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Some relations involving a generalized fractional derivative operator

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Abstract

Recently, Katugampola (Appl. Math. Comput. 218:860-865, 2011) studied a special case of the Erdélyi-Kober generalized fractional derivative. This special case generalized the well-known Riemann-Liouville and the Hadamard fractional integrals into a single form. Katugampola denoted this special case by the operator ${}^{\rho}D_x^{\alpha}$. Some properties and examples for this fractional derivative operator was given. In this paper, we present some additional properties for this operator defined, this time, in the complex plane. In particular, we express this fractional derivative operator in terms of the classical Riemann-Liouville fractional derivative operator. A generalized Leibniz rule is obtained.

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1 Introduction

One of the most frequently encountered operators of fractional derivatives, that is, calculus of integrals and derivatives of an arbitrary real or complex order is provided by the Riemann-Liouville operator [1, 2] denoted D_z^{α} and defined by

$$D_z^{\alpha} z^p f(z) := \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_0^z f(\zeta) \zeta^p (z - \zeta)^{-\alpha-1} d\zeta & \text{if } \operatorname{Re}(\alpha) < 0, \\ \frac{d^m}{dz^m} D_z^{\alpha-m} z^p f(z) & \text{if } m - 1 \leq \operatorname{Re}(\alpha) < m \end{cases} \quad (1)$$

($m \in \mathbb{N}$) which is valid for $\operatorname{Re}(p) > -1$, $z \neq 0$ on a simply connected region \mathcal{R} containing the origin. Note that $f(z)$ must be analytic in \mathcal{R} and specially at the origin.

Many other representations for fractional derivative operator exists. For example, using a single-loop contour of integration, we can obtain a less restrictive definition for the fractional derivative operator D_z^{α} which holds for $\operatorname{Re}(p) > -1$ and α not a negative integer. This representation has been widely used in many interesting papers [3–8]. The less restrictive representation for the fractional derivative operator in the complex plane is the one introduced by Lavoie, Osler and Tremblay [9] in 1976 based on Pochhammer's contour of integration [10–12]. This representation is valid when α is not a negative integer and p is not an integer.

Many interesting applications of Pochhammer's contour based definition for fractional derivatives can be found in [13–17].

Another interesting and largely studied fractional derivative operator is the Erdélyi-Kober operator [2, 18] defined as follows:

$$D_{0+;\rho;\eta}^\alpha f(x) := \begin{cases} \frac{\rho x^{-\rho(-\alpha+\eta)}}{\Gamma(-\alpha)} \int_0^x \frac{f(t)t^{\rho\eta+\rho-1}}{(x^\rho-t^\rho)^{\alpha+1}} dt & \text{if } \operatorname{Re}(\alpha) < 0, \\ x^{-\rho\eta} \left(\frac{1}{\rho x^{\rho-1}} \frac{d}{dx}\right)^m x^{\rho(m+\eta)} D_{0+;\rho;\eta-\alpha}^{\alpha-m} f(x) & \text{if } m-1 \leq \operatorname{Re}(\alpha) < m, \end{cases} \quad (2)$$

where $\eta \in \mathcal{R}$ and $\rho > 0$. This operator has been introduced to solve some integral equations of particular type. For many interesting applications of fractional calculus operators, the reader is referred to the book of Kilbas *et al.* [19].

Recently, Katugampola in [20] studied a special case of the Erdélyi-Kober generalized fractional derivative operator which contains as special cases the Riemann-Liouville fractional operator (1) and the Hadamard fractional operator [21]. In particular, the author considered the fractional derivative operator ${}^\rho_0 D_x^\alpha$ defined by

$${}^\rho_0 D_x^\alpha f(x) := \begin{cases} \frac{(\rho+1)^{\alpha+1}}{\Gamma(-\alpha)} \int_0^x \frac{f(t)t^\rho}{(x^{\rho+1}-t^{\rho+1})^{\alpha+1}} dt & \text{if } \operatorname{Re}(\alpha) < 0, \\ (x^{-\rho} \frac{d}{dx})^m {}^\rho_0 D_x^{\alpha-m} f(x) & \text{if } m-1 \leq \operatorname{Re}(\alpha) < m \end{cases} \quad (3)$$

($m \in \mathbb{N}$) which is valid for $\rho \in \mathbb{R}$ with $\rho > -1$.

It is easy to verify that they are related by the following relation:

$${}^\rho_0 D_x^\alpha f(x) = (\rho+1)^\alpha x^{-(\rho+1)\alpha} D_{0+;\rho+1;0}^\alpha f(x). \quad (4)$$

Now, substituting $\rho = 0$ in (3) gives the standard Riemann-Liouville derivative (1). Moreover, considering functions belonging to a particular space of function, namely $X_c^p(a, b)$ ($c \in \mathbb{R}$, $1 \leq p \leq \infty$) (see, [21]) and taking the limit when $\rho \rightarrow -1^+$ in (3), the author in [20] obtained, with the help of the L'Hospital's rule, the following relationship:

$$\begin{aligned} & \lim_{\rho \rightarrow -1^+} \frac{(\rho+1)^{\alpha+1}}{\Gamma(-\alpha)} \int_0^x \frac{f(t)t^\rho}{(x^{\rho+1}-t^{\rho+1})^{\alpha+1}} dt \\ &= \frac{1}{\Gamma(-\alpha)} \int_0^x \lim_{\rho \rightarrow -1^+} f(t)t^\rho \left(\frac{x^{\rho+1}-t^{\rho+1}}{\rho+1}\right)^{-\alpha-1} dt \\ &= \frac{1}{\Gamma(-\alpha)} \int_0^x \left(\log \frac{x}{t}\right)^{-\alpha-1} \frac{f(t)}{t} dt. \end{aligned} \quad (5)$$

This last relation is the well-known and extensively studied Hadamard fractional integral [21-23].

Applying the generalized fractional derivative operator defined in (3) to the function $f(x) = x^v$, $v \in \mathbb{R}$ in the case $\operatorname{Re}(\alpha) < 0$ yields the following formula:

$$\begin{aligned} {}^\rho_0 D_x^\alpha x^v &= \frac{(\rho+1)^{\alpha+1}}{\Gamma(-\alpha)} \int_0^x t^{\rho+v} (x^{\rho+1}-t^{\rho+1})^{-\alpha-1} dt \\ &= \frac{(\rho+1)^{\alpha+1}}{\Gamma(-\alpha)} \int_0^1 \frac{x^{v-\alpha(\rho+1)} u^{\frac{v}{\rho+1}} (1-u)^{-\alpha-1}}{\rho+1} du \\ &= \frac{(\rho+1)^\alpha x^{v-\alpha(\rho+1)}}{\Gamma(-\alpha)} B\left(\frac{v}{\rho+1}+1, -\alpha\right) = \frac{(\rho+1)^\alpha \Gamma\left(\frac{v}{\rho+1}+1\right)}{\Gamma\left(\frac{v}{\rho+1}+1-\alpha\right)} x^{v-\alpha(\rho+1)}. \end{aligned} \quad (6)$$

The last relation is valid for $\frac{\nu}{\rho+1} > -1$. Note that we used the following change of variables $u = \frac{t^{\rho+1}}{x^{\rho+1}}$ in (6) and that $B(x, y)$ holds for the familiar Beta function [2, p.17, Eq. 1.68]. Replacing ρ by 0 in (6) gives the classical fractional integral formula for the function x^ν , that is,

$$D_x^\alpha x^\nu = \frac{\Gamma(1 + \nu)}{\Gamma(1 + \nu - \alpha)} x^{\nu-\alpha} \tag{7}$$

with $\text{Re}(\alpha) < 0$ and $\nu > -1$. In this paper, we consider the fractional derivative operator ${}^\rho_0 D_z^\alpha$ defined in the complex plane. In Section 2, we obtain two representations for this fractional derivative operator in terms of the classical Riemann-Liouville fractional operator. The second representation is obtained by making use of a new transformation formula for the fractional derivative recently published by Tremblay *et al.* [16]. Some special cases are computed. Finally, in Section 3, we derive a generalized Leibniz rule for the fractional derivative operator ${}^\rho_0 D_z^\alpha$. Some applications of this Leibniz rule are also obtained.

2 Representations of the generalized fractional derivative operator in terms of the classical Riemann-Liouville fractional derivative operator

The purpose of this section is to develop two representations of the generalized fractional derivative operator ${}^\rho_0 D_z^\alpha$ in terms of the common Riemann-Liouville fractional operator (1). These representations are found by making use of a new transformation formula for fractional derivatives published recently by the authors [16]. Some examples of possible new relationships are also given.

Note that for the remainder of this paper, $(\lambda)_n$ will denote the Pochhammer's symbol defined by

$$(\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)}; \quad (\lambda)_0 = 1. \tag{8}$$

Moreover, we adopt the following notation to denote the generalized hypergeometric function

$${}_pF_q \left[\begin{matrix} a_1, & a_2, & \dots, & a_p; \\ b_1, & b_2, & \dots, & b_q; \end{matrix} \middle| z \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_p)_k z^k}{(b_1)_k (b_2)_k \dots (b_q)_k k!}. \tag{9}$$

The first representation for the fractional derivative operator ${}^\rho_0 D_z^\alpha$ is contained in the next theorem.

Theorem 1

- (i) Let \mathcal{R} be a simply connected region containing the origin.
- (ii) Let $f(z)$ be analytic in \mathcal{R} . Then, for α and $\rho \in \mathbb{C}$ with $\text{Re}(\rho) > -1$, the following relation holds true

$${}^\rho_0 D_z^\alpha f(z) = \frac{(\rho + 1)^{\alpha+1}}{\Gamma(-\alpha)} z^{-(\rho+1)(\alpha+1)} \sum_{n=0}^{\infty} \frac{(\alpha + 1)_n}{n!} z^{-n(\rho+1)} \times \Gamma((\rho + 1)n + \rho + 1) D_z^{-(\rho+1)n - \rho - 1} f(w - z) \Big|_{w=z}. \tag{10}$$

Proof Consider the integral representation of the operator ${}_0^\rho D_z^\alpha$ in the complex plane with $\text{Re}(\alpha) < 0$:

$${}_0^\rho D_z^\alpha f(z) = \frac{(\rho + 1)^{\alpha+1}}{\Gamma(-\alpha)} \int_0^z \frac{f(t)t^\rho}{(z^{\rho+1} - t^{\rho+1})^{\alpha+1}} dt. \tag{11}$$

Making the following change of variables $t = z - \xi$ ($dt = -d\xi$), we have

$$\begin{aligned} {}_0^\rho D_z^\alpha f(z) &= -\frac{(\rho + 1)^{\alpha+1}}{\Gamma(-\alpha)} \int_z^0 \frac{f(z - \xi)(z - \xi)^\rho}{(z^{\rho+1} - (z - \xi)^{\rho+1})^{\alpha+1}} d\xi \\ &= \frac{(\rho + 1)^{\alpha+1}}{\Gamma(-\alpha)} z^{-(\rho+1)(\alpha+1)} \int_0^z \frac{f(z - \xi)(z - \xi)^\rho}{(1 - (\frac{z-\xi}{z})^{\rho+1})^{\alpha+1}} d\xi \\ &= \frac{(\rho + 1)^{\alpha+1} \Gamma(1 + \rho)}{\Gamma(-\alpha)} z^{-(\rho+1)(\alpha+1)} \\ &\quad \times D_z^{-\rho-1} \left(1 - \left(\frac{w-z}{w} \right)^{\rho+1} \right)^{-\alpha-1} f(w-z) \Big|_{w=z}. \end{aligned} \tag{12}$$

Note that we must have $w \rightarrow z$ in the right side of (12) after the evaluation of the fractional derivative. So, the point w must be near the point z . Using this fact, we can expand the expression $(1 - (\frac{w-z}{w})^{\rho+1})^{-\alpha-1}$ in power series. We thus have

$$\begin{aligned} {}_0^\rho D_z^\alpha f(z) &= \frac{(\rho + 1)^{\alpha+1}}{\Gamma(-\alpha)} z^{-(\rho+1)(\alpha+1)} \int_0^z \sum_{n=0}^{\infty} \frac{(\alpha + 1)_n}{n!} \left(\frac{z - \xi}{z} \right)^{(\rho+1)n} f(z - \xi)(z - \xi)^\rho d\xi \\ &= \frac{(\rho + 1)^{\alpha+1}}{\Gamma(-\alpha)} z^{-(\rho+1)(\alpha+1)} \sum_{n=0}^{\infty} \frac{(\alpha + 1)_n}{n!} z^{-n(\rho+1)} \int_0^z f(z - \xi)(z - \xi)^{(\rho+1)n+\rho} d\xi. \end{aligned} \tag{13}$$

Rewriting the integral in (13) in terms of the Riemann-Liouville fractional derivative operator yields the desired result. \square

Corollary 2 *Substituting $\rho = 0$ in Theorem 1, we obtain*

$$D_z^\alpha f(z) = \frac{z^{-(\alpha+1)}}{\Gamma(-\alpha)} \sum_{n=0}^{\infty} (\alpha + 1)_n z^{-n} D_z^{-n-1} f(w-z) \Big|_{w=z}. \tag{14}$$

Example 1 Let $f(z) = (1 - z)^{-\gamma}$ in Theorem 1. We obtain for the l.h.s. of (10) by making use of the power series expansion of $(1 - z)^{-\gamma}$

$${}_0^\rho D_z^\alpha (1 - z)^{-\gamma} = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k!} {}_0^\rho D_z^\alpha z^k = (\rho + 1)^\alpha z^{-\alpha(\rho+1)} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k!} \frac{\Gamma(\frac{k}{\rho+1} + 1)}{\Gamma(\frac{k}{\rho+1} + 1 - \alpha)} z^k. \tag{15}$$

For the fractional derivative operator involved in the r.h.s. of (10), we have

$$\begin{aligned} &D_z^{-(\rho+1)n-\rho-1} (1 - (w - z))^{-\gamma} \Big|_{w=z} \\ &= D_z^{-(\rho+1)n-\rho-1} (1 - w)^{-\gamma} \left(1 - \left(\frac{z}{w-1} \right) \right)^{-\gamma} \Big|_{w=z} \end{aligned}$$

$$\begin{aligned}
 &= (1-z)^{-\gamma} D_z^{-(\rho+1)n-\rho-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k!} \frac{z^k}{(w-1)^k} \Big|_{w=z} \\
 &= (1-z)^{-\gamma} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k!(z-1)^k} D_z^{-(\rho+1)n-\rho-1} z^k \\
 &= (1-z)^{-\gamma} z^{(\rho+1)n+\rho+1} \sum_{k=0}^{\infty} \frac{(\gamma)_k \Gamma(1+k)}{\Gamma(2+k+(\rho+1)n+\rho)k!} \left(\frac{z}{z-1}\right)^k \\
 &= \frac{(1-z)^{-\gamma} z^{(\rho+1)n+\rho+1}}{\Gamma(2+(\rho+1)n+\rho)} {}_2F_1 \left[\begin{matrix} \gamma, & 1; \\ & 2+(\rho+1)n+\rho; \end{matrix} \frac{z}{z-1} \right]. \tag{16}
 \end{aligned}$$

Combining (15) and (16) in (10), we obtain after some simple calculations

$$\begin{aligned}
 &\sum_{k=0}^{\infty} \frac{(\gamma)_k \Gamma(\frac{k}{\rho+1} + 1)}{\Gamma(-\frac{k}{\rho+1} + 1 - \alpha)} \frac{z^k}{k!} \\
 &= \frac{(1-z)^{-\gamma}}{\Gamma(-\alpha)} \sum_{n=0}^{\infty} \frac{(\rho+1)(\alpha+1)_n}{n!((\rho+1)n+\rho+1)} {}_2F_1 \left[\begin{matrix} \gamma, & 1; \\ & 2+(\rho+1)n+\rho; \end{matrix} \frac{z}{z-1} \right], \tag{17}
 \end{aligned}$$

where ${}_2F_1 \left[\begin{matrix} a, b; \\ c; \end{matrix} z \right]$ denotes the Gauss hypergeometric function [24].

Moreover, setting $\rho = 0$ in (17) gives the following relationship involving the Gauss hypergeometric function:

$${}_2F_1 \left[\begin{matrix} \gamma, & 1; \\ & 1-\alpha; \end{matrix} z \right] = \frac{\Gamma(1-\alpha)(1-z)^{-\gamma}}{\Gamma(-\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha+1)_n}{(n+1)!} {}_2F_1 \left[\begin{matrix} \gamma, & 1; \\ & 2+n; \end{matrix} \frac{z}{z-1} \right]. \tag{18}$$

Recently, Tremblay *et al.* [16] discovered a new transformation formula for the fractional derivatives. Many interesting applications of this formula has also been given. Especially, they proved the next result.

Theorem 3 *Let $f(z)$ be an analytic function in the simply connected region \mathcal{R} containing the origin. For $\text{Re}(\mu) > -1$, we have*

$$D_z^\alpha z^\mu f(z) = \frac{\Gamma(1+\mu)}{\Gamma(-\alpha)} D_z^{-\mu-1} (z)^{-\alpha-1} f(w-z) \Big|_{w=z}. \tag{19}$$

Note that we must have $w \rightarrow z$ in the right side of (19) after the evaluation of the fractional derivative since the point w must be near the point z .

With the help of this new result, we can easily obtain the next theorem.

Theorem 4

(i) *Let \mathcal{R} be a simply connected region containing the origin.*

(ii) Let $f(z)$ be analytic in \mathcal{R} . Then, for α and $\rho \in \mathbb{C}$ with $\text{Re}(\rho) > -1$, the following relation holds true

$${}^{\rho}D_z^{\alpha}f(z) = \frac{(\rho + 1)^{\alpha+1}}{\Gamma(-\alpha)} z^{-(\rho+1)(\alpha+1)} \sum_{n=0}^{\infty} \frac{(\alpha + 1)_n}{n!} z^{-n(\rho+1)} D_z^{-1} z^{(\rho+1)n+\rho} f(z). \quad (20)$$

Proof Applying Theorem 3 to Theorem 1, the result follows easily. \square

Corollary 5 Substituting $\rho = 0$ in the Theorem 4, we obtain

$$D_z^{\alpha}f(z) = \frac{z^{-(\alpha+1)}}{\Gamma(-\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha + 1)_n}{n!} z^{-n} D_z^{-1} z^n f(z). \quad (21)$$

Example 2 Putting $f(z) = e^z$ in Theorem 4 gives for the l.h.s. of (20)

$${}^{\rho}D_z^{\alpha}e^z = \sum_{k=0}^{\infty} \frac{{}^{\rho}D_z^{\alpha}z^k}{k!} = (\rho + 1)^{\alpha} z^{-\alpha(\rho+1)} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{k}{\rho+1} + 1)}{\Gamma(\frac{k}{\rho+1} + 1 - \alpha)} \frac{z^k}{k!}. \quad (22)$$

Now, for the r.h.s. of (20), the computation of the fractional derivative operator applied to the exponential function yields

$$D_z^{-1} z^{(\rho+1)n+\rho} e^z = \sum_{k=0}^{\infty} \frac{D_z^{-1} z^{(\rho+1)n+\rho+k}}{k!} = z^{(\rho+1)n+\rho+1} \sum_{k=0}^{\infty} \frac{z^k}{(1 + (\rho + 1)n + \rho + k)k!} \quad (23)$$

Replacing (22) and (23) in (20), we arrive, after some simplifications, to the next formula:

$$\sum_{k=0}^{\infty} \frac{\Gamma(\frac{k}{\rho+1} + 1)z^k}{\Gamma(\frac{k}{\rho+1} + 1 - \alpha)k!} = \frac{(\rho + 1)}{\Gamma(-\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha + 1)_n}{n!} \sum_{k=0}^{\infty} \frac{z^k}{(1 + (\rho + 1)n + \rho + k)k!}. \quad (24)$$

Note that by making use of Theorem 1, we also have that

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{k}{\rho+1} + 1)z^k}{\Gamma(\frac{k}{\rho+1} + 1 - \alpha)k!} &= \frac{(\rho + 1)}{\Gamma(-\alpha)} z^{-(\rho+1)} \sum_{n=0}^{\infty} \frac{(\alpha + 1)_n}{n!} z^{-n(\rho+1)} \\ &\quad \times \Gamma((\rho + 1)n + \rho + 1) e^z D_z^{-(\rho+1)n-\rho-1} e^{-z} \\ &= \frac{(\rho + 1)e^z}{\Gamma(-\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha + 1)_n}{(1 + (\rho + 1)n + \rho)n!} \sum_{k=0}^{\infty} \frac{(1)_k (-z)^k}{(2 + (\rho + 1)n + \rho)k!}. \end{aligned}$$

3 A Leibniz rule for the fractional derivative operator ${}^{\rho}D_z^{\alpha}$

This section is devoted to obtain a generalized Leibniz-type rule for the fractional derivative operator ${}^{\rho}D_z^{\alpha}$. Two special cases are also computed. This is done by using the first representation (10) and the generalized Leibniz rule for fractional derivatives obtained by Osler [7] and given in the next theorem.

Theorem 6

(i) Let \mathcal{R} be a simply connected region containing the origin.

(ii) Let $f(z)$ and $g(z)$ be analytic in \mathcal{R} . Then, for $0 < a \leq 1$, $\alpha \in \mathbb{C}$ with α not a negative integer, $\gamma \in \mathbb{C}$, the following Leibniz rule holds true

$$D_z^\alpha f(z)g(z) = a \sum_{n=-\infty}^{\infty} \binom{\alpha}{\gamma + an} D_z^{\alpha-\gamma-an} f(z) D_z^{\gamma+an} g(z). \tag{25}$$

We are now able to establish the generalized Leibniz rule for the operator ${}^\rho_0 D_z^\alpha$.

Theorem 7

(i) Let \mathcal{R} be a simply connected region containing the origin.
 (ii) Let $f(z)$ and $g(z)$ be analytic in \mathcal{R} . Then, for $0 < a \leq 1$, $\alpha \in \mathbb{C}$ with α not a negative integer, $\gamma \in \mathbb{C}$, $\text{Re}(\rho) > -1$ and for all $k \in \mathbb{N}_0$, $(-1 - \rho - (\rho + 1)k)$ not a negative integer, the following Leibniz rule

$$\begin{aligned} {}^\rho_0 D_z^\alpha f(z)g(z) &= a \frac{(\rho + 1)^{\alpha+1}}{\Gamma(-\alpha)} z^{-(\rho+1)(\alpha+1)} \\ &\times \sum_{k=0}^{\infty} \frac{(\alpha + 1)_k \Gamma(1 + \rho + (\rho + 1)k)}{k! z^{(\rho+1)k}} \sum_{n=-\infty}^{\infty} \binom{-1 - \rho - (\rho + 1)k}{\gamma + an} \\ &\times D_z^{-1-\rho-(\rho+1)k-\gamma-an} f(w-z) \Big|_{w=z} D_z^{\gamma+an} g(w-z) \Big|_{w=z} \end{aligned} \tag{26}$$

holds true.

Proof We know from Theorem 1 that the fractional derivative operator ${}^\rho_0 D_z^\alpha$ applied to the product of the functions $f(z)g(z)$ can be written in the following form:

$$\begin{aligned} {}^\rho_0 D_z^\alpha f(z)g(z) &= \frac{(\rho + 1)^{\alpha+1}}{\Gamma(-\alpha)} z^{-(\rho+1)(\alpha+1)} \sum_{n=0}^{\infty} \frac{(\alpha + 1)_n}{n!} z^{-n(\rho+1)} \\ &\times \Gamma((\rho + 1)n + \rho + 1) D_z^{-(\rho+1)n-\rho-1} f(w-z)g(w-z) \Big|_{w=z}, \end{aligned} \tag{27}$$

where $w \rightarrow z$ in the right side of (27) after the evaluation of the fractional derivative since the point w must be near the point z . Applying Theorem 6 to the r.h.s. of (27) gives the desired result. □

We end this paper by computing two special cases of Theorem 7. Note that calculations have been done with the help of standard formulas for fractional derivative operator acting on common functions (see [25, 26]).

Example 3 Setting $f(z) = 1$ and $g(z) = e^z$ in Theorem 7 gives after some simple calculations:

$$\begin{aligned} \sum_{j=0}^{\infty} \frac{\Gamma(\frac{j}{\rho+1} + 1) z^j}{\Gamma(\frac{j}{\rho+1} + 1 - \alpha) j!} &= a \frac{(\rho + 1) e^z}{\Gamma(-\alpha)} \sum_{k=0}^{\infty} \frac{(\alpha + 1)_k \Gamma(1 + \rho + (\rho + 1)k)}{k!} \\ &\times \sum_{n=-\infty}^{\infty} \frac{(-1-\rho-(\rho+1)k)_{\gamma+an} {}_1F_1 \left[\begin{matrix} 1; \\ 1-\gamma-an; \end{matrix} \right] (-z)}{\Gamma(2 + \rho + (\rho + 1)k + \gamma + an) \Gamma(1 - \gamma - an)}. \end{aligned} \tag{28}$$

Example 4 If we put $f(z) = (1 - z)^{-\mu}$ and $g(z) = (1 - z)^{-\nu}$ in Theorem 7, we obtain, after simplifications, the following relation:

$$\begin{aligned} & \sum_{j=0}^{\infty} \frac{(\mu + \nu)_j \Gamma(\frac{j}{\rho+1} + 1) z^j}{\Gamma(\frac{j}{\rho+1} + 1 - \alpha) j!} \\ &= a \frac{(\rho + 1)(1 - z)^{-\mu-\nu}}{\Gamma(-\alpha)} \sum_{k=0}^{\infty} \frac{(\alpha + 1)_k \Gamma(1 + \rho + (\rho + 1)k)}{k!} \sum_{n=-\infty}^{\infty} \binom{-1 - \rho - (\rho + 1)k}{\gamma + an} \\ & \times \frac{{}_2F_1 \left[\begin{matrix} \mu, & 1; \\ & 2 + \rho + (\rho + 1)k + \gamma + an; \end{matrix} \frac{z}{z-1} \right] {}_2F_1 \left[\begin{matrix} \nu, & 1; \\ & 1 - \gamma - an; \end{matrix} \frac{z}{z-1} \right]}{\Gamma(2 + \rho + (\rho + 1)k + \gamma + an) \Gamma(1 - \gamma - an)}. \end{aligned} \tag{29}$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors completed the paper together. Both authors read and approved the final manuscript.

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