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Neighborhoods and partial sums of certain subclass of starlike functions

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Abstract
The main purpose of the present paper is to derive the neighborhoods and partial sums of a certain subclass of starlike functions.
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1 Introduction

Let $\mathcal{A}_m$ denote the class of functions f of the form

$$f(z) = z + \sum_{k=m+1}^{\infty} a_k z^k \quad \bigl(m \in \mathbb{N} := \{1,2,3,\ldots\}\bigr), \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \bigl\{z : z \in \mathbb{C} \text{ and } |z| < 1\bigr\}.$$

A function $f \in \mathcal{A}_m$ is said to be in the class $\mathcal{S}_m^*(\beta)$ of *starlike functions of order* β if it satisfies the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \beta \quad (z \in \mathbb{U}; 0 \leqq \beta < 1). \tag{1.2}$$

Assuming that $\alpha \geqq 0$, $0 \leqq \beta < 1$ and $f \in \mathcal{A}_m$, we say that a function $f \in \mathcal{H}_m(\alpha,\beta)$ if it satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)}\right) > \alpha\beta\left(\beta + \frac{m}{2} - 1\right) + \beta - \frac{m\alpha}{2} \quad (z \in \mathbb{U}). \tag{1.3}$$

For convenience, throughout this paper, we write

$$\gamma_m := \alpha\beta\left(\beta + \frac{m}{2} - 1\right) + \beta - \frac{m\alpha}{2}. \tag{1.4}$$

Recently, Ravichandran *et al.* [1] proved that $\mathcal{H}_m(\alpha,\beta) \subset \mathcal{S}_m^*(\beta)$. Subsequently, Liu *et al.* [2] derived various properties and characteristics such as inclusion relationships, Hadamard products, coefficient estimates, distortion theorems and cover theorems for the class $\mathcal{H}_m(\alpha,\beta)$ and a subclass of $\mathcal{H}_m(\alpha,\beta)$ with negative coefficients. Furthermore, Singh *et al.* [3] generalized the class $\mathcal{H}_m(\alpha,\beta)$ and found several sufficient conditions for starlikeness. In the present paper, we aim at proving the neighborhoods and partial sums of the class $\mathcal{H}_m(\alpha,\beta)$.

2 Main results

Following the earlier works (based upon the familiar concept of a neighborhood of analytic functions) by Goodman [4] and Ruscheweyh [5], and (more recently) by Altintaş *et al.* [6–9], Cătaş [10], Frasin [11], Keerthi *et al.* [12] and Srivastava *et al.* [13], we begin by introducing here the δ-neighborhood of a function $f \in \mathcal{A}_m$ of the form (1.1) by means of the definition

$$\begin{aligned}\mathcal{N}_\delta(f) := \Biggl\{ & g \in \mathcal{A}_m : g(z) = z + \sum_{k=m+1}^{\infty} b_k z^k \text{ and} \\ & \sum_{k=m+1}^{\infty} \frac{k(1+k\alpha-\alpha)-\gamma_m}{1-\gamma_m} |a_k - b_k| \leqq \delta \ (\delta, \alpha \geqq 0; 0 \leqq \beta < 1; \gamma_m < 1) \Biggr\}. \end{aligned} \tag{2.1}$$

By making use of the definition (2.1), we now derive the following result.

Theorem 1 *If $f \in \mathcal{A}_m$ satisfies the condition*

$$\frac{f(z)+\varepsilon z}{1+\varepsilon} \in \mathcal{H}_m(\alpha,\beta) \quad \bigl(\varepsilon \in \mathbb{C}; |\varepsilon| < \delta; \delta > 0\bigr), \tag{2.2}$$

then

$$\mathcal{N}_\delta(f) \subset \mathcal{H}_m(\alpha,\beta). \tag{2.3}$$

Proof By noting that the condition (1.3) can be rewritten as follows:

$$\left| \frac{\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} - (2\gamma_m - 1)} \right| < 1 \quad (z \in \mathbb{U}), \tag{2.4}$$

we easily find from (2.4) that a function $g \in \mathcal{H}_m(\alpha,\beta)$ if and only if

$$\frac{zg'(z) + \alpha z^2 g''(z) - g(z)}{zg'(z) + \alpha z^2 g''(z) - (2\gamma_m - 1)g(z)} \neq \sigma \quad \bigl(z \in \mathbb{U}; \sigma \in \mathbb{C}; |\sigma| = 1\bigr),$$

which is equivalent to

$$\frac{(g*h)(z)}{z} \neq 0 \quad (z \in \mathbb{U}), \tag{2.5}$$

where

$$h(z) = z + \sum_{k=m+1}^{\infty} c_k z^k \quad \left(c_k := \frac{k + \alpha k(k-1) - 1 - [k + \alpha k(k-1) - (2\gamma_m - 1)]\sigma}{2(\gamma_m - 1)\sigma} \right). \tag{2.6}$$

It follows from (2.6) that

$$\begin{aligned} |c_k| &= \left| \frac{k + \alpha k(k-1) - 1 - [k + \alpha k(k-1) - (2\gamma_m - 1)]\sigma}{2(\gamma_m - 1)\sigma} \right| \\ &\leqq \frac{k + \alpha k(k-1) - 1 + [k + \alpha k(k-1) - (2\gamma_m - 1)]|\sigma|}{2(1-\gamma_m)|\sigma|} \\ &= \frac{k(1 + k\alpha - \alpha) - \gamma_m}{1 - \gamma_m} \quad \bigl(|\sigma| = 1\bigr). \end{aligned}$$

If $f \in \mathcal{A}_m$ satisfies the condition (2.2), we deduce from (2.5) that

$$\frac{(f * h)(z)}{z} \neq -\varepsilon \quad \bigl(|\varepsilon| < \delta; \delta > 0\bigr),$$

or, equivalently,

$$\left| \frac{(f * h)(z)}{z} \right| \geqq \delta \quad (z \in \mathbb{U}; \delta > 0). \tag{2.7}$$

We now suppose that

$$q(z) = z + \sum_{k=m+1}^{\infty} d_k z^k \in \mathcal{N}_\delta(f).$$

It follows from (2.1) that

$$\begin{aligned} \left| \frac{((q-f) * h)(z)}{z} \right| &= \left| \sum_{k=m+1}^{\infty} (d_k - a_k) c_k z^{k-1} \right| \\ &\leqq |z| \sum_{k=m+1}^{\infty} \frac{k(1 + k\alpha - \alpha) - \gamma_m}{1 - \gamma_m} |d_k - a_k| < \delta. \end{aligned} \tag{2.8}$$

Combining (2.7) and (2.8), we easily find that

$$\left| \frac{(q * h)(z)}{z} \right| = \left| \frac{([f + (q-f)] * h)(z)}{z} \right| \geqq \left| \frac{(f * h)(z)}{z} \right| - \left| \frac{((q-f) * h)(z)}{z} \right| > 0,$$

which implies that

$$\frac{(q * h)(z)}{z} \neq 0 \quad (z \in \mathbb{U}).$$

Therefore, we conclude that

$$q(z) \in \mathcal{N}_\delta(f) \subset \mathcal{H}_m(\alpha, \beta).$$

We thus complete the proof of Theorem 1. □

Next, we derive the partial sums of the class $\mathcal{H}_m(\alpha, \beta)$. For some recent investigations involving the partial sums in analytic function theory, one can refer to [14–16] and the references cited therein.

Theorem 2 *Let $f \in \mathcal{A}_m$ be given by* (1.1) *and define the partial sums $f_n(z)$ of f by*

$$f_n(z) = z + \sum_{k=m+1}^{n} a_k z^k \quad (n \in \mathbb{N}; n \geqq m+1). \tag{2.9}$$

If

$$\sum_{k=m+1}^{\infty} \frac{k(1 + k\alpha - \alpha) - \gamma_m}{1 - \gamma_m} |a_k| \leqq 1 \quad (\alpha \geqq 0; 0 \leqq \beta < 1; \gamma_m < 1), \tag{2.10}$$

then

(1) $f \in \mathcal{H}_m(\alpha, \beta)$;

(2)

$$\Re\left(\frac{f(z)}{f_n(z)}\right) \geqq \frac{n(1 + \alpha + n\alpha)}{(n+1)(1 + n\alpha) - \gamma_m} \quad (n \in \mathbb{N}; n \geqq m+1; z \in \mathbb{U}) \tag{2.11}$$

and

$$\Re\left(\frac{f_n(z)}{f(z)}\right) \geqq \frac{(n+1)(1 + n\alpha) - \gamma_m}{(n+1)(1 + n\alpha) + 1 - 2\gamma_m} \quad (n \in \mathbb{N}; n \geqq m+1; z \in \mathbb{U}). \tag{2.12}$$

The bounds in (2.11) *and* (2.12) *are sharp.*

Proof (1) Suppose that $f_1(z) = z$. We know that $z \in \mathcal{H}_m(\alpha, \beta)$, which implies that

$$\frac{f_1(z) + \varepsilon z}{1 + \varepsilon} = z \in \mathcal{H}_m(\alpha, \beta).$$

From (2.10), we easily find that

$$\sum_{k=m+1}^{\infty} \frac{k(1 + k\alpha - \alpha) - \gamma_m}{1 - \gamma_m} |a_k - 0| \leqq 1,$$

which implies that $f \in \mathcal{N}_1(z)$. In view of Theorem 1, we deduce that

$$f \in \mathcal{N}_1(z) \subset \mathcal{H}_m(\alpha, \beta).$$

(2) It is easy to verify that

$$\begin{aligned}\frac{(n+1)[1 + (n+1)\alpha - \alpha] - \gamma_m}{1 - \gamma_m} &= \frac{(n+1)(1 + n\alpha) - \gamma_m}{1 - \gamma_m} \\ &> \frac{n(1 + n\alpha - \alpha) - \gamma_m}{1 - \gamma_m} > 1 \quad (n \in \mathbb{N}).\end{aligned}$$

Therefore, we have

$$\sum_{k=m+1}^{n} |a_k| + \frac{(n+1)(1 + n\alpha) - \gamma_m}{1 - \gamma_m} \sum_{k=n+1}^{\infty} |a_k| \leqq \sum_{k=m+1}^{\infty} \frac{k(1 + k\alpha - \alpha) - \gamma_m}{1 - \gamma_m} |a_k| \leqq 1. \tag{2.13}$$

We now suppose that

$$\begin{aligned}\psi(z) &= \frac{(n+1)(1+n\alpha)-\gamma_m}{1-\gamma_m}\left(\frac{f(z)}{f_n(z)} - \frac{n(1+\alpha+n\alpha)}{(n+1)(1+n\alpha)-\gamma_m}\right) \\ &= 1 + \frac{\frac{(n+1)(1+n\alpha)-\gamma_m}{1-\gamma_m}\sum_{k=n+1}^{\infty} a_k z^{k-1}}{1+\sum_{k=m+1}^{n} a_k z^{k-1}}. \end{aligned} \tag{2.14}$$

It follows from (2.13) and (2.14) that

$$\left|\frac{\psi(z)-1}{\psi(z)+1}\right| \leqq \frac{\frac{(n+1)(1+n\alpha)-\gamma_m}{1-\gamma_m}\sum_{k=n+1}^{\infty}|a_k|}{2-2\sum_{k=m+1}^{n}|a_k| - \frac{(n+1)(1+n\alpha)-\gamma_m}{1-\gamma_m}\sum_{k=n+1}^{\infty}|a_k|} \leqq 1 \quad (z \in \mathbb{U}),$$

which shows that

$$\Re\big(\psi(z)\big) \geqq 0 \quad (z \in \mathbb{U}). \tag{2.15}$$

Combining (2.14) and (2.15), we deduce that the assertion (2.11) holds true.

Moreover, if we put

$$f(z) = z + \frac{1-\gamma_m}{(n+1)(1+n\alpha)-\gamma_m} z^{n+1} \quad \big(n \in \mathbb{N}\setminus\{1,2,\ldots,m-1\}; m \in \mathbb{N}\big), \tag{2.16}$$

then for $z = re^{i\pi/n}$, we have

$$\frac{f(z)}{f_n(z)} = 1 + \frac{1-\gamma_m}{(n+1)(1+n\alpha)-\gamma_m} z^n \to \frac{n(1+\alpha+n\alpha)}{(n+1)(1+n\alpha)-\gamma_m} \quad \big(r \to 1^-\big),$$

which implies that the bound in (2.11) is the best possible for each $n \in \mathbb{N}\setminus\{1,2,\ldots,m-1\}$.

Similarly, we suppose that

$$\begin{aligned}\varphi(z) &= \frac{(n+1)(1+n\alpha)+1-2\gamma_m}{1-\gamma_m}\left(\frac{f_n(z)}{f(z)} - \frac{(n+1)(1+n\alpha)-\gamma_m}{(n+1)(1+n\alpha)+1-2\gamma_m}\right) \\ &= 1 - \frac{\frac{(n+1)(1+n\alpha)+1-2\gamma_m}{1-\gamma_m}\sum_{k=n+1}^{\infty} a_k z^{k-1}}{1+\sum_{k=m+1}^{\infty} a_k z^{k-1}}. \end{aligned} \tag{2.17}$$

In view of (2.13) and (2.17), we conclude that

$$\left|\frac{\varphi(z)-1}{\varphi(z)+1}\right| \leqq \frac{\frac{(n+1)(1+n\alpha)+1-2\gamma_m}{1-\gamma_m}\sum_{k=n+1}^{\infty}|a_k|}{2-2\sum_{k=m+1}^{n}|a_k| - \frac{n(1+\alpha+n\alpha)}{1-\gamma_m}\sum_{k=n+1}^{\infty}|a_k|} \leqq 1 \quad (z \in \mathbb{U}),$$

which implies that

$$\Re\big(\varphi(z)\big) \geqq 0 \quad (z \in \mathbb{U}). \tag{2.18}$$

Combining (2.17) and (2.18), we readily get the assertion (2.12) of Theorem 2. The bound in (2.12) is sharp with the extremal function f given by (2.16).

The proof of Theorem 2 is thus completed. □

Finally, we turn to ratios involving derivatives. The proof of Theorem 3 below is much akin to that of Theorem 2, we here choose to omit the analogous details.

Theorem 3 *Let $f \in \mathcal{A}_m$ be given by* (1.1) *and define the partial sums $f_n(z)$ of f by* (2.9). *If the condition* (2.10) *holds, then*

$$\Re\left(\frac{f'(z)}{f_n'(z)}\right) \geqq \frac{(n+1)(n\alpha+\gamma_m)-\gamma_m}{(n+1)(1+n\alpha)-\gamma_m} \quad (n \in \mathbb{N}; n \geqq m+1; z \in \mathbb{U}) \tag{2.19}$$

and

$$\Re\left(\frac{f_n'(z)}{f'(z)}\right) \geqq \frac{(n+1)(1+n\alpha)-\gamma_m}{(n+1)(2+n\alpha-\gamma_m)-\gamma_m} \quad (n \in \mathbb{N}; n \geqq m+1; z \in \mathbb{U}). \tag{2.20}$$

The bounds in (2.19) *and* (2.20) *are sharp with the extremal function given by* (2.16).

Remark By setting $\alpha = 0$ and $m = 1$ in Theorems 2 and 3, we get the corresponding results obtained by Silverman [16].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors completed the paper together. They also read and approved the final manuscript.

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