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Fixed points and stability of functional equations in fuzzy ternary Banach algebras

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Abstract

By using Diaz and Margolis fixed point theorem, we establish the generalized Hyers-Ulam-Rassias stability of the ternary homomorphisms and ternary derivations between fuzzy ternary Banach algebras associated to the following (m,n)-Cauchy-Jensen additive functional equation:

$$\sum_{\substack{1\le i_1<\cdots<i_m\le n\\ 1\le k_l\le n\\ k_l\ne i_j,\ \forall j\in\{1,\ldots,m\}}} f\left(\frac{\sum_{j=1}^m x_{i_j}}{m}+\sum_{l=1}^{n-m} x_{k_l}\right)=\frac{(n-m+1)}{n}\binom{n}{m}\sum_{i=1}^{n} f(x_i).$$

MSC: 39B52; 46S40; 26E50

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1 Introduction

A classical question in the theory of functional equations is the following:

When is it true that a function which approximately satisfies a functional equation $\mathcal{E}$ must be close to an exact solution of $\mathcal{E}$?

If the problem admits a solution, we say that the equation $\mathcal{E}$ is stable. Such a problem was formulated by Ulam [1] in 1940 and solved in the next year for the Cauchy functional equation by Hyers [2]. Since Hyers, many authors have studied the stability theory for functional equations. The result of Hyers was extended by Aoki [3] in 1950, by considering the unbounded Cauchy differences. Also, Hyers' theorem was generalized by Rassias [4] for linear mappings by considering an unbounded Cauchy difference.

Theorem 1.1 (TM Rassias) *Let $f: E\to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the following inequality*:

$$\left\|f(x+y)-f(x)-f(y)\right\|\le\epsilon\left(\|x\|^p+\|y\|^p\right)$$

for all $x,y\in E$, where ϵ and p are constants with $\epsilon>0$ and $0\le p<1$. Then the limit $L(x)=\lim_{n\to\infty}\frac{f(2^n x)}{2^n}$ exists for all $x\in E$, and $L: E\to E'$ is the unique additive mapping which *satisfies*

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2-2^p}\|x\|^p$$

for all $x \in E$*. Also, if for each* $x \in E$*, the function* $f(tx)$ *is continuous in* $t \in \mathbb{R}$*, then* L *is linear.*

Găvruta [5] generalized the Rassias' result. Beginning around the year 1980, the stability problems of several functional equations and approximate homomorphisms have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [6–29]).

Katsaras [30] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view (see [31, 32]). In particular, Bag and Samanta [33], following Cheng and Mordeson [34], gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Karmosil and Michalek type [35]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [36].

Now, we consider a mapping $f: X \to Y$ satisfying the following functional equation, which is introduced by Rassias and Kim [37] (see also [38]):

$$\sum_{\substack{1\leq i_1<\cdots<i_m\leq n\\ 1\leq k_l\leq n\\ k_l\neq i_j,\ \forall j\in\{1,\dots,m\}}} f\left(\frac{\sum_{j=1}^m x_{i_j}}{m} + \sum_{l=1}^{n-m} x_{k_l}\right) = \frac{(n-m+1)}{n}\binom{n}{m}\sum_{i=1}^n f(x_i) \tag{1.1}$$

for all $x_1, \dots, x_n \in X$, where $m, n \in \mathbb{N}$ are fixed integers with $n \geq 2$ and $1 \leq m \leq n$. Especially, we observe that, in the case $m = 1$, equation (1.1) yields the Cauchy additive equation

$$f\left(\sum_{l=1}^n x_{k_l}\right) = \sum_{l=1}^n f(x_i).$$

Also, we observe that, in the case $m = n$, equation (1.1) yields the Jensen additive equation

$$f\left(\frac{\sum_{j=1}^n x_j}{n}\right) = \frac{1}{n}\sum_{l=1}^n f(x_i).$$

Therefore, equation (1.1) is a generalized form of the Cauchy-Jensen additive equation and thus every solution of equation (1.1) may be analogously called the general (m,n)-Cauchy-Jensen additive. For the case $m = 2$, the authors have established new theorems about the Ulam-Hyers-Rassias stability in quasi-β-normed spaces [37].

Let X and Y be linear spaces. For each m with $1 \leq m \leq n$, a mapping $f: X \to Y$ satisfies equation (1.1) for all $n \geq 2$ if and only if $f(x) - f(0) = A(x)$ is Cauchy additive, where $f(0) = 0$ if $m < n$. In particular, we have $f((n-m+1)x) = (n-m+1)f(x)$ and $f(mx) = mf(x)$ for all $x \in X$.

Definition 1.1 Let X be a real vector space. A function $N: X \times \mathbb{R} \to [0,1]$ is called a fuzzy norm on X if for all $x, y \in X$ and $s, t \in \mathbb{R}$,

(N1) $N(x,t) = 0$ for $t \leq 0$;
(N2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
(N5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;
(N6) for any $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

Example 1.1 Let $(X, \|\cdot\|)$ be a normed linear space and $\beta > 0$. Then

$$N(x,t) = \begin{cases} \frac{t}{t+\beta\|x\|}, & t > 0, x \in X, \\ 0, & t \leq 0, x \in X \end{cases}$$

is a fuzzy norm on X.

Definition 1.2 Let (X,N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent if there exists $x \in X$ such that

$$\lim_{n\to\infty} N(x_n - x, t) = 1$$

for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ in X, which is denoted by $N - \lim_{t\to\infty} x_n = x$.

Definition 1.3 Let (X,N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\epsilon > 0$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$ and $p > 0$,

$$N(x_{n+p} - x_n, t) > 1 - \epsilon.$$

It is well known that every convergent sequence in a fuzzy normed vector space is a Cauchy sequence. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x \in X$ if for each sequence $\{x_n\}$ converging to $x_0 \in X$, the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on X (see [36]).

Ternary algebraic operations were considered in the nineteenth century by several mathematicians such as Cayley [39] who introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii in 1990 [40]. The comments on physical applications of ternary structures can be found in [41–45].

Definition 1.4 Let X be a ternary algebra and (X,N) be a fuzzy normed space.
(1) The fuzzy normed space (X,N) is called a ternary fuzzy normed algebra if

$$N\big([xyz], stu\big) \geq N(x,s)N(y,t)N(z,u)$$

for all $x, y, z \in X$ and $s, t, u > 0$;

(2) A complete ternary fuzzy normed algebra is called a ternary fuzzy Banach algebra.

Example 1.2 Let $(X, \|\cdot\|)$ be a ternary normed (Banach) algebra. Let

$$N(x,t) = \begin{cases} \frac{t}{t+\|x\|}, & t > 0, x \in X, \\ 0, & t \leq 0, x \in X. \end{cases}$$

Then $N(x,t)$ is a fuzzy norm on X and (X,N) is a ternary fuzzy normed (Banach) algebra.

Definition 1.5 Let (X,N) and (Y,N') be two ternary fuzzy normed algebras.

(1) A $\mathbb{C}$-linear mapping $H : (X,N) \to (Y,N')$ is called a ternary homomorphism if

$$H([xyz]) = [H(x)H(y)H(z)]$$

for all $x,y,z \in X$;

(2) A $\mathbb{C}$-linear mapping $D : (X,N) \to (X,N)$ is called a ternary fuzzy derivation if

$$D([xyz]) = [D(x)yz] + [xD(y)z] + [xyD(z)]$$

for all $x,y,z \in X$.

We apply the following theorem on weighted spaces (see [46–49]).

Theorem 1.2 (The generalized fixed point theorem of Diaz and Margolis) *Let (X,d) be a complete metric space and $T : X \to X$ be a contraction, i.e., there exists $\alpha \in [0,1)$ such that*

$$d(Tx, Ty) \leq \alpha d(x,y)$$

for all $x,y \in X$. Then there exists a unique $a \in X$ such that $Ta = a$. Moreover, $a = \lim_{n\to\infty} T^n x$ and

$$d(a,x) \leq \frac{1}{1-\alpha} d(x, Tx)$$

for all $x \in X$.

Throughout this paper, we suppose that X is a ternary fuzzy normed algebra and Y is a ternary fuzzy Banach algebra. Moreover, we assume that $n_0 \in \mathbb{N}$ is a positive integer and $\mathbb{T}^1_{\frac{1}{n_o}} := \{e^{i\theta} : 0 \leq \theta \leq \frac{2\pi}{n_o}\}$. For the convenience, we use the following abbreviation for a given mapping $f : X \to Y$:

$$\begin{aligned} &\Delta f(x_1, \ldots, x_n) \\ &\quad = \sum_{\substack{1 \leq i_1 < \cdots < i_m \leq n \\ 1 \leq k_l \leq n \\ k_l \neq i_j,\ \forall j \in \{1,\ldots,m\}}} f\left(\frac{\sum_{j=1}^m \mu x_{i_j}}{m} + \sum_{l=1}^{n-m} \mu x_{k_l} \right) - \frac{(n-m+1)\binom{n}{m} \sum_{i=1}^n \mu f(x_i)}{n}. \end{aligned}$$

2 Main results

In this section, by using the idea of Gavruta and Gavruta [14], we prove the generalized Hyers-Ulam-Rassias stability of ternary homomorphisms related to functional equation (1.1) on ternary fuzzy Banach algebras (see also [50]).

Theorem 2.1 *Let $n \geq 3$ and $\varphi : X^n \to [0,\infty)$ be a mapping such that there exists $L < \frac{1}{(n-m+1)^{n-2}}$ such that*

$$\varphi\left(\frac{x_1}{n-m+1},\dots,\frac{x_n}{n-m+1}\right) \leq \frac{L\varphi(x_1,x_2,\dots,x_n)}{n-m+1}$$

for all $x_1,\dots,x_n \in X$. Let $f : X \to Y$ with $f(0) = 0$ be a mapping satisfying

$$N\big(\Delta f(x_1,\dots,x_n),t\big) \geq \frac{t}{t+\varphi(x_1,\dots,x_n)} \tag{2.1}$$

and

$$N\big(f\big([abc]\big) - \big[f(a)f(b)f(c)\big],t\big) \geq \frac{t}{t+\varphi(a,b,c,0,\dots,0)} \tag{2.2}$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$, $x_1,\dots,x_n,a,b,c \in X$ and $t > 0$. Then there exists a unique ternary homomorphism $H : X \to Y$ such that

$$N\big(f(x) - H(x),t\big) \geq \frac{(n-m+1)\binom{n}{m}(1-L)t}{(n-m+1)\binom{n}{m}(1-L)t + L\varphi(x,\dots,x)} \tag{2.3}$$

for all $x \in X$ and $t > 0$.

Proof Letting $\mu = 1$ and putting $x_1 = x_2 = \cdots = x_n = x$ in (2.1), we have

$$N\left(\binom{n}{m}f\big((n-m+1)x\big) - \binom{n}{m}(n-m+1)f(x),t\right) \geq \frac{t}{t+\varphi(x,\dots,x)} \tag{2.4}$$

for all $x \in X$ and $t > 0$. Set $S_0 := \{h : X \to Y : h(0) = 0\}$ and define a mapping $d_0 : S_0 \times S_0 \to [0,\infty]$ by

$$d_0(f,g) = \inf\left\{\mu \in \mathbb{R}^+ : N\big(g(x) - h(x),\mu t\big) \geq \frac{t}{t+\varphi(x,\dots,x)}, \forall x \in X, t > 0\right\},$$

where $\inf \emptyset = +\infty$. Also, put $S := \{h \in S_0 : d_0(h,f) < \infty\}$. Suppose that d is the restriction of d_0 on $S \times S$. By using the same technique in the proof of Theorem 3.2 [50], we can show that (S,d) is a complete metric space. Now, we define a mapping $J : S \to S$ by

$$Jg(x) := (n-m+1)g\left(\frac{x}{n-m+1}\right)$$

for all $x \in X$. It is easy to see that $d(Jg,Jh) \leq Ld(g,h)$ for all $g,h \in S$. This implies that

$$d(f,Jf) \leq \frac{L}{(n-m+1)\binom{n}{m}}.$$

Thus, by Banach's fixed point theorem (Theorem 1.2), J has a unique fixed point $H : X \to Y$ in S satisfying

$$H\left(\frac{x}{n-m+1}\right) = \frac{H(x)}{n-m+1} \tag{2.5}$$

for all $x \in X$. This implies that H is a unique mapping with (2.5) such that there exists $\mu \in (0,\infty)$ satisfying

$$N\big(f(x) - H(x), \mu t\big) \geq \frac{t}{t + \varphi(x,\dots,x)}$$

for all $x \in X$ and $t > 0$.

Moreover, we have $d(J^p f, H) \to 0$ as $p \to \infty$, which implies

$$N\text{-}\lim_{p\to\infty} \frac{f(\frac{x}{(n-m+1)^p})}{(n-m+1)^{-p}} = H(x) \tag{2.6}$$

for all $x \in X$. Thus it follows from (2.1) and (2.6) that

$$\sum_{\substack{1\le i_1<\cdots<i_m\le n\\ 1\le k_l\le n\\ k_l\ne i_j,\ \forall j\in\{1,\dots,m\}}} H\left(\frac{\sum_{j=1}^m \mu x_{i_j}}{m} + \sum_{l=1}^{n-m} \mu x_{k_l}\right) = \frac{(n-m+1)}{n}\binom{n}{m}\sum_{i=1}^n \mu H(x_i)$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$ and $x_1,\dots,x_n \in X$. This means that $H : X \to Y$ is additive. By using the same technique as in the proof of Theorem 2.1 [51], we can show that H is $\mathbb{C}$-linear. On the other hand, by (2.2), we have

$$N(\alpha,\beta) \geq \frac{t}{t + \varphi(\frac{a}{(n-m+1)^p}, \frac{b}{(n-m+1)^p}, \frac{c}{(n-m+1)^p}, 0, 0, \dots, 0)}$$

for all $a, b, c \in X$ and $t > 0$, where

$$\alpha = \frac{f(\frac{[abc]}{(n-m+1)^{(n-1)p}})}{(n-m+1)^{-(n-1)p}} - \frac{[f(\frac{a}{(n-m+1)^p})f(\frac{b}{(n-m+1)^p})f(\frac{c}{(n-m+1)^p})]}{(n-m+1)^{-(n-1)p}},$$
$$\beta = \frac{t}{(n-m+1)^{-(n-1)p}}.$$

Then we have, as $p \to +\infty$,

$$\begin{aligned} &N\left(\frac{f(\frac{[abc]}{(n-m+1)^{(n-1)p}})}{(n-m+1)^{-(n-1)p}} - \frac{[f(\frac{a}{(n-m+1)^p})f(\frac{b}{(n-m+1)^p})f(\frac{c}{(n-m+1)^p})]}{(n-m+1)^{-(n-1)p}}, t\right) \\ &\quad\geq \frac{\frac{t}{(n-m+1)^{(n-1)p}}}{\frac{t}{(n-m+1)^{(n-1)p}} + \varphi(\frac{a}{(n-m+1)^p}, \frac{b}{(n-m+1)^p}, \frac{c}{(n-m+1)^p}, 0, 0, \dots, 0)} \\ &\quad\geq \frac{\frac{t}{(n-m+1)^{(n-1)p}}}{\frac{t}{(n-m+1)^{(n-1)p}} + \frac{L^p\varphi(a,b,c,0,0,\dots,0)}{(n-m+1)^p}} \to 1 \end{aligned}$$

for all $a, b, c \in X$ and $t > 0$. So, it follows that

$$N\big(H([abc]) - [H(a)H(b)H(c)], t\big) = 1$$

for all $a, b, c \in X$ and $t > 0$. Hence we have $H([abc]) = [H(a)H(b)H(c)]$ for all $a, b, c \in X$. This means that H is a ternary homomorphism. This completes the proof. □

Theorem 2.2 *Let $\varphi : X^n \to [0, \infty)$ be a mapping such that there exists $L < 1$ with*

$$\varphi(x_1, \dots, x_n) \leq (n - m + 1)L\varphi\left(\frac{x_1}{n - m + 1}, \dots, \frac{x_n}{n - m + 1}\right)$$

for all $x_1, x_2, \dots, x_n \in X$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying (2.1). *Then the limit $H(x) := N\text{-}\lim_{p\to\infty} \frac{f((n-m+1)^p x)}{(n-m+1)^p}$ exists for all $x \in X$ and $H : X \to Y$ is defined as a ternary homomorphism such that*

$$N\big(f(x) - H(x), t\big) \geq \frac{(n - m + 1)\binom{n}{m}(1 - L)t}{(n - m + 1)\binom{n}{m}(1 - L)t + \varphi(x, \dots, x)} \tag{2.7}$$

for all $x \in X$ and $t > 0$.

Proof Let (S, d) be the metric space defined as in the proof of Theorem 2.1. Consider the mapping $T : S \to S$ defined by $Tg(x) := \frac{g((n-m+1)x)}{n-m+1}$ for all $x \in X$. One can show that $d(g, h) = \epsilon$ implies that $d(Tg, Th) \leq L\epsilon$ for all positive real numbers ϵ. This means that T is a contraction on (S, d). The mapping

$$H(x) := N\text{-}\lim_{p\to\infty} \frac{f((n - m + 1)^p x)}{(n - m + 1)^p}$$

is the unique fixed point of T in S and H has the following property:

$$(n - m + 1)H(x) = H\big((n - m + 1)x\big) \tag{2.8}$$

for all $x \in X$. This implies that H is a unique mapping satisfying (2.8) such that there exists $\mu \in (0, \infty)$ satisfying $N(f(x) - H(x), \mu t) \geq \frac{t}{t+\varphi(x,\dots,x)}$ for all $x \in X$ and $t > 0$.

The rest of the proof is similar to the proof of Theorem 2.1. This completes the proof. □

Now, we investigate the Hyers-Ulam-Rassias stability of ternary derivations in ternary fuzzy Banach algebras.

Theorem 2.3 *Let X be a fuzzy Banach algebra. Let $\varphi : X^n \to [0, \infty)$ be a function such that there exists $L < \frac{1}{(n-m+1)^{n-2}}$ with*

$$\varphi\left(\frac{x_1}{n - m + 1}, \dots, \frac{x_n}{n - m + 1}\right) \leq \frac{L\varphi(x_1, x_2, \dots, x_n)}{n - m + 1}$$

for all $x_1, \dots, x_n \in X$. Let $f : X \to X$ be a mapping with $f(0) = 0$ satisfying (2.1) *and*

$$N\big(f([abc]) - [f(a)bc] - [af(b)c] - [abf(c)], t\big) \geq \frac{t}{t + \varphi(a, b, c, 0, 0, \dots, 0)} \tag{2.9}$$

for all $a, b, c \in X$ *and* $t > 0$. *Then* $D(x) := N\text{-}\lim_{p\to\infty} \frac{f(\frac{x}{(n-m+1)^p})}{(n-m+1)^{-p}}$ *exists for all* $x \in X$ *and* $D : X \to X$ *is defined as a unique ternary derivation such that*

$$N\big(f(x) - D(x), t\big) \geq \frac{(n-m+1)\binom{n}{m}(1-L)t}{(n-m+1)\binom{n}{m}(1-L)t + L\varphi(x, \dots, x)} \tag{2.10}$$

for all $x \in X$ *and* $t > 0$.

Proof By the same reasoning as that in the proof of Theorem 2.1, the mapping $D : X \to X$ is a unique $\mathbb{C}$-linear mapping which satisfies (2.10).

Now, we show that D is a ternary derivation. By (2.9), we have

$$\begin{aligned}
&N\left(\frac{f(\frac{[abc]}{(n-m+1)^{(n-1)p}})}{(n-m+1)^{-(n-1)p}} - \frac{[f(\frac{a}{(n-m+1)^p})bc] - [af(\frac{b}{(n-m+1)^p})c] - [abf(\frac{c}{(n-m+1)^p})]}{(n-m+1)^{-(n-1)p}},\right.\\
&\left.\frac{t}{(n-m+1)^{-(n-1)p}}\right)\\
&\geq \frac{t}{t + \varphi(\frac{a}{(n-m+1)^p}, \frac{b}{(n-m+1)^p}, \frac{c}{(n-m+1)^p}, 0, 0, \dots, 0)}
\end{aligned} \tag{2.11}$$

for all $a, b, c \in X$ and $t > 0$. Then we have

$$\begin{aligned}
&N\left(\frac{f(\frac{[abc]}{(n-m+1)^{(n-1)p}})}{(n-m+1)^{-(n-1)p}} - \frac{[f(\frac{a}{(n-m+1)^p})bc] - [af(\frac{b}{(n-m+1)^p})c] - [abf(\frac{c}{(n-m+1)^p})]}{(n-m+1)^{-(n-1)p}}, t\right)\\
&\geq \frac{\frac{t}{(n-m+1)^{(n-1)p}}}{\frac{t}{(n-m+1)^{(n-1)p}} + \varphi(\frac{a}{(n-m+1)^p}, \frac{b}{(n-m+1)^p}, \frac{c}{(n-m+1)^p}, 0, 0, \dots, 0)}\\
&\geq \frac{\frac{t}{(n-m+1)^{(n-1)p}}}{\frac{t}{(n-m+1)^{(n-1)p}} + \frac{L^p\varphi(a,b,c,0,0,\dots,0)}{(n-m+1)^p}} \to 1 \quad \text{when } p \to +\infty
\end{aligned}$$

for all $a, b, c \in X$ and $t > 0$. So, we have

$$N\big(D([abc]) - [D(a)bc] - [aD(b)c] - [abH(c)], t\big) = 1$$

for all $a, b, c \in X$ and $t > 0$. Hence we have $D([abc]) = [D(a)bc] + [aD(b)c] + [abD(c)]$ for all $a, b, c \in X$. This means that D is a ternary derivation. This completes the proof. □

Theorem 2.4 *Let* X *be a fuzzy Banach algebra. Let* $\varphi : X^n \to [0, \infty)$ *be a function such that there exists* $L < 1$ *with*

$$\varphi(x_1, \dots, x_n) \leq (n-m+1)L\varphi\left(\frac{x_1}{n-m+1}, \dots, \frac{x_n}{n-m+1}\right)$$

for all $x_1, x_2, \dots, x_n \in X$. *Let* $f : X \to X$ *be a mapping with* $f(0) = 0$ *satisfying* (2.1) *and* (2.9). *Then the limit* $D(x) := N\text{-}\lim_{p\to\infty} \frac{f((n-m+1)^p x)}{(n-m+1)^p}$ *exists for all* $x \in X$ *and* $D : X \to X$ *is defined as a ternary derivation such that*

$$N\big(f(x) - D(x), t\big) \geq \frac{(n-m+1)\binom{n}{m}(1-L)t}{(n-m+1)\binom{n}{m}(1-L)t + \varphi(x, \dots, x)} \tag{2.12}$$

for all $x \in X$ *and* $t > 0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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References

1. Ulam, SM: A Collection of the Mathematical Problems. Interscience, New York (1960)
2. Hyers, DH: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. USA **27**, 222-224 (1941)
3. Aoki, T: On the stability of the linear transformation in Banach spaces. J. Math. Soc. Jpn. **2**, 64-66 (1950)
4. Rassias, TM: On the stability of the linear mapping in Banach spaces. Proc. Am. Math. Soc. **72**, 297-300 (1978)
5. Găvruta, P: A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. J. Math. Anal. Appl. **184**, 431-436 (1994)
6. Bourgin, DG: Classes of transformations and bordering transformations. Bull. Am. Math. Soc. **57**, 223-237 (1951)
7. Cădariu, L, Radu, V: Fixed points and the stability of Jensen's functional equation. J. Inequal. Pure Appl. Math. **4**(1), Article ID 4 (2003)
8. Cădariu, L, Radu, V: Fixed points and the stability of quadratic functional equations. An. Univ. Timişoara, Ser. Mat. Inform **41**, 25-48 (2003)
9. Czerwik, S: The stability of the quadratic functional equation. In: Rassias, TM Tabor, J (eds.) Stability of Mappings of Hyers-Ulam Type, pp. 81-91. Hadronic Press, Florida (1994)
10. Eshaghi Gordji, M, Rassias, JM, Ghobadipour, N: Generalized Hyers-Ulam stability of generalized (N,K)-derivations. Abstr. Appl. Anal. **2009**, Article ID 437931 (2009)
11. Eshaghi Gordji, M, Ghaemi, MB, Kaboli Gharetapeh, S, Shams, S, Ebadian, A: On the stability of J^*-derivations. J. Geom. Phys. **60**, 454-459 (2010)
12. Eshaghi Gordji, M, Najati, A: Approximately J^*-homomorphisms: a fixed point approach. J. Geom. Phys. **60**, 809-814 (2010)
13. Rassias, JM, Jun, KW, Kim, HM: Approximate (m,n)-Cauchy-Jensen additive mappings in C^*-algebras. Acta Math. Sin. **27**(10), 1907-1922 (2011)
14. Găvruta, P, Găvruta, L: A new method for the generalized Hyers-Ulam-Rassias stability. Int. J. Nonlinear Anal. Appl. **1**, 11-18 (2010)
15. Khodaei, H, Rassias, TM: Approximately generalized additive functions in several variables. Int. J. Nonlinear Anal. Appl. **1**, 22-41 (2010)
16. Cholewa, PW: Remarks on the stability of functional equations. Aequ. Math. **27**, 76-86 (1984)
17. Czerwik, S: On the stability of the quadratic mapping in normed spaces. Abh. Math. Semin. Univ. Hamb. **62**, 239-248 (1992)
18. Saadati, R, Vaezpour, M, Cho, YJ: A note to paper 'On the stability of cubic mappings and quartic mappings in random normed spaces'. J. Inequal. Appl. **2009**, Article ID 214530 (2009)
19. Saadati, R, Zohdi, MM, Vaezpour, SM: Nonlinear L-random stability of an ACQ functional equation. J. Inequal. Appl. **2011**, Article ID 194394 (2011)
20. Azadi Kenary, H, Rezaei, H, Ghaffaripour, A, Talebzadeh, S, Park, C, Lee, JR: Fuzzy Hyers-Ulam stability of an additive functional equation. J. Inequal. Appl. **2011**, 140 (2011)
21. Park, C, Lee, JR, Rassias, TM, Saadati, R: Fuzzy *-homomorphisms and fuzzy *-derivations in induced fuzzy C^*-algebras. Math. Comput. Model. **54**, 2027-2039 (2011)
22. Park, C, Eshaghi Gordji, M, Cho, YJ: Stability and superstability of generalized quadratic ternary derivations on non-Archimedean ternary Banach algebras: a fixed point approach. Fixed Point Theory Appl. **2012**, 97 (2012)
23. Cho, YJ, Kang, SM, Sadaati, R: Nonlinear random stability via fixed-point method. J. Appl. Math. **2012**, Article ID 902931 (2012)
24. Cho, YJ, Park, C, Saadati, R: Functional inequalities in non-Archimedean Banach spaces. Appl. Math. Lett. **23**, 1238-1242 (2010)
25. Cho, YJ, Saadati, R, Vahidi, J: Approximation of homomorphisms and derivations on non-Archimedean Lie C^*-algebras via fixed point method. Discrete Dyn. Nat. Soc. **2012**, Article ID 373904 (2012)
26. Saadati, R, Cho, YJ, Vahidi, J: The stability of the quartic functional equation in various spaces. Comput. Math. Appl. **60**, 1994-2002 (2010)
27. Saadati, R, Park, C: Non-Archimedean L-fuzzy normed spaces and stability of functional equations. Comput. Math. Appl. **60**, 2488-2496 (2010)
28. Saadati, R, Vaezpour, SM, Park, C: The stability of the cubic functional equation in various spaces. Math. Commun. **16**, 131-145 (2011)
29. Rassias, TM: On the stability of functional equations in Banach spaces. J. Math. Anal. Appl. **251**, 264-284 (2000)
30. Katsaras, AK: Fuzzy topological vector spaces. Fuzzy Sets Syst. **12**, 143-154 (1984)
31. Felbin, C: Finite-dimensional fuzzy normed linear space. Fuzzy Sets Syst. **48**, 239-248 (1992)

32. Park, C: Fuzzy stability of a functional equation associated with inner product spaces. Fuzzy Sets Syst. **160**, 1632-1642 (2009)
33. Bag, T, Samanta, SK: Finite dimensional fuzzy normed linear spaces. J. Fuzzy Math. **11**, 687-705 (2003)
34. Cheng, SC, Mordeson, JN: Fuzzy linear operators and fuzzy normed linear spaces. Bull. Calcutta Math. Soc. **86**, 429-436 (1994)
35. Karmosil, I, Michalek, J: Fuzzy metric and statistical metric spaces. Kybernetica **11**, 326-334 (1975)
36. Bag, T, Samanta, SK: Fuzzy bounded linear operators. Fuzzy Sets Syst. **151**, 513-547 (2005)
37. Rassias, JM, Kim, H: Generalized Hyers-Ulam stability for general additive functional equations in quasi β-normed spaces. J. Math. Anal. Appl. **356**, 302-309 (2009)
38. Azadi Kenary, H: Non-Archimedean stability of Cauchy-Jensen type functional equation. J. Nonlinear Anal. Appl. **1**, 1-10 (2011)
39. Cayley, A: On the 34 concomitants of the ternary cubic. Am. J. Math. **4**, 1-15 (1981)
40. Kapranov, M, Gelfand, IM, Zelevinskii, A: Discrimininants, Resultants and Multidimensional Determinants. Birkhauser, Berlin (1994)
41. Abramov, V, Kerner, R, Liivapuu, O, Shitov, S: Algebras with ternary law of composition and their realization by cubic matrices. J. Gen. Lie Theory Appl. **3**, 77-94 (2009)
42. Kerner, R: Ternary and non-associative structures. Int. J. Geom. Methods Mod. Phys. **5**, 1265-1294 (2008)
43. Kerner, R: The cubic chessboard. Geometry and physics. Class. Quantum Gravity **14**, 203-225 (1997)
44. Sewell, GL: Quantum Mechanics and Its Emergent Macrophysics. Princeton University Press, Princeton (2002)
45. Zettl, H: A characterization of ternary rings of operators. Adv. Math. **48**, 117-143 (1983)
46. Diaz, J, Margolis, B: A fixed point theorem of the alternative for contractions on a generalized complete metric space. Bull. Am. Math. Soc. **74**, 305-309 (1968)
47. Mihet, D, Radu, V: On the stability of the additive Cauchy functional equation in random normed spaces. J. Math. Anal. Appl. **343**, 567-572 (2008)
48. Park, C, Lee, JR, Rassias, TM, Saadati, R: Fuzzy *-homomorphisms and fuzzy *-derivations in induced fuzzy C^*-algebras. Math. Comput. Model. **54**, 2027-2039 (2011)
49. Radu, V: The fixed point alternative and the stability of functional equations. Fixed Point Theory **4**, 91-96 (2003)
50. Eshaghi Gordji, M, Moradlou, F: Approximate Jordan derivations on Hilbert C^*-moduls. Fixed Point Theory (to appear)
51. Eshaghi Gordji, M: Nearly involutions on Banach algebras; a fixed point approach. Fixed Point Theory **12**, 341-348 (2011)

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