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On some Diophantine equations

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Abstract

We consider the sequences (u_n) and (v_n) which are the generalizations of Fibonacci and Lucas sequences, respectively. Then we determine some identities involving these generalized sequences to present all solutions of the equations

$$x^2 - v_n xy + y^2 = -(p^2 - 4)u_n^2,$$
$$x^2 - v_n xy + y^2 = u_n^2,$$

and

$$x^2 - (p^2 - 4)u_n xy - (p^2 - 4)y^2 = v_n^2,$$

for $p \geq 3$ and a square-free integer $p^2 - 4$. In addition to these, all solutions of some different Diophantine equations such as $x^2 - v_{2n}xy + y^2 = -(p^2 - 4)u_n^2$, $x^2 - v_n xy + y^2 = -(p^2 - 4)$, $x^2 - v_n xy + y^2 = 1$, $x^2 - v_{2n}xy + y^2 = u_n^2$, $x^2 - v_{2n}xy + y^2 = v_n^2$, $x^2 - (p^2 - 4)u_n xy - (p^2 - 4)y^2 = 1$ are identified, by using divisibility rules of the sequences (u_n) and (v_n).

MSC: 11B37; 11B39; 11C20; 11D09; 11D45

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1 Introduction

In this paper, we consider the generalized Fibonacci sequence (u_n) and the generalized Lucas sequence (v_n). Let $p \geq 3$ be an integer. For any $n \geq 2$, (u_n) is defined by the recurrence relation $u_n = pu_{n-1} - u_{n-2}$ with the initial conditions $u_0 = 0$, $u_1 = 1$. The generalized Lucas sequence (v_n) is defined by the recurrence relation $v_n = pv_{n-1} - v_{n-2}$ for any $n \geq 2$ with the initial conditions $v_0 = 2$ and $v_1 = p$. The terms u_n and v_n are called the nth generalized Fibonacci and Lucas numbers, respectively.

Moreover, generalized Fibonacci and Lucas numbers can be extended to negative indices. In general, for all $n \in \mathbb{N}$, $u_{-n} = -u_n$ and $v_{-n} = v_n$. Furthermore it is known that $v_n = u_{n+1} - u_{n-1}$. For more detailed information about these sequences, one can consult [1–5] and [6].

In [3], McDaniel showed that the solutions of the equation $x^2 - (p^2 - 4)y^2 = 4$ are given by $(x, y) = (v_n, u_n)$ with $n \geq 1$. Moreover, in [7–9] and [10], Jones investigated whether the equations $x^2 - (p^2 \mp 4)y^2 = \mp 4$, $x^2 - (p^2 \mp 1)y^2 = \mp 4$, $x^2 - (p^2 \mp 1)y^2 = \mp 1$ and $x^2 - (p^2 \mp 4)y^2 = \mp 1$ have solutions or not. In his proofs, he used Fermat's method of infinite descent.

In [11], Demirtürk and Keskin determined all solutions of the known Diophantine equations $x^2 - L_n xy - y^2 = \mp 1$, $x^2 - L_n xy + (-1)^n y^2 = \mp 5$ and new Diophantine equations

$$x^2 - 5F_n xy - 5(-1)^n y^2 = \mp 1,$$
$$x^2 - L_{2n} xy + y^2 = \mp 5F_n^2,$$
$$x^2 - L_{2n} xy + y^2 = \mp F_n^2,$$
$$x^2 - L_{2n} xy + y^2 = \mp L_n^2$$

and

$$x^2 - L_{2n} xy + y^2 = \mp 5L_n^2.$$

In this paper, our main purpose is to determine all (x, y) solutions of the Diophantine equations

$$x^2 + axy + by^2 = c,$$

where a, b, c are generalized Fibonacci and generalized Lucas numbers. These equations can be listed as follows:

$$x^2 - v_n xy + y^2 = -(p^2 - 4)u_n^2,$$
$$x^2 - v_n xy + y^2 = u_n^2,$$
$$x^2 - (p^2 - 4)u_n xy - (p^2 - 4)y^2 = v_n^2,$$
$$x^2 - v_{2n} xy + y^2 = -(p^2 - 4)u_n^2,$$
$$x^2 - v_n xy + y^2 = -(p^2 - 4),$$
$$x^2 - v_n xy - y^2 = 1,$$
$$x^2 - v_{2n} xy + y^2 = u_n^2,$$
$$x^2 - v_{2n} xy + y^2 = v_n^2$$

and

$$x^2 - (p^2 - 4)u_n xy - (p^2 - 4)y^2 = 1.$$

2 Divisibility rules of sequences (u_n) and (v_n)

In this section, we recall some divisibility rules related to generalized Fibonacci and Lucas sequences (u_n) and (v_n). Since these rules are proved in [12–16], we omit their proofs. Using these divisibility rules, in the last section, we will find all solutions of Diophantine equations mentioned above.

Theorem 1 *Let $m, n \in \mathbb{N}$. Then $v_n | u_m$ if and only if $n | m$ and m/n is an even integer.*

Theorem 2 *Let $m, n \in \mathbb{N}$. Then $u_n | u_m$ if and only if $n | m$.*

Theorem 3 *Let $m, n \in \mathbb{N}$. Then $v_n | v_m$ if and only if $n | m$ and m/n is an odd integer.*

Theorem 4 *Let $m, n \in \mathbb{N}$ and $n > 1$. Then $u_n | v_m$ if and only if $n = 2$ and m is an odd integer.*

3 Some identities of the sequences (u_n) and (v_n)

In this section, we obtain some identities by using special matrices including generalized Fibonacci and Lucas numbers.

Now we compile some identities to use in the proofs of the following theorems. These identities can be found in [5, 17, 18] and [19].

$$u_n^2 - p u_n u_{n-1} + u_{n-1}^2 = 1, \tag{3.1}$$

$$v_n^2 - p v_n v_{n-1} + v_{n-1}^2 = -(p^2 - 4), \tag{3.2}$$

$$v_m u_n - u_m v_n = 2u_{n-m}, \tag{3.3}$$

$$v_m v_n - (p^2 - 4) u_m u_n = 2v_{n-m}, \tag{3.4}$$

$$u_n u_{m+1} - u_m u_{n-1} = u_{n+m}, \tag{3.5}$$

$$v_m v_n + (p^2 - 4) u_m u_n = 2v_{n+m}, \tag{3.6}$$

$$u_m v_n + v_m u_n = 2u_{n+m}, \tag{3.7}$$

$$u_{n+1} - u_{n-1} = v_n, \tag{3.8}$$

$$v_{n+1} - v_{n-1} = (p^2 - 4) u_n, \tag{3.9}$$

$$v_n^2 - (p^2 - 4) u_n^2 = 4, \tag{3.10}$$

$$u_{m+1} v_n - u_m v_{n-1} = v_{n+m} \tag{3.11}$$

for all $m, n \in \mathbb{Z}$.

Theorem 5

$$v_{n+m}^2 - (p^2 - 4) u_{n-t} v_{n+m} u_{m+t} - (p^2 - 4) u_{m+t}^2 = v_{n-t}^2,$$

for all $m, n, t \in \mathbb{Z}$.

Proof If we consider identities (3.6) and (3.7), then the matrix multiplication

$$\begin{bmatrix} v_n/2 & (p^2-4)u_n/2 \\ u_t/2 & v_t/2 \end{bmatrix} \begin{bmatrix} v_m \\ u_m \end{bmatrix} = \begin{bmatrix} v_{n+m} \\ u_{m+t} \end{bmatrix}$$

can be written. By identity (3.4), we get

$$\begin{bmatrix} v_m \\ u_m \end{bmatrix} = \begin{bmatrix} v_n/2 & (p^2-4)u_n/2 \\ u_t/2 & v_t/2 \end{bmatrix}^{-1} \begin{bmatrix} v_{n+m} \\ u_{m+t} \end{bmatrix} = \frac{2}{v_{n-t}} \begin{bmatrix} v_t/2 & -(p^2-4)u_n/2 \\ -u_t/2 & v_n/2 \end{bmatrix} \begin{bmatrix} v_{n+m} \\ u_{m+t} \end{bmatrix},$$

since

$$\begin{vmatrix} v_n/2 & (p^2-4)u_n/2 \\ u_t/2 & v_t/2 \end{vmatrix} = \frac{v_n v_t - (p^2-4)u_n u_t}{4} = \frac{v_{n-t}}{2} \neq 0.$$

Thus, it follows that

$$v_m = \frac{v_t v_{n+m} - (p^2 - 4) u_n u_{m+t}}{v_{n-t}}$$

and

$$u_m = \frac{v_n u_{m+t} - u_t v_{n+m}}{v_{n-t}}.$$

Since $v_m^2 - (p^2 - 4)u_m^2 = 4$, we have

$$\left(v_t v_{n+m} - (p^2 - 4) u_n u_{m+t}\right)^2 - (p^2 - 4)(v_n u_{m+t} - u_t v_{n+m})^2 = 4 v_{n-t}^2.$$

Therefore, we obtain $(v_t^2 - (p^2 - 4)u_t^2)v_{n+m}^2 - 2(p^2 - 4)(v_t u_n - v_n u_t)v_{n+m}u_{m+t} - (p^2 - 4)(v_n^2 - (p^2 - 4)u_n^2)u_{m+t}^2 = 4v_{n-t}^2$. By using (3.3) and (3.10) in this equation, it is seen that

$$4v_{n+m}^2 - 4(p^2 - 4)u_{n-t}v_{n+m}u_{m+t} - 4(p^2 - 4)u_{m+t}^2 = 4v_{n-t}^2.$$

Thus, we get

$$v_{n+m}^2 - (p^2 - 4)u_{n-t}v_{n+m}u_{m+t} - (p^2 - 4)u_{m+t}^2 = v_{n-t}^2. \qquad (3.12)$$

$\square$

Theorem 6 *Let $m, n, t \in \mathbb{Z}$ and $t \neq n$. Then*

$$v_{n+m}^2 - v_{n-t}v_{n+m}v_{m+t} + v_{m+t}^2 = -(p^2 - 4)u_{n-t}^2.$$

Proof By using (3.6), we can consider the matrix multiplication

$$\begin{bmatrix} v_n/2 & (p^2 - 4)u_n/2 \\ v_t/2 & (p^2 - 4)u_t/2 \end{bmatrix} \begin{bmatrix} v_m \\ u_m \end{bmatrix} = \begin{bmatrix} v_{n+m} \\ v_{m+t} \end{bmatrix}.$$

Since $t \neq n$, we get

$$\begin{vmatrix} v_n/2 & (p^2 - 4)u_n/2 \\ v_t/2 & (p^2 - 4)u_t/2 \end{vmatrix} = \frac{(p^2 - 4)(v_n u_t - v_t u_n)}{4} = \frac{(p^2 - 4)u_{n-t}}{2} \neq 0,$$

by (3.3). Therefore, we have

$$\begin{bmatrix} v_m \\ u_m \end{bmatrix} = \begin{bmatrix} v_n/2 & (p^2 - 4)u_n/2 \\ v_t/2 & (p^2 - 4)u_t/2 \end{bmatrix}^{-1} \begin{bmatrix} v_{n+m} \\ v_{m+t} \end{bmatrix}$$

$$= \frac{2}{(p^2 - 4)u_{n-t}} \begin{bmatrix} (p^2 - 4)u_t/2 & -(p^2 - 4)u_n/2 \\ -v_t/2 & v_n/2 \end{bmatrix} \begin{bmatrix} v_{n+m} \\ v_{m+t} \end{bmatrix}.$$

Thus, it follows that

$$v_m = \frac{u_t v_{n+m} - u_n v_{m+t}}{u_{n-t}}$$

and

$$u_m = \frac{v_n v_{m+t} - v_t v_{n+m}}{(p^2 - 4)u_{n-t}}.$$

Since $v_m^2 - (p^2 - 4)u_m^2 = 4$, we get

$$(p^2 - 4)(u_t v_{n+m} - u_n v_{m+t})^2 - (v_n v_{m+t} - v_t v_{n+m})^2 = 4(p^2 - 4)u_{n-t}^2.$$

Hence, it is seen that $-(v_t^2 - (p^2 - 4)u_t^2)v_{n+m}^2 + 2(v_n v_t - (p^2 - 4)u_t u_n)v_{n+m}v_{m+t} - (v_n^2 - (p^2 - 4)u_n^2)v_{m+t}^2 = 4(p^2 - 4)u_{n-t}^2$. By using identities (3.4) and (3.10), we obtain

$$4v_{n+m}^2 - 4v_{n-t}v_{n+m}v_{m+t} + 4v_{m+t}^2 = -4(p^2 - 4)u_{n-t}^2,$$

that is,

$$v_{n+m}^2 - v_{n-t}v_{n+m}v_{m+t} + v_{m+t}^2 = -(p^2 - 4)u_{n-t}^2. \tag{3.13}$$

$\square$

Using (3.7) and the matrix multiplication

$$\begin{bmatrix} u_n/2 & v_n/2 \\ u_t/2 & v_t/2 \end{bmatrix} \begin{bmatrix} v_m \\ u_m \end{bmatrix} = \begin{bmatrix} u_{n+m} \\ u_{m+t} \end{bmatrix},$$

we can give the following theorem.

Theorem 7 *Let $m, n, t \in \mathbb{Z}$ and $t \neq n$. Then*

$$u_{n+m}^2 - v_{n-t}u_{n+m}u_{m+t} + u_{m+t}^2 = u_{n-t}^2. \tag{3.14}$$

4 Solutions of some Diophantine equations

In [4], Melham proved that all solutions of the equations $y^2 - v_m xy + x^2 = \mp u_m^2$ are $(x, y) = \mp(u_n, u_{n+m})$ with $n \in \mathbb{Z}$. Moreover, he showed that if $m \in \mathbb{Z}$ and $p^2 - 4$ is a square-free integer, then all solutions of the equation $y^2 - v_m xy + x^2 = -(p^2 - 4)u_m^2$ are given by $(x, y) = \mp(v_n, v_{n+m})$ with $n \in \mathbb{Z}$. These theorems of Melham are generalized forms of the theorems given in [3], by McDaniel. In [2], Kılıç and Ömür examined more general situations of the conics that McDaniel and Melham dealt in [3] and [4], respectively.

In this section, using the identities given in (3.13) and (3.14), we will obtain all solutions of the equations

$$x^2 - v_n xy + y^2 = -(p^2 - 4)u_n^2$$

and

$$x^2 - v_n xy + y^2 = u_n^2$$

with $n \geq 1$, $p \geq 3$ and $p^2 - 4$ is a square-free integer. The solutions of these equations were explored by Kimberling, McDaniel and Melham, respectively in [3, 20] and [4], but we will

give different proofs of them here. Moreover, for $p \geq 3$, we will obtain all solutions of the Diophantine equation

$$x^2 - (p^2 - 4)u_n xy - (p^2 - 4)y^2 = v_n^2,$$

by using (3.12). Subsequently, if $p \geq 3$ and $p^2 - 4$ is a square-free integer, then we will find all solutions of Diophantine equations

$$x^2 - v_{2n}xy + y^2 = -(p^2 - 4)u_n^2$$

and

$$x^2 - v_n xy + y^2 = -(p^2 - 4).$$

Moreover, all solutions of the equations

$$x^2 - v_n xy + y^2 = 1, \qquad x^2 - v_{2n}xy + y^2 = u_n^2$$

and

$$x^2 - v_{2n}xy + y^2 = v_n^2$$

will be determined. Addition to this, if $p \geq 3$, then all solutions of the equation

$$x^2 - (p^2 - 4)u_n xy - (p^2 - 4)y^2 = 1$$

will be found.

Now we will remind some Diophantine equations with their solutions. The solutions of these equations are explored in [3] and [6]. We will use these equations for determining all solutions of other Diophantine equations.

Theorem 8 *Let $p > 3$. All solutions of the equation $x^2 - pxy + y^2 = 1$ are given by $(x, y) = \mp(u_m, u_{m-1})$ with $m \in \mathbb{Z}$.*

Since Corollary 1 can be seen from Theorem 8 and Corollary 2 is stated in [10], we will give them without proof.

Corollary 1 *All solutions of the equation $x^2 - 3xy + y^2 = 1$ are given by $(x, y) = \mp(F_{2m+2}, F_{2m})$ with $m \in \mathbb{Z}$.*

Corollary 2 *Let $p > 3$. All nonnegative solutions of the equation $u^2 - (p^2 - 4)v^2 = 4$ are given by $(u, v) = (v_m, u_m)$ with $m \geq 1$.*

Theorem 9 and Theorem 10 are stated in [21], so will give them without proof.

Theorem 9 *Let $p > 3$. Then the equation $x^2 - pxy + y^2 = -1$ has no solutions.*

Theorem 10 *All solutions of the equation $x^2 - 3xy + y^2 = -1$ are given by $(x, y) = \mp(F_{2m+1}, F_{2m-1})$ with $m \in \mathbb{Z}$.*

From now on we will assume that n is an integer such that $n \geq 1$.

Theorem 11 *If $p \geq 3$, then all solutions of the equation $x^2 - (p^2 - 4)u_n xy - (p^2 - 4)y^2 = v_n^2$ are given by $(x, y) = \mp(v_{n+m}, u_m)$ with $m \in \mathbb{Z}$.*

Proof Assume that $x^2 - (p^2 - 4)u_n xy - (p^2 - 4)y^2 = v_n^2$ for some integers x and y. Hence, we can write

$$\left(2x - (p^2 - 4)u_n y\right)^2 - \left((p^2 - 4)^2 u_n^2 + 4(p^2 - 4)\right)y^2 = 4v_n^2.$$

Thus, it follows that $(2x - (p^2 - 4)u_n y)^2 - (p^2 - 4)((p^2 - 4)u_n^2 + 4)y^2 = 4v_n^2$. By using (3.10) in this equation, we get $(2x - (p^2 - 4)u_n y)^2 - (p^2 - 4)v_n^2 y^2 = 4v_n^2$. Therefore it can be seen that $v_n | 2x - (p^2 - 4)u_n y$. Then taking

$$u = \frac{\left(\frac{(2x - (p^2 - 4)u_n y)}{v_n} + py\right)}{2} \quad \text{and} \quad v = y,$$

we obtain $u = (x + v_{n-1}y)/v_n$, by (3.9). From here we get

$$
\begin{aligned}
u^2 - puv + v^2 &= \left((x + v_{n-1}y)/v_n\right)^2 - p\left((x - v_{n-1}y)/v_n\right)y + y^2 \\
&= \left(x^2 - (v_{n+1} - v_{n-1})xy + y^2\left(v_n^2 - pv_n v_{n-1} + v_{n-1}^2\right)\right)/v_n^2.
\end{aligned}
$$

Hence, it follows that

$$u^2 - puv + v^2 = \left(x^2 - (p^2 - 4)u_n xy - (p^2 - 4)y^2\right)/v_n^2 = v_n^2/v_n^2 = 1,$$

by using (3.2) and (3.9). From Theorem 8, we obtain $(u, v) = \mp(u_{m+1}, u_m)$ for some $m \in \mathbb{Z}$. Thus, it is seen that

$$(x + v_{n-1}y)/v_n = \mp u_{m+1} \quad \text{and} \quad y = \mp u_m,$$

so we get $x = \mp(u_{m+1}v_n - v_{n-1}u_m)$ and $y = \mp u_m$. Now using (3.11), we obtain

$$(x, y) = \mp(v_{n+m}, u_m).$$

Conversely, if $(x, y) = \mp(v_{n+m}, u_m)$ with $m \in \mathbb{Z}$, then it can be seen that $x^2 - (p^2 - 4)u_n xy - (p^2 - 4)y^2 = v_n^2$, by (3.12). □

Using Theorem 9 in the same manner with Theorem 11, the following corollary can be given.

Corollary 3 *If $p > 3$, then the equation $x^2 - (p^2 - 4)u_n xy - (p^2 - 4)y^2 = -v_n^2$ has no solutions.*

Proof Assume that $x^2 - (p^2 - 4)u_n xy - (p^2 - 4)y^2 = -v_n^2$ for some integers x and y. Similar with the proof of Theorem 11, taking $u = (x + v_{n-1}y)/v_n$ and $v = y$, it can be seen that

$$u^2 - puv + v^2 = \left(x^2 - (p^2 - 4)u_n xy - (p^2 - 4)y^2\right)/v_n^2 = -v_n^2/v_n^2 = -1,$$

which is impossible by Theorem 9. Thus, it follows that the equation $x^2 - (p^2 - 4)u_n xy - (p^2 - 4)y^2 = -v_n^2$ has no integer solutions. □

The following corollary is a result of Theorem 11. Since it is proved in [11], we will give it without proof.

Corollary 4 *All solutions of the equation* $x^2 - 5F_{2n}xy - 5y^2 = -L_{2n}^2$ *are given by* $(x, y) = \mp(L_{2n+2m+1}, F_{2m+1})$ *with* $m \in \mathbb{Z}$.

Theorem 12 and Theorem 13 are stated by Melham, Kılıç and Ömür without proof in [4] and [2], respectively. Now we will prove them.

Theorem 12 *Let* $p \geq 3$ *and* $p^2 - 4$ *be a square-free integer. Then all solutions of the equation* $x^2 - v_n xy + y^2 = -(p^2 - 4)u_n^2$ *are given by* $(x, y) = \mp(v_{n+m}, v_m)$ *with* $m \in \mathbb{Z}$.

Proof Assume that $x^2 - v_n xy + y^2 = -(p^2 - 4)u_n^2$ for some integers x and y. Then multiplying both sides of this equation by 4 and using (3.10), we get $(2x - v_n y)^2 - (p^2 - 4)u_n^2 y^2 = -4(p^2 - 4)u_n^2$. Since $p^2 - 4$ is square-free, it follows that $u_n | 2x - v_n y$. Therefore, there is an integer z such that $2x - v_n y = u_n z$. From here we get $(u_n z)^2 - (p^2 - 4)u_n^2 y^2 = -4(p^2 - 4)u_n^2$, and then $z^2 - (p^2 - 4)y^2 = -4(p^2 - 4)$. This implies that $(p^2 - 4)|z$. Then there is an integer a such that $z = (p^2 - 4)a$, and we have $2x - v_n y = (p^2 - 4)u_n a$. Thus, it follows that

$$y^2 - (p^2 - 4)a^2 = 4.$$

Since

$$y^2 - p^2 a^2 = 4 + 4a^2,$$

we have $y^2 - p^2 a^2$ is even. Then we can see that y and pa have the same parity. Taking $u = (y + pa)/2$ and $v = a$, we obtain

$$u = \frac{y + p\left(\frac{2x - v_n y}{(p^2 - 4)u_n}\right)}{2} = \frac{2px + ((p^2 - 4)u_n - pv_n)y}{2(p^2 - 4)u_n} = \frac{px - v_{n-1}y}{(p^2 - 4)u_n}$$

and

$$v = \frac{2x - v_n y}{(p^2 - 4)u_n}.$$

Hence, we get

$$u^2 - puv + v^2 = \left(\frac{px - v_{n-1}y}{(p^2 - 4)u_n}\right)^2 - p\left(\frac{px - v_{n-1}y}{(p^2 - 4)u_n}\right)\left(\frac{2x - v_n y}{(p^2 - 4)u_n}\right) + \left(\frac{2x - v_n y}{(p^2 - 4)u_n}\right)^2$$

$$= -(p^2 - 4)(x^2 - v_n xy + y^2)/(p^2 - 4)^2 u_n^2 = 1.$$

Therefore it follows that $(u, v) = \mp(u_{m+1}, u_m)$ with $m \in \mathbb{Z}$ from Theorem 8. Thus, we obtain

$$(px - v_{n-1}y)/(p^2 - 4)u_n = \mp u_{m+1} \quad \text{and} \quad (2x - v_n y)/(p^2 - 4)u_n = \mp u_m. \tag{4.1}$$

Using the identities (3.6), (3.8) and (3.9) in (4.1), we get $(x, y) = \mp(v_{n+m}, v_m)$.

Conversely, if $(x, y) = \mp(v_{n+m}, v_m)$, then it follows that $x^2 - v_n xy + y^2 = -(p^2 - 4)u_n^2$, by (3.13). □

Using Theorem 9 in the same manner with Theorem 12, we can give the following corollary.

Corollary 5 *Let $p > 3$ and $p^2 - 4$ be a square-free integer. Then the equation $x^2 - v_n xy + y^2 = (p^2 - 4)u_n^2$ has no solutions.*

We can give the following corollary from Corollary 5.

Corollary 6 *Let $p > 3$ and $p^2 - 4$ be a square-free integer. Then the equation $x^2 - v_n xy + y^2 = (p^2 - 4)$ has no solutions.*

When $p = 3$, the equation $x^2 - v_n xy + y^2 = (p^2 - 4)u_n^2$ has solutions. In this case we have the equation $x^2 - L_{2n}xy + y^2 = 5F_{2n}^2$. Now we can give all solutions of these equations in the following lemma. Since this lemma is proved in [11], we will give it without proof.

Lemma 1 *All solutions of the equation $x^2 - L_{2n}xy + y^2 = 5F_{2n}^2$ are given by $(x, y) = \mp(L_{2n+2m+1}, L_{2m+1})$ with $m \in \mathbb{Z}$.*

Theorem 13 *All solutions of the equation $x^2 - v_n xy + y^2 = u_n^2$ are given by $(x, y) = \mp(u_{n+m}, u_m)$ with $m \in \mathbb{Z}$.*

Proof Suppose that $x^2 - v_n xy + y^2 = u_n^2$ for some integers x and y. Completing the square gives $(2x - v_n y)^2 - (p^2 - 4)u_n^2 y^2 = 4u_n^2$, and it is seen that $u_n | 2x - v_n y$. Thus, it follows that

$$\left((2x - v_n y)/u_n\right)^2 - (p^2 - 4)y^2 = 4.$$

Taking $u = (((2x - v_n y)/u_n) + py)/2 = (x + u_{n-1}y)/u_n$ and $v = y$, we have $u^2 - puv + v^2 = 1$. Therefore, from Theorem 8, we get $(u, v) = \mp(u_{m+1}, u_m)$ with $m \in \mathbb{Z}$. From here, we obtain $(x, y) = \mp(u_n u_{m+1} - u_{n-1}u_m, u_m)$. Then by (3.5), it follows that $(x, y) = \mp(u_{n+m}, u_m)$.

Conversely, if $(x, y) = \mp(u_{n+m}, u_m)$, then it can be seen that $x^2 - v_n xy + y^2 = u_n^2$, by (3.14). □

Using Theorem 9 in the same manner with Theorem 13, we can give the following corollaries.

Corollary 7 *The equation $x^2 - v_n xy + y^2 = -u_n^2$ has no solutions.*

The following corollary is a generalized form of Theorem 10. Since it is proved in [11], we will give it without proof.

Corollary 8 *All solutions of the equation $x^2 - L_{2n}xy + y^2 = -F_{2n}^2$ are given by $(x, y) = \mp(F_{2n+2m+1}, F_{2m+1})$ with $m \in \mathbb{Z}$.*

Now, let us examine all solutions of the following equations by using Diophantine equations given in Theorem 11, Theorem 12, Theorem 13 and the divisibility rules of the sequences (u_n) and (v_n).

Theorem 14 *All solutions of the equation $x^2 - v_n xy + y^2 = 1$ are given by $(x,y) = \mp(u_{(t+1)n}/u_n, u_{tn}/u_n)$ with $t \in \mathbb{Z}$.*

Proof Assume that $x^2 - v_n xy + y^2 = 1$ for some integers x and y. Multiplying both sides of this equation by u_n^2, we get

$$(u_n x)^2 - v_n(u_n x)(u_n y) + (u_n y)^2 = u_n^2.$$

From Theorem 13, it follows that $u_n x = \mp u_{n+m}$ and $u_n y = \mp u_m$ for some integer m. Hence, we get $x = \mp u_{n+m}/u_n$ and $y = \mp u_m/u_n$. Since x and y are integers, it is clear that $n|m$. Therefore, it follows that $m = tn$ for some $t \in \mathbb{Z}$. Then we obtain

$$(x,y) = \mp(u_{(t+1)n}/u_n, u_{tn}/u_n).$$

Conversely, if $(x,y) = \mp(u_{(t+1)n}/u_n, u_{tn}/u_n)$ with $t \in \mathbb{Z}$, then it follows that $x^2 - v_n xy + y^2 = 1$, by (3.14). □

Multiplying both sides of the equation $x^2 - v_n xy + y^2 = -1$ by u_n^2 and using Corollary 7, the following corollary can be given.

Corollary 9 *The equation $x^2 - v_n xy + y^2 = -1$ has no solutions.*

Theorem 15 *If $p \geq 3$, then all solutions of the equation $x^2 - (p^2 - 4)u_n xy - (p^2 - 4)y^2 = 1$ are given by $(x,y) = \mp(v_{(2t+1)n}/v_n, u_{2tn}/v_n)$ with $t \in \mathbb{Z}$.*

Proof Assume that $x^2 - (p^2 - 4)u_n xy - (p^2 - 4)y^2 = 1$ for some integers x and y. Multiplying both sides of this equation by v_n^2, we get

$$(v_n x)^2 - (p^2 - 4)u_n(v_n x)(v_n y) - (p^2 - 4)(v_n y)^2 = v_n^2.$$

Thus, it follows that $v_n x = \mp v_{n+m}$ and $u_n y = \mp u_m$ according to Theorem 11. Hence, we get $(x,y) = \mp(v_{n+m}/v_n, u_m/v_n)$. From Theorem 1 and Theorem 3, it can be seen that $n|m$ and m/n is an even integer. This implies that $m = 2tn$ for some $t \in \mathbb{Z}$. Therefore, we obtain $(x,y) = \mp(v_{(2t+1)n}/v_n, u_{2tn}/v_n)$.

Conversely, if $(x,y) = \mp(v_{(2t+1)n}/v_n, u_{2tn}/v_n)$ for some $t \in \mathbb{Z}$, then it follows that $x^2 - (p^2 - 4)u_n xy - (p^2 - 4)y^2 = 1$, by (3.12). □

The following corollary can be given from Corollary 3.

Corollary 10 *If $p \geq 3$, then the equation $x^2 - (p^2 - 4)u_n xy - (p^2 - 4)y^2 = -1$ has no solutions.*

Theorem 16 *If $p \geq 3$ and $p^2 - 4$ is a square-free integer, then all solutions of the equation $x^2 - v_{2n}xy + y^2 = -(p^2 - 4)u_n^2$ are given by $(x,y) = \mp(v_{(2t+3)n}/v_n, v_{(2t+1)n}/v_n)$ with $t \in \mathbb{Z}$.*

Proof Suppose that $x^2 - v_{2n}xy + y^2 = -(p^2 - 4)u_n^2$ for some integers x and y. Multiplying both sides of this equation by v_n^2 and considering the fact that $u_{2n} = u_n v_n$, we get

$$(v_n x)^2 - v_{2n}(v_n x)(v_n y) + (v_n y)^2 = -(p^2 - 4)u_{2n}^2.$$

From Theorem 12, it follows that $v_n x = \mp v_{2n+m}$ and $v_n y = \mp v_m$. Hence we get $(x, y) = \mp(v_{2n+m}/v_n, v_m/v_n)$. Moreover, since x and y are integers, it follows that $n \mid m$ and m/n is an odd integer from Theorem 3. Then there is an integer t such that $m = (2t + 1)n$. Therefore, we obtain $(x, y) = \mp(v_{(2t+3)n}/v_n, v_{(2t+1)n}/v_n)$.

Conversely, if $(x, y) = \mp(v_{(2t+3)n}/v_n, v_{(2t+1)n}/v_n)$ for some $t \in \mathbb{Z}$, then it follows that $x^2 - v_{2n}xy + y^2 = -(p^2 - 4)u_n^2$, by (3.13). $\qquad \square$

The following corollary can be proved similar to Theorem 16, by using Corollary 5. So, we omit its proof.

Corollary 11 *If $p > 3$ and $p^2 - 4$ is a square-free integer, then the equation $x^2 - v_{2n}xy + y^2 = (p^2 - 4)u_n^2$ has no solutions.*

Theorem 17 *All solutions of the equations $x^2 - v_{2n}xy + y^2 = u_n^2$ and $x^2 - v_{2n}xy + y^2 = v_n^2$ are given by $(x, y) = \mp(u_{(2t+2)n}/v_n, u_{2tn}/v_n)$ and $(x, y) = \mp(u_{(t+2)n}/u_n, u_{tn}/u_n)$ with $t \in \mathbb{Z}$, respectively.*

Proof Assume that $x^2 - v_{2n}xy + y^2 = u_n^2$ for some integers x and y. Multiplying both sides of this equation by v_n^2, we get

$$(v_n x)^2 - v_{2n}(v_n x)(v_n y) + (v_n y)^2 = u_{2n}^2.$$

Then from Theorem 13, it follows that $(x, y) = \mp(u_{2n+m}/v_n, u_m/v_n)$ for some $m \in \mathbb{Z}$. Hence, using Theorem 1 it is seen that $n \mid m$ and m/n is an even integer. Thus, we have $m = 2tn$ for some $t \in \mathbb{Z}$. Therefore, $(x, y) = \mp(u_{(2t+2)n}/v_n, u_{2tn}/v_n)$.

Conversely, if $(x, y) = \mp(u_{(2t+2)n}/v_n, u_{2tn}/v_n)$ for some $t \in \mathbb{Z}$, then by (3.14) it follows that $x^2 - v_{2n}xy + y^2 = u_n^2$.

Now suppose that $x^2 - v_{2n}xy + y^2 = v_n^2$ for some integers x and y. Multiplying both sides of this equation by u_n^2 and considering Theorem 13, we get $(x, y) = \mp(u_{2n+m}/u_n, u_m/u_n)$ for some $m \in \mathbb{Z}$. Furthermore, since x and y are integers, it follows that $n \mid m$ from Theorem 2. Then we have $m = tn$ for some $t \in \mathbb{Z}$. Thus, we obtain $(x, y) = \mp(u_{(t+2)n}/u_n, u_{tn}/u_n)$.

Conversely, if $(x, y) = \mp(u_{(t+2)n}/u_n, u_{tn}/u_n)$, then by (3.14) it follows that $x^2 - v_{2n}xy + y^2 = v_n^2$. $\qquad \square$

The proof of the following corollary is similar to that of Theorem 17, by using Corollary 7. So, we omit its proof.

Corollary 12 *The equations $x^2 - L_{4n}xy + y^2 = -F_{2n}^2$ and $x^2 - L_{4n}xy + y^2 = -L_{2n}^2$ have no solutions.*

Theorem 18 *Let $p \geq 3$, $p^2 - 4$ be a square-free integer. If $n > 2$, then the equation $x^2 - v_n xy + y^2 = -(p^2 - 4)$ has no solutions and all solutions of the equation $x^2 - (p^2 - 2)xy + y^2 = -(p^2 - 4)$ are given by $(x, y) = \mp(\frac{v_{2t+3}}{p}, \frac{v_{2t+1}}{p})$ with $t \in \mathbb{Z}$.*

Proof Assume that $x^2 - v_n xy + y^2 = -(p^2 - 4)$ for some integers x and y. Multiplying both sides of this equation by u_n^2, we get

$$(u_n x)^2 - v_n(u_n x)(u_n y) + (u_n y)^2 = -(p^2 - 4)u_n^2.$$

From Theorem 12, it follows that $(x, y) = \mp(v_{n+m}/u_n, v_m/u_n)$ with $m \in \mathbb{Z}$. If $n > 2$, then $u_n \nmid v_m$ from Theorem 4. Therefore, the equation $x^2 - v_n xy + y^2 = -(p^2 - 4)$ has no solutions. If $n = 2$, then we get that all solutions of the equation $x^2 - (p^2 - 2)xy + y^2 = -(p^2 - 4)$ are given by $(x, y) = \mp(v_{m+2}/u_2, v_m/u_2)$ with $m \in \mathbb{Z}$. Hence, it is seen that m is an odd integer according to Theorem 4. Thus, $m = 2t + 1$ for some $t \in \mathbb{Z}$. Therefore, it follows that $(x, y) = \mp(\frac{v_{2t+3}}{p}, \frac{v_{2t+1}}{p})$. □

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