holds. Let $\{f_k\}$ be a bounded sequence in $\mathcal{Z}$ which converges to 0 uniformly on compact subsets of $\mathbb{D}$. We need to show that $\{C_\varphi f_k\}$ converges to 0 in Q_K norm. By (13), for given $\varepsilon>0$, there is an $r\in(0,1)$, such that

$$\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}\left|\varphi'(z)\right|^2\left(\ln\frac{e}{1-|\varphi(z)|^2}\right)^2 K\big(g(z,a)\big)\,dA(z)<\varepsilon.$$

Therefore, by (8), we have

$$\begin{aligned}
&\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left|(C_\varphi f_k)'(z)\right|^2K\big(g(z,a)\big)\,dA(z)\\
&\quad=\sup_{a\in\mathbb{D}}\left\{\int_{|\varphi(z)|>r}+\int_{|\varphi(z)|\le r}\right\}\left|f_k'\big(\varphi(z)\big)\right|^2\left|\varphi'(z)\right|^2K\big(g(z,a)\big)\,dA(z)\\
&\quad\le C\|f_k\|_{\mathcal{Z}}^2\varepsilon+\sup_{|w|\le r}\left|f_k'(w)\right|^2\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left|\varphi'(z)\right|^2K\big(g(z,a)\big)\,dA(z).
\end{aligned} \tag{15}$$

From the assumption, we see that $\{f_k'\}$ also converges to 0 uniformly on compact subsets of $\mathbb{D}$ by Cauchy's estimates. It follows that $\|C_\varphi f_k\|_{Q_K}\to 0$ since $|f_k(\varphi(0))|\to 0$ and $\sup_{|w|\le r}|f_k'(w)|\to 0$ as $k\to\infty$. By Lemma 1, $C_\varphi:\mathcal{Z}\to Q_K$ is compact, and hence $C_\varphi:\mathcal{Z}_0\to Q_K$ is also compact.

(ii) We only need to prove the case of $C_\varphi:\mathcal{Z}_0\to Q_K$. Assume that $C_\varphi:\mathcal{Z}_0\to Q_K$ is compact. By taking $g(z)=z\in\mathcal{Z}_0$ we get $\varphi\in Q_K$. Now we choose the function $f(z)$ given in the proof of Theorem 1. Then $f\in\mathcal{Z}$. Choose a sequence $\{\lambda_j\}$ in $\mathbb{D}$ which converges to 1 as $j\to\infty$, and let $f_j(z)=f(\lambda_j z)$ for $j\in\mathbb{N}$. Then, $f_j\in\mathcal{Z}_0$ for all $j\in\mathbb{N}$ and $\|f_j\|_{\mathcal{Z}}\le C$. Let $f_{j,\theta}(z)=f_j(e^{i\theta}z)$. Then $f_{j,\theta}\in\mathcal{Z}_0$. Replace f by $f_{j,\theta}$ in (5) and then integrate both sides with respect to θ. By Fubini's theorem, we obtain

$$\begin{aligned}
\varepsilon&>\sup_{a\in\mathbb{D}}\frac{1}{2\pi}\int_{|\varphi(z)|>r}\left(\int_0^{2\pi}\left|f_j'\big(e^{i\theta}\varphi(z)\big)\right|^2d\theta\right)\left|\varphi'(z)\right|^2K\big(g(z,a)\big)\,dA(z)\\
&=\sup_{a\in\mathbb{D}}\frac{1}{2\pi}\int_{|\varphi(z)|>r}\int_0^{2\pi}\left|\sum_{k=1}^{\infty}\big(\lambda_j\varphi(z)e^{i\theta}\big)^{2^k}\right|^2d\theta\,|\lambda_j|^2\left|\varphi'(z)\right|^2K\big(g(z,a)\big)\,dA(z)\\
&=\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}\left(\sum_{k=1}^{\infty}\left|\lambda_j\varphi(z)\right|^{2^{k+1}}\right)|\lambda_j|^2\left|\varphi'(z)\right|^2K\big(g(z,a)\big)\,dA(z).
\end{aligned} \tag{16}$$

From the proof of Theorem 1, for $1/\sqrt{e} < r < 1$ and for sufficiently large j, (16) gives

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |\lambda_j|^2 \left|\varphi'(z)\right|^2 \ln \frac{1}{1-|\lambda_j\varphi(z)|^2} K\bigl(g(z,a)\bigr)\, dA(z) < \varepsilon.$$

By Fatou's lemma, we get (14). □

Theorem 3 *Let K be a nonnegative nondecreasing function on $[0,\infty)$. Assume that φ is an analytic self-map of $\mathbb{D}$. Then the following statements hold*:

(i) *If $C_\varphi : \mathcal{Z}_0 \to Q_{K,0}$ is bounded, then $\varphi \in Q_{K,0}$ and*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|\varphi'(z)\right|^2 \ln \frac{1}{1-|\varphi(z)|^2} K\bigl(g(z,a)\bigr)\, dA(z) < \infty. \tag{17}$$

(ii) *If $\varphi \in Q_{K,0}$ and*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|\varphi'(z)\right|^2 \left(\ln \frac{e}{1-|\varphi(z)|^2}\right)^2 K\bigl(g(z,a)\bigr)\, dA(z) < \infty, \tag{18}$$

then $C_\varphi : \mathcal{Z}_0 \to Q_{K,0}$ is bounded.

Proof (i) Assume that $C_\varphi : \mathcal{Z}_0 \to Q_{K,0}$ is bounded. Then it is obvious that $C_\varphi : \mathcal{Z}_0 \to Q_K$ is bounded. By Theorem 1, (17) holds. Taking $g(z) = z$ and using the boundedness of $C_\varphi : \mathcal{Z}_0 \to Q_{K,0}$, we get $\varphi \in Q_{K,0}$.

(ii) Suppose that $\varphi \in Q_{K,0}$ and (18) holds. From Theorem 1, we see that $C_\varphi : \mathcal{Z}_0 \to Q_K$ is bounded. To prove that $C_\varphi : \mathcal{Z}_0 \to Q_{K,0}$ is bounded, it suffices to prove that $C_\varphi f \in Q_{K,0}$ for any $f \in \mathcal{Z}_0$. Let $f \in \mathcal{Z}_0$. By Lemma 4, for every $\varepsilon > 0$, we can choose $\rho \in (0,1)$ such that $|f'(w)| < \varepsilon \ln \frac{e}{1-|w|^2}$ for all $w \in \mathbb{D} \setminus \rho\overline{\mathbb{D}}$. Then by (8), we have

$$\begin{aligned}
&\lim_{|a|\to 1} \int_{\mathbb{D}} \left|(C_\varphi f)'(z)\right|^2 K\bigl(g(z,a)\bigr)\, dA(z) \\
&\quad = \lim_{|a|\to 1} \left(\int_{|\varphi(z)|>\rho} + \int_{|\varphi(z)|\le\rho}\right) \left|f'\bigl(\varphi(z)\bigr)\right|^2 \left|\varphi'(z)\right|^2 K\bigl(g(z,a)\bigr)\, dA(z) \\
&\quad \le \varepsilon^2 \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>\rho} \left|\varphi'(z)\right|^2 \left(\ln \frac{e}{1-|\varphi(z)|^2}\right)^2 K\bigl(g(z,a)\bigr)\, dA(z) \\
&\qquad + C\|f\|_{\mathcal{Z}}^2 \left(\ln \frac{e}{1-\rho^2}\right)^2 \lim_{|a|\to 1} \int_{|\varphi(z)|\le\rho} \left|\varphi'(z)\right|^2 K\bigl(g(z,a)\bigr)\, dA(z) \\
&\quad \le \varepsilon^2 \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>\rho} \left|\varphi'(z)\right|^2 \left(\ln \frac{e}{1-|\varphi(z)|^2}\right)^2 K\bigl(g(z,a)\bigr)\, dA(z) \\
&\qquad + C\|f\|_{\mathcal{Z}}^2 \left(\ln \frac{e}{1-\rho^2}\right)^2 \lim_{|a|\to 1} \int_{\mathbb{D}} \left|\varphi'(z)\right|^2 K\bigl(g(z,a)\bigr)\, dA(z),
\end{aligned}$$

which together with the assumed conditions imply the desired result. □

Theorem 4 *Let K be a nonnegative nondecreasing function on $[0,\infty)$. Assume that φ is an analytic self-map of $\mathbb{D}$. Then the following statements holds*:

(i) *If*

$$\lim_{|a|\to 1}\int_{\mathbb{D}}\left|\varphi'(z)\right|^2\left(\ln\frac{e}{1-|\varphi(z)|^2}\right)^2 K\big(g(z,a)\big)\,dA(z)=0, \tag{19}$$

then $C_\varphi:\mathcal{Z}(\mathcal{Z}_0)\to Q_{K,0}$ *is compact.*

(ii) *If* $C_\varphi:\mathcal{Z}(\mathcal{Z}_0)\to Q_{K,0}$ *is compact, then*

$$\lim_{|a|\to 1}\int_{\mathbb{D}}\left|\varphi'(z)\right|^2\ln\frac{1}{1-|\varphi(z)|^2}K\big(g(z,a)\big)\,dA(z)=0. \tag{20}$$

Proof (i) Assume that (19) holds. Set

$$h_{\varphi,K}(a)=\int_{\mathbb{D}}\left|\varphi'(z)\right|^2\left(\ln\frac{e}{1-|\varphi(z)|^2}\right)^2 K\big(g(z,a)\big)\,dA(z).$$

From the assumption, we have that for every $\varepsilon>0$, there is a $s\in(0,1)$ such that for $|a|>s$, $h_{\varphi,K}(a)<\varepsilon$. Similarly to the proof of Lemma 2.3 of [25], we see that $h_{\varphi,K}$ is continuous on $|a|\le s$, hence is bounded on $|a|\le s$. Therefore, $h_{\varphi,K}$ is bounded on $\mathbb{D}$. From Theorem 1, we see that $C_\varphi:\mathcal{Z}\to Q_K$ is bounded.

For any $f\in\mathcal{Z}$, by (8), we have

$$\begin{aligned}&\int_{\mathbb{D}}\left|(C_\varphi f)'(z)\right|^2K\big(g(z,a)\big)\,dA(z)\\&\quad\le C\|f\|_{\mathcal{Z}}^2\int_{\mathbb{D}}\left|\varphi'(z)\right|^2\left(\ln\frac{e}{1-|\varphi(z)|^2}\right)^2K\big(g(z,a)\big)\,dA(z),\end{aligned} \tag{21}$$

which together with (19) imply that $C_\varphi:\mathcal{Z}\to Q_{K,0}$ is bounded. Fix $f\in\mathbb{B}_{\mathcal{Z}}$. The right-hand side of (21) tends to 0, as $|a|\to 1$ by (19). From Lemma 3, we see that $C_\varphi:\mathcal{Z}\to Q_{K,0}$ is compact, and hence $C_\varphi:\mathcal{Z}_0\to Q_K$ is compact.

(ii) From the assumption, we see that $C_\varphi:\mathcal{Z}_0\to Q_{K,0}$ is bounded and $C_\varphi:\mathcal{Z}_0\to Q_K$ is compact. From Theorems 2 and 3, we have $\varphi\in Q_{K,0}$ and

$$\lim_{r\to 1}\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}\left|\varphi'(z)\right|^2\ln\frac{1}{1-|\varphi(z)|^2}K\big(g(z,a)\big)\,dA(z)=0. \tag{22}$$

Hence, for any given $\varepsilon>0$, there exists a $s\in(0,1)$ such that

$$\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>s}\left|\varphi'(z)\right|^2\ln\frac{1}{1-|\varphi(z)|^2}K\big(g(z,a)\big)\,dA(z)<\varepsilon. \tag{23}$$

Therefore, by (23) and the fact that $\varphi\in Q_{K,0}$, we have

$$\begin{aligned}&\lim_{|a|\to 1}\int_{\mathbb{D}}\left|\varphi'(z)\right|^2\ln\frac{1}{1-|\varphi(z)|^2}K\big(g(z,a)\big)\,dA(z)\\&\quad\le\lim_{|a|\to 1}\int_{|\varphi(z)|>s}\left|\varphi'(z)\right|^2\ln\frac{1}{1-|\varphi(z)|^2}K\big(g(z,a)\big)\,dA(z)\\&\qquad+\lim_{|a|\to 1}\int_{|\varphi(z)|\le s}\left|\varphi'(z)\right|^2\ln\frac{1}{1-|\varphi(z)|^2}K\big(g(z,a)\big)\,dA(z)\end{aligned}$$

$$\begin{aligned}&\leq \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>s} \left|\varphi'(z)\right|^2 \ln \frac{1}{1-|\varphi(z)|^2} K\bigl(g(z,a)\bigr)\,dA(z)\\&\quad + \ln \frac{1}{1-s^2} \lim_{|a|\to 1} \int_D \left|\varphi'(z)\right|^2 K\bigl(g(z,a)\bigr)\,dA(z)\\&< \varepsilon.\end{aligned}$$

By the arbitrary of ε, we get the desired result. The proof of the theorem is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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