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A generalizations of Simpson's type inequality for differentiable functions using (α, m)-convex functions and applications

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Abstract
In this paper, we establish some new inequalities of Simpson's type based on (α, m)-convexity for differentiable mappings. This contributes to new better estimates than presented already. Some applications for special means of real numbers and error estimates for some numerical quadrature rules are also given.

MSC: 26D15; 26D10

Keywords: Simpson's inequality; mid point inequality; Holder's inequality; special means; (α, m)-convex function

1 Introduction

Hudzik and Maligranda [1] defined the s-convex function as: A function $f : [0,\infty) \to \mathrm{R}$ is said to be s-convex or f belongs to the class K_s^i, if for all $x, y \in [0,\infty)$ and $\mu, \nu \in [0,1]$, the following inequality holds:

$$f(\mu x + \nu y) \le \mu^s f(x) + \nu^s f(y),$$

for some fixed $s \in (0,1]$.

Note that, if $\mu^s + \nu^s = 1$, the above class of convex functions is called s-convex functions in first sense and represented by K_s^1 and if $\mu + \nu = 1$, the above class is called s-convex in second sense and represented by K_s^2.

It may be noted that every 1-convex function is convex. In [1], they also discussed a few results connecting with s-convex functions in second sense and some new results about Hadamard's inequality for s-convex functions are discussed in [2], while on the other hand there are many important inequalities connecting with 1-convex (convex) functions [2].

The Simpson's inequality is very important and well known in the literature. This inequality is stated as: If $f : [a,b] \to \mathrm{R}$ be four times continuously differentiable mapping on (a,b) and $\|f^4\|_\infty = \mathrm{Sup}_{x\in(a,b)} |f^4(x)| < \infty$. Then

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{1}{2{,}880}\|f^4\|_\infty (b-a)^4.$$

Recently, many others [1–6] developed and discussed error estimates of the Simpson's inequality interms of refinement, counterparts, generalizations and new Simpson's type inequalities.

In [3], Dragomir *et al.* proved the following recent developments on Simpson's inequality for which the remainder is expressed interms of lower derivatives than the fourth.

Theorem 1.1 *Suppose $f : [a,b] \to \mathrm{R}$ is a differentiable mapping whose derivative is continuous on (a,b) and $f' \in L[a,b]$. Then*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{3}\|f'\|_1, \tag{1.1}$$

where $\|f'\|_1 = \int_a^b |f'(x)|\,dx$.

Note that the bound of (1.1) for L-Lipschitzian mapping is $\frac{5}{36}L(b-a)$ [3].

Theorem 1.2 *Suppose $f : [a,b] \to \mathrm{R}$ is an absolutely continuous mapping on $[a,b]$ whose derivative belongs to $L_p[a,b]$. Then the following inequality holds*:

$$\begin{aligned} &\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\quad \le \frac{1}{6}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\|f'\|_p, \end{aligned} \tag{1.2}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $p>1$.

In [7], Kirmaci established the following Hermite-Hadamard type inequality for differentiable convex functions as the following.

Theorem 1.3 *Let $f : I \subset \mathrm{R} \to \mathrm{R}$ be a differentiable function on I^0 (interior of I), where $a,b \in I$ with $a<b$. If the mapping $|f'|$ is convex on $[a,b]$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{8}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \tag{1.3}$$

For generalizations of (1.3), we refer to [8–10].

In [4] and [5], Dragomir and Fitzpatrick presented the following inequalities.

Theorem 1.4 [4] *Let $f : [a,b] \to \mathrm{R}$ be a L-Lipschitzian mapping on $[a,b]$. Then*

$$\left|\int_a^b f(x) - \frac{(b-a)}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right| \le \frac{5}{36}L(b-a)^2. \tag{1.4}$$

Theorem 1.5 [5] *Suppose that $f : [0,\infty) \to [0,\infty)$ is a convex function in the second sense, where $s \in (0,1)$ and let $a,b \in [0,\infty)$, $a<b$. If $f \in L^1[a,b]$, then*

$$2^{s-1}f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{s+1}. \tag{1.5}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.5). *The above inequalities are sharp.*

In [11], Mishen presented the class of (α,m)-convex functions as the following.

Definition 1.6 A function $f : [0, b) \to \mathrm{R}$ is said to be (α, m)-convex, where $(\alpha, m) \in [0, 1]^2$, if for every $x, y \in [0, b]$ and $\lambda \in [0, 1]$, the following inequality holds:

$$f\big(\lambda y + m(1-\lambda)x\big) \le \lambda^\alpha f(y) + m\big(1-\lambda^\alpha\big)f(x),$$

where $(\alpha, m) \in [0, 1]^2$, for some fixed $m \in (0, 1]$.

Note that $(\alpha, m) \in \{(0, 0), (\alpha, 0), (1, 0), (1, m)(1, 1), (\alpha, 1)\}$. One receives the following classes of functions respectively: increasing, α-starshaped, starshaped, m-convex, convex and α-convex. Denote by $K_m^\alpha(b)$, the set of all (α, m)-convex function on $[0, b]$ with $f(0) \le 0$. For recent results and generalizations referring m-convex and (α, m)-convex functions, we refer to [12, 13] and [14].

In this paper, we establish some new inequalities of Simpson's type based on (α, m)-convexity for differentiable mappings. This contributes to new better estimates than presented already. Some applications for special means of real numbers and error estimates for some numerical quadrature rules are also given. By using these results, without discussing the higher derivatives, which may not exist, not be bounded and may be difficult to investigate, we find the error estimate of Simpson's formula.

2 Main results

Before proceeding toward our main theorem regarding generalization of Simpson's inequality using (α, m)-convex function, we begin with the following lemma.

Lemma 2.1 *Let $f : I \subset \mathrm{R} \to \mathrm{R}$ be differentiable mapping on I^0 (interior of I), where $a, b \in I$ such that $a < b$. Then we have the following inequality*:

$$\begin{aligned}
&\frac{1}{6}\left[f(ma) + 4f\left(\frac{ma+b}{2}\right) + f(b)\right] - \frac{1}{b-ma}\int_{ma}^{b} f(x)\,dx \\
&\quad = (b-ma)\int_0^1 k(\lambda) f'\big(\lambda b + m(1-\lambda)a\big)\,d\lambda,
\end{aligned} \tag{2.1}$$

where

$$k(\lambda) = \begin{cases} \lambda - \frac{1}{6}, & \lambda \in [0, \frac{1}{2}], \\ \lambda - \frac{5}{6}, & \lambda \in [\frac{1}{2}, 1]. \end{cases}$$

Proof Consider

$$\begin{aligned}
I &= \int_0^1 k(\lambda) f'\big(\lambda b + m(1-\lambda)a\big)\,d\lambda \\
&= \int_0^{1/2} \left(\lambda - \frac{1}{6}\right) f'\big(\lambda b + m(1-\lambda)a\big)\,d\lambda \\
&\quad + \int_{1/2}^{1} \left(\lambda - \frac{5}{6}\right) f'\big(\lambda b + m(1-\lambda)a\big)\,d\lambda.
\end{aligned}$$

Using integration by parts, we have

$$\begin{aligned} I &= \left(\lambda - \frac{1}{6}\right)\frac{f(\lambda b + m(1-\lambda)a)}{b-ma}\bigg|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{f(\lambda b + m(1-\lambda)a)}{b-ma}\,d\lambda \\ &\quad + \left(\lambda - \frac{5}{6}\right)\frac{f(\lambda b + m(1-\lambda)a)}{b-ma}\bigg|_{\frac{1}{2}}^{1} - \int_{\frac{1}{2}}^{1} \frac{f(\lambda b + m(1-\lambda)a)}{b-ma}\,d\lambda \\ &= \frac{1}{6(b-ma)}\left[f(ma) + 4f\left(\frac{ma+b}{2}\right) + f(b)\right] - \int_0^1 \frac{f(\lambda b + m(1-\lambda)a)}{b-ma}\,d\lambda. \end{aligned}$$

Let we substitute, $x = \lambda b + m(1-\lambda)a$, and $dx = (b-ma)\,d\lambda$, which gives $(b-ma)$. $I = \frac{1}{6}[f(ma) + 4f(\frac{ma+b}{2}) + f(b)] - \frac{1}{b-ma}\int_{ma}^{b} f(x)\,d\lambda$. This proves as required. □

In the following result, we have another refinement of the Simpson's inequality *via* (α, m)-convex functions.

Theorem 2.2 *Let f be defined as in Lemma* 2.1. *If the mapping $|f'|$ is (α, m)-convex on $[a, b]$, for $(\alpha, m) \in [0,1]^2$. Then we have the following inequality*:

$$\begin{aligned} &\left|\frac{1}{6}\left[f(ma) + 4f\left(\frac{ma+b}{2}\right) + f(b)\right] - \frac{1}{b-ma}\int_{ma}^{b} f(x)\,dx\right| \\ &\quad \le (b-ma)\left[\nu_1\left|f'(b)\right| + m\nu_2\left|f'(a)\right|\right], \end{aligned} \tag{2.2}$$

where $\nu_1 = \frac{6^{-\alpha} - 9(2)^{-\alpha} + (5)^{\alpha+2}(6)^{-\alpha} + 3\alpha - 12}{18(\alpha+1)(\alpha+2)}$ *and* $\nu_2 = (\frac{5}{36} - \nu_1)$.

Proof Using Lemma 2.1 and (α, m)-convexity of $|f'|$, we have

$$\begin{aligned} &\left|\frac{1}{6}\left[f(ma) + 4f\left(\frac{ma+b}{2}\right) + f(b)\right] - \frac{1}{b-ma}\int_{ma}^{b} f(x)\,dx\right| \\ &\quad \le (b-ma)\int_0^1 \left|k(\lambda)\right|\left|f'\left(\lambda b + m(1-\lambda)a\right)\right|\,d\lambda \\ &\quad \le (b-ma)\int_0^{\frac{1}{2}} \left|\lambda - \frac{1}{6}\right|\left|f'\left(\lambda b + m(1-\lambda)a\right)\right|\,d\lambda \\ &\qquad + (b-ma)\int_{\frac{1}{2}}^{1} \left|\lambda - \frac{5}{6}\right|\left|f'\left(\lambda b + m(1-\lambda)a\right)\right|\,d\lambda \\ &\quad \le (b-ma)\int_0^{\frac{1}{2}} \left|\lambda - \frac{1}{6}\right|\left|\lambda^\alpha\left|f'(b)\right| + m\left(1-\lambda^\alpha\right)\left|f'(a)\right|\right|\,d\lambda \\ &\qquad + (b-ma)\int_{\frac{1}{2}}^{1} \left|\lambda - \frac{5}{6}\right|\left|\lambda^\alpha\left|f'(b)\right| + m\left(1-\lambda^\alpha\right)\left|f'(a)\right|\right|\,d\lambda. \end{aligned}$$

By simple calculations, we have

$$\int_0^{1/2} \lambda^\alpha\left|\lambda - \frac{1}{6}\right|\,d\lambda + \int_{1/2}^{1} \lambda^\alpha\left|\lambda - \frac{5}{6}\right|\,d\lambda = \frac{6^{-\alpha} - 9(2)^{-\alpha} + (5)^{\alpha+2}(6)^{-\alpha} + 3\alpha - 12}{18(\alpha+1)(\alpha+2)}.$$

Also,

$$\begin{aligned}&\int_0^{1/2}(1-\lambda^\alpha)\left|\lambda-\frac{1}{6}\right|d\lambda+\int_{1/2}^{1}(1-\lambda^\alpha)\left|\lambda-\frac{5}{6}\right|d\lambda\\&\quad=\frac{5}{36}-\frac{6^{-\alpha}-9(2)^{-\alpha}+(5)^{\alpha+2}(6)^{-\alpha}+3\alpha-12}{18(\alpha+1)(\alpha+2)}.\end{aligned}$$

The proof is completed. □

Now, we conclude the following corollaries.

Corollary 2.3 *Let f be defined as in Theorem 2.2. If the mapping $|f'|$ is (α,m)-convex on $[a,b]$, $(\alpha,m)\in[0,1]^2$. Then we have the following inequality*:

$$\begin{aligned}&\left|\frac{1}{6}\left[f(ma)+4f\left(\frac{ma+b}{2}\right)+f(b)\right]-\frac{1}{b-ma}\int_{ma}^{b}f(x)\,dx\right|\\&\quad\le\frac{5(b-ma)}{72}\left[m\left|f'(a)\right|+\left|f'(b)\right|\right].\end{aligned}\tag{2.3}$$

Observation 1 It is observed that the above midpoint inequality (2.3) is better than the inequality (1.1) presented by Dragomir [12].

The upper bound of the midpoint inequality for the first derivative is presented as:

Corollary 2.4 *By substituting $f(ma)=f(\frac{ma+b}{2})=f(b)$, in inequality (2.2), we get*

$$\left|\frac{1}{b-ma}\int_{ma}^{b}f(x)\,dx-f\left(\frac{ma+b}{2}\right)\right|\le(b-ma)\left[\nu_1\left|f'(b)\right|+m\nu_2\left|f'(a)\right|\right],\tag{2.4}$$

where $\nu_1=\frac{6^{-\alpha}-9(2)^{-\alpha}+(5)^{\alpha+2}(6)^{-\alpha}+3\alpha-12}{18(\alpha+1)(\alpha+2)}$ and $\nu_2=(\frac{5}{36}-\nu_1)$.

Corollary 2.5 *Putting $\alpha=1$, and $m=1$, in the above inequality (2.4), we get*

$$\left|\frac{1}{b-a}\int_{a}^{b}f(x)\,dx-f\left(\frac{a+b}{2}\right)\right|\le\frac{5(b-a)}{72}\left[\left|f'(a)\right|+\left|f'(b)\right|\right].\tag{2.5}$$

Observation 2 It is observed that the above midpoint inequality (2.5) seems better than the inequality (1.3) presented by Kiramic [7].

By applying Holder's inequality, we obtain the following theorem.

Theorem 2.6 *Let f be defined as in Theorem 2.2 with $\frac{1}{p}+\frac{1}{q}=1$. If the mapping $|f'|^{p/(p-1)}$ is (α,m)-convex on $[a,b]$, for $(\alpha,m)\in[0,1]^2$ and $p>1$. Then we have the following inequality*:

$$\begin{aligned}&\left|\frac{1}{6}\left[f(ma)+4f\left(\frac{ma+b}{2}\right)+f(b)\right]-\frac{1}{b-ma}\int_{ma}^{b}f(x)\,dx\right|\\&\quad\le(b-ma)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{1/p}\frac{1}{(\alpha+1)^{1/q}}\\&\qquad\times\left[\left(\left|f'(ma)\right|^q+\left|f'\left(\frac{ma+b}{2}\right)\right|^q\right)^{1/q}+\left(\left|f'\left(\frac{ma+b}{2}\right)\right|^q+\left|f'(b)\right|^q\right)^{1/q}\right].\end{aligned}\tag{2.6}$$

Proof Using Holder's inequality and by Lemma 2.1, we get

$$
\begin{aligned}
&\left|\frac{1}{6}\left[f(ma)+4f\left(\frac{ma+b}{2}\right)+f(b)\right]-\frac{1}{b-ma}\int_{ma}^{b}f(x)\,dx\right| \\
&\leq (b-ma)\int_0^1 |k(\lambda)|\left|f'\big(\lambda b+m(1-\lambda)a\big)\right|d\lambda \\
&\leq (b-ma)\int_0^{\frac{1}{2}}\left|\lambda-\frac{1}{6}\right|\left|f'\big(\lambda b+m(1-\lambda)a\big)\right|d\lambda \\
&\quad+(b-ma)\int_{\frac{1}{2}}^{1}\left|\lambda-\frac{5}{6}\right|\left|f'\big(\lambda b+m(1-\lambda)a\big)\right|d\lambda \\
&\leq (b-ma)\left(\int_0^{1/2}\left|\left(\lambda-\frac{1}{6}\right)\right|^p d\lambda\right)^{1/p}\left(\int_0^{1/2}\left|f'\big(\lambda b+m(1-\lambda)a\big)\right|^q d\lambda\right)^{1/q} \\
&\quad+(b-ma)\left(\int_{1/2}^{1}\left|\left(\lambda-\frac{5}{6}\right)\right|^p d\lambda\right)^{1/p}\left(\int_{1/2}^{1}\left|f'\big(\lambda b+m(1-\lambda)a\big)\right|^q d\lambda\right)^{1/q}.
\end{aligned}
$$

By simple calculations, we get

$$
\int_0^{1/2}\left|\left(\lambda-\frac{1}{6}\right)\right|^p d\lambda=\int_{1/2}^{1}\left|\left(\lambda-\frac{5}{6}\right)\right|^p d\lambda=\frac{1+2^{p+1}}{6^{p+1}(p+1)}. \tag{2.7}
$$

Also the (α,m)-convexity of $|f'|^{p/(p-1)}$ implies that

$$
\int_0^{1/2}\left|f'\big(\lambda b+m(1-\lambda)a\big)\right|^q d\lambda\leq\frac{|f'(ma)|^q+|f'(\frac{ma+b}{2})|^q}{\alpha+1}, \tag{2.8}
$$

$$
\int_{1/2}^{1}\left|f'\big(\lambda b+m(1-\lambda)a\big)\right|^q d\lambda\leq\frac{|f'(\frac{ma+b}{2})|^q+|f'(b)|^q}{\alpha+1}. \tag{2.9}
$$

Therefore, by combining (2.7), (2.8) and (2.9), we get the required result. The proof is completed. □

Corollary 2.7 *Let f be defined as in Theorem* 2.6. *If the mapping $|f'|^{p/(p-1)}$ is (α,m)-convex on $[a,b]$, for $(\alpha,m)\in[0,1]^2$ with $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. Then we have the following inequality*:

$$
\begin{aligned}
&\left|\frac{1}{6}\left[f(ma)+4f\left(\frac{ma+b}{2}\right)+f(b)\right]-\frac{1}{b-ma}\int_{ma}^{b}f(x)\,dx\right| \\
&\quad\leq (b-ma)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{1/p}\left(\frac{1}{2^{\frac{1}{q}}}\right) \\
&\qquad\times\left[\left(\left|f'(ma)\right|^q+\left|f'\left(\frac{ma+b}{2}\right)\right|^q\right)^{1/q}+\left(\left|f'\left(\frac{ma+b}{2}\right)\right|^q+\left|f'(b)\right|^q\right)^{1/q}\right].
\end{aligned} \tag{2.10}
$$

Corollary 2.8 *By putting $|f'(ma)|=|f'(b)|=0$, in Theorem* 2.6, *we get*

$$
\begin{aligned}
&\left|\frac{1}{6}\left[f(ma)+4f\left(\frac{ma+b}{2}\right)+f(b)\right]-\frac{1}{b-ma}\int_{ma}^{b}f(x)\,dx\right| \\
&\qquad\leq 2\frac{(b-ma)}{(\alpha+1)^{\frac{1}{q}}}\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{1/p}\left|f'\left(\frac{ma+b}{2}\right)\right|.
\end{aligned} \tag{2.11}
$$

In the following corollary, we have the mid point inequality for powers in terms of the first derivative.

Corollary 2.9 *By substituting $f(ma) = f(\frac{ma+b}{2}) = f(b)$ in Theorem* 2.6, *we get*

$$\begin{aligned}
&\left|\frac{1}{b-ma}\int_{ma}^{b} f(x)\,dx - f\left(\frac{ma+b}{2}\right)\right| \\
&\quad\le (b-ma)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{1/p}\frac{1}{(\alpha+1)^{1/q}} \\
&\qquad\times\left[\left(\left|f'(ma)\right|^q + \left|f'\left(\frac{ma+b}{2}\right)\right|^q\right)^{1/q} + \left(\left|f'\left(\frac{ma+b}{2}\right)\right|^q + \left|f'(b)\right|^q\right)^{1/q}\right]. \qquad (2.12)
\end{aligned}$$

In the following theorem, we obtain another form of Simpson inequality for powers in term of the first derivative.

Theorem 2.10 *Let f be defined as in Theorem* 2.6. *If the mapping $|f'|^q$ is (α,m)-convex on $[a,b]$, for $(\alpha,m)\in[0,1]^2$ and $q\ge 1$. We have the following inequality*:

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(ma) + 4f\left(\frac{ma+b}{2}\right) + f(b)\right] - \frac{1}{b-ma}\int_{ma}^{b} f(x)\,dx\right| \\
&\quad\le (b-ma)\left(\frac{5}{72}\right)^{1-1/q} \\
&\qquad\times\left[\left(u_1\left|f'(b)\right|^q + mu_2\left|f'(a)\right|^q\right)^{1/q} + \left(u_3\left|f'(b)\right|^q + mu_4\left|f'(a)\right|^q\right)^{1/q}\right], \qquad (2.13)
\end{aligned}$$

where $u_1 = \frac{(3^{-\alpha})(2^{1-\alpha})+3(\alpha)(2^{1-\alpha})+3(2^{-\alpha})}{6^3(\alpha+1)(\alpha+2)}$, $u_2 = (\frac{5}{72} - u_1)$, $u_3 = \frac{(5^{\alpha+2})(3^{-\alpha})(2^{1-\alpha})-3(\alpha)(2^{1-\alpha})-21(2^{-\alpha})+6(\alpha)-24}{6^3(\alpha+1)(\alpha+2)}$ *and* $u_4 = (\frac{5}{72} - u_3)$.

Proof From Lemma 2.1, and using power mean inequality, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(ma) + 4f\left(\frac{ma+b}{2}\right) + f(b)\right] - \frac{1}{b-ma}\int_{ma}^{b} f(x)\,dx\right| \\
&\quad\le (b-ma)\int_0^1 \left|k(\lambda)\right|\left|f'\left(\lambda b + m(1-\lambda)a\right)\right|d\lambda \\
&\quad\le (b-ma)\int_0^{\frac{1}{2}}\left|\lambda - \frac{1}{6}\right|\left|f'\left(\lambda b + m(1-\lambda)a\right)\right|d\lambda \\
&\qquad + (b-ma)\int_{\frac{1}{2}}^1\left|\lambda - \frac{5}{6}\right|\left|f'\left(\lambda b + m(1-\lambda)a\right)\right|d\lambda \\
&\quad\le (b-ma)\left(\int_0^{\frac{1}{2}}\left|\left(\lambda - \frac{1}{6}\right)\right|d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|\lambda - \frac{1}{6}\right|\left|f'\left(\lambda b + m(1-\lambda)a\right)\right|d\lambda\right)^{\frac{1}{q}} \\
&\qquad + (b-ma)\left(\int_{\frac{1}{2}}^1\left|\left(\lambda - \frac{5}{6}\right)\right|d\lambda\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\left|\lambda - \frac{5}{6}\right|\left|f'\left(\lambda b + m(1-\lambda)a\right)\right|d\lambda\right)^{\frac{1}{q}}.
\end{aligned}$$

The (α,m)-convexity of $|f'|^q$ gives that

$$\int_0^{\frac{1}{2}}\left|\lambda - \frac{1}{6}\right|\left|f'\left(\lambda b + m(1-\lambda)a\right)\right|^q d\lambda \le u_1\left|f'(b)\right|^q + mu_2\left|f'(a)\right|^q. \qquad (2.14)$$

Also

$$\int_{1/2}^{1}\left|\left(\lambda-\frac{5}{6}\right)\right|\left|f'\left(\lambda b+m(1-\lambda)a\right)\right|^{q}d\lambda\leq mu_4\left|f'(a)\right|^{q}+u_3\left|f'(b)\right|^{q}. \tag{2.15}$$

By simple calculations, we have

$$\int_{0}^{1/2}\left|\left(\lambda-\frac{1}{6}\right)\right|d\lambda=\int_{1/2}^{1}\left|\left(\lambda-\frac{5}{6}\right)\right|d\lambda=\frac{5}{72}. \tag{2.16}$$

Our required result is obtained by combining inequalities (2.14), (2.15) and (2.16). The proof is completed. □

Corollary 2.11 *Let f be as in Theorem* 2.10 *and $\alpha=1$, the inequality holds for s-convex functions*:

$$\begin{aligned}&\left|\frac{1}{6}\left[f(ma)+4f\left(\frac{ma+b}{2}\right)+f(b)\right]-\frac{1}{b-ma}\int_{ma}^{b}f(x)\,dx\right|\\&\quad\leq(b-ma)\left(\frac{5}{72}\right)^{1-1/q}\left[\left(\frac{29}{1{,}296}\left|f'(b)\right|^{q}+\frac{61}{1{,}296}m\left|f'(a)\right|^{q}\right)^{\frac{1}{q}}\right.\\&\qquad\left.+\left(\frac{61}{1{,}296}\left|f'(b)\right|^{q}+\frac{29}{1{,}296}m\left|f'(a)\right|^{q}\right)^{\frac{1}{q}}\right].\end{aligned} \tag{2.17}$$

Moreover, if $\alpha=1$, $m=1$, the inequality holds for convex function. If $|f'(x)|\leq Q$, $\forall x\in I$, then we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_{a}^{b}f(x)\,dx\right|\leq\frac{5(b-a)}{36}\cdot Q. \tag{2.18}$$

Observation 3 It is observed that the inequality (2.18) with $m=1$ gives an improvement for the inequality (1.4).

The following corollary gives the refinement of inequality (2.13).

Corollary 2.12 *Let f be as in Theorem* 2.10, *then we have the following inequality*:

$$\begin{aligned}&\left|\frac{1}{6}\left[f(ma)+4f\left(\frac{ma+b}{2}\right)+f(b)\right]-\frac{1}{b-ma}\int_{ma}^{b}f(x)\,dx\right|\\&\quad\leq\frac{b-ma}{(6^{3}(\alpha+1)(\alpha+2))^{\frac{1}{q}}}\left(\frac{5}{72}\right)^{1-1/q}\left[(\nu_1)^{1/q}+(\nu_2)^{1/q}\right]\times\left(m\left|f'(a)\right|+\left|f'(b)\right|\right),\end{aligned} \tag{2.19}$$

where $\nu_1=(3^{-\alpha})(2^{1-\alpha})+3(\alpha)(2^{1-\alpha})+3(2^{-\alpha})$, *and* $\nu_2=(5^{\alpha+2})(3^{-\alpha})(2^{1-\alpha})-3(\alpha)(2^{1-\alpha})-21(2^{-\alpha})+6(\alpha)-24$. *Further, if $\alpha=1$, we get*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(ma)+4f\left(\frac{ma+b}{2}\right)+f(b)\right]-\frac{1}{b-ma}\int_{ma}^{b}f(x)\,dx\right|\\&\quad\leq\frac{5[(732)^{1/q}+(348)^{1/q}]}{72(6{,}480)^{1/q}}(b-ma)\left(m\left|f'(a)\right|+\left|f'(b)\right|\right).\end{aligned} \tag{2.20}$$

Proof Let us take inequality (2.13), for $p > 1$, $q = p/(p-1)$. Suppose that

$$\psi_1 = \nu_1 \left|f'(b)\right|^q, \qquad \xi_1 = \nu_2 m \left|f'(a)\right|^q,$$
$$\psi_2 = \nu_1 m \left|f'(a)\right|^q, \qquad \xi_2 = \nu_2 \left|f'(b)\right|^q.$$

Take $0 < 1/q < 1$, for $q > 1$ and by using the well-known fact,

$$\sum_{j=1}^{n} (\xi_j + \psi_j)^r \leq \sum_{j=1}^{n} \xi_j^r + \sum_{j=1}^{n} \psi_j^r,$$

we consider $0 < r < 1$, $\xi_1, \xi_2, \ldots, \xi_n \geq 0$ and $\psi_1, \psi_2, \ldots, \psi_n \geq 0$. For $n = 2$, we have

$$\left|\frac{1}{6}\left[f(ma) + 4f\left(\frac{ma+b}{2}\right) + f(b)\right] - \frac{1}{b-ma}\int_{ma}^{b} f(x)\,dx\right|$$
$$\leq \frac{5[(732)^{1/q} + (348)^{1/q}]}{72(6{,}480)^{1/q}}(b-ma)\left(m\left|f'(a)\right| + \left|f'(b)\right|\right).$$

The proof is completed. □

3 Application to Simpson's formula

Suppose D be the partition of the interval $[a,b]$, with $h_i = (x_{i+1} - x_i)/2$ and suppose that $D: a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. Since the Simpson's formula is:

$$S_n(f,D) = \sum_{i=0}^{n-1} \frac{f(x_i) + 4f(x_i + h_i) + f(x_{i+1})}{6}(x_{i+1} - x_i). \tag{3.1}$$

We know that if the function $f : [a,b] \to \mathrm{R}$, is differentiable such that the fourth derivative of $f(x)$ exists on (a,b) and $K = \max_{x\in(a,b)} |f^{(4)}(x)| < \infty$, we have

$$I = \int_a^b f(x)\,dx = S_n(f,D) + E_n^S(f,D), \tag{3.2}$$

where the error term $E_n^S(f,D)$ of the integral I by Simpson's formula $S_n(f,D)$ fulfils the following:

$$\left|E_n^S(f,D)\right| \leq \frac{K}{90}\sum_{i=0}^{n-1}(x_{i+1} - x_i)^5. \tag{3.3}$$

Clearly, (3.2) cannot be applied, if the fourth derivative of f is not bounded on (a,b). Some new error estimates for the Simpson's rule in terms of first and second derivative are presented as follows.

Proposition 3.1 *Let f be defined as in Corollary* 2.3. *If the mapping $|f'|$ is (α,m)-convex on $[a,b]$, then for every division D of $[a,b]$, in* (3.2), *we have*

$$\left|E_n^S(f,D)\right| \leq \frac{5}{72}\sum_{i=0}^{n-1}(x_{i+1} - mx_i)^2\left[\left|f'(mx_i)\right| + \left|f'(x_{i+1})\right|\right].$$

Proof Let D be the division of the subintervals $[x_{i+1} - x_i]$ $(i = 0, 1, \ldots, n-1)$. By applying Corollary 2.3 on the subintervals, we get

$$\left|\frac{(x_{i+1} - mx_i)}{6}\left(f(mx_i) + 4f\left(\frac{x_{i+1} - mx_i}{2}\right) + f(x_{i+1})\right) - \int_{mx_i}^{x_{i+1}} f(x)\,dx\right| \\ \leq \frac{5(x_{i+1} - mx_i)^2}{72}\left[\left|mf'(x_i)\right| + \left|f'(x_{i+1})\right|\right].$$

Using the (α, m)-convexity of $|f'|$, by summing over i from 0 to $n-1$, and by triangle inequality, we get

$$\left|S_n(f, D) - \int_a^b f(x)\,dx\right| \leq \frac{5}{72}\sum_{i=0}^{n-1}(x_{i+1} - mx_i)^2\left[\left|mf'(x_i)\right| + \left|f'(x_{i+1})\right|\right].$$

The proof is completed. □

The proof of following proposition is same as of Proposition 3.1 and by using Corollary 2.9.

Proposition 3.2 *Let f be defined as in Proposition* 3.1. *If $|f'|^{p/(p-1)}$ is (α, m)-convex on* $[a, b]$, $p > 1$, *then for every division D of* $[a, b]$, *in* (3.2), *we have*

$$\left|E_n^S(f, D)\right| \leq \left(\frac{1 + 2^{p+1}}{6^{p+1}(p+1)}\right)^{1/p}\left(\frac{1}{2^{\frac{1}{q}}}\right) \\ \times \sum_{i=0}^{n-1}(x_{i+1} - mx_i)^2\left[\left(\left|f'(mx_i)\right|^q + \left|f'\left(\frac{mx_i + x_{i+1}}{2}\right)\right|^q\right)^{1/q} \right. \\ \left. + \left(\left|f'\left(\frac{mx_i + x_{i+1}}{2}\right)\right|^q + \left|f'(x_{i+1})\right|^q\right)^{1/q}\right].$$

4 Application to the midpoint formula

Suppose D be the partition of the interval $[a, b]$, with $h_i = (x_{i+1} - x_i)/2$ and suppose that $D : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. Since the midpoint formula is:

$$M(f, D) = \sum_{i=0}^{n-1}(x_{i+1} - x_i)f\left(\frac{x_i + x_{i+1}}{2}\right). \tag{4.1}$$

We know that, if the function $f : [a, b] \to \mathrm{R}$, is differentiable such that second derivative of $f(x)$ on (a, b) exists and $K = \max_{x \in (a,b)} |f^2(x)| < \infty$, then

$$I = \int_a^b f(x)\,dx = M(f, D) + E_M(f, D). \tag{4.2}$$

Where the error term $E_M(f, D)$ of the integral I by the mid point formula $M(f, D)$ fulfils the following:

$$\left|E_M(f, D)\right| \leq \frac{\widetilde{K}}{24}\sum_{i=0}^{n-1}(x_{i+1} - x_i)^3. \tag{4.3}$$

Here, we derive some new better error estimates for the remainder term $E_M(f,D)$ in terms of the first derivative which are refined estimates as compared to presented in [10].

Proposition 4.1 *Let f be defined as in Corollary 2.5. If the mapping $|f'|$ is (α,m)-convex on $[a,b]$, then for every division D of $[a,b]$, in (4.2), we have*

$$\left|E_M(f,D)\right| \le \frac{5}{72}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left[\left|f'(x_i)\right| + \left|f'(x_{i+1})\right|\right].$$

Proof Let D be the division of the subintervals $[x_{i+1}-x_i]$ $(i=0,1,\ldots,n-1)$. By applying Corollary 2.5 on the subintervals, we get

$$\left|(x_{i+1}-x_i)f\left(\frac{x_{i+1}+x_i}{2}\right) - \int_{x_i}^{x_{i+1}} f(x)\,dx\right| \le \frac{5(x_{i+1}-x_i)^2}{72}\left[\left|f'(x_i)\right| + \left|f'(x_{i+1})\right|\right].$$

Using the (α,m)-convexity of $|f'|$, by summing over i from 0 to $n-1$, and by triangle inequality, we get

$$\left|E_M(f,D)\right| \le \frac{5}{72}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left[\left|f'(x_i)\right| + \left|f'(x_{i+1})\right|\right].$$

The proof is completed. □

The proof of following proposition is same as of Proposition 4.1, by putting $m=1$ in Corollary 2.9.

Proposition 4.2 *Let f be defined as in Proposition 4.1. If $|f'|^{p/(p-1)}$ is (α,m)-convex on $[a,b]$, $p>1$, then for every division D of $[a,b]$, in (4.2), we have*

$$\begin{aligned}\left|E_M(f,D)\right| \le{}& \left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{1/p}\left(\frac{1}{2^{\frac{1}{q}}}\right) \\ &\times\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left[\left(\left|f'(x_i)\right|^q + \left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q\right)^{1/q}\right. \\ &\left.+\left(\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q + \left|f'(x_{i+1})\right|^q\right)^{1/q}\right].\end{aligned}$$

5 Application to some special means

We now consider the applications of our main theorem to the special means.

(a) The arithmetic mean:

$$A = A(a,b) := \frac{a+b}{2}, \quad a,b>0.$$

(b) The logarithmic mean:

$$L = L(a,b) = \begin{cases} a, & \text{if } a=b, \\ \frac{b-a}{\ln b-\ln a}, & \text{if } a\neq b, \end{cases} \quad a,b>0.$$

(c) The p-logarithmic mean:

$$L_p \equiv L_p(a,b) = \begin{cases} a, & \text{if } a = b, \\ \frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}, & \text{if } a \neq b, \end{cases} \quad p \in \Re\backslash\{-1,0\}: \; a,b > 0.$$

It is well known that L_P is monotonic nondecreasing over $p \in \mathrm{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have $L \le A$.

Suppose that $\alpha \in (0,1]$ and $\varphi, \phi, \chi \in \mathrm{R}$. Consider the function $g : [0,\infty) \to [0,\infty)$ as

$$g(t) = \begin{cases} \varphi, & t = 0, \\ \phi t^{\alpha} + \chi, & t > 0 \end{cases}$$

for $\phi \ge 0$ and $0 \le \chi \le \varphi$, we have $g \in K_{\alpha}^{2}$ [1]. Thus, by taking $\varphi = \chi = 0$, $\phi = 1$, we get $g : [0,\infty) \to [0,\infty)$ implies: $g(t) = t^{\alpha}$, $g \in K_{\alpha}^{2}$.

Consider $f : [a,b] \to \mathrm{R}$ $(0 < a < b)$, $f(x) = x^{\alpha}$, $\alpha \in (0,1]$. Then we have the following means:

$$\frac{1}{b-a}\int_a^b f(x)\,dx = L_{\alpha}^{\alpha}(a,b),$$

$$\frac{f(a)+f(b)}{2} = A\bigl(a^{\alpha}, b^{\alpha}\bigr),$$

$$f\left(\frac{a+b}{2}\right) = A^{\alpha}(a,b).$$

Now using the results of Section 2, some new inequalities are derived for the above means.

Proposition 5.1 *Let $f : [a,b] \to \mathrm{R}$, $0 < a < b$ and $n \in N$. Then we have*

$$\left|\frac{1}{3}A\bigl(a^{\alpha}, b^{\alpha}\bigr) + \frac{2}{3}A^{\alpha}(a,b) - L_{\alpha}^{\alpha}(a,b)\right| \le \alpha(b-a)\frac{6^{-\alpha} - 9(2)^{-\alpha} + (5)^{\alpha+2}(6)^{-\alpha} + 3\alpha - 12}{18(\alpha+1)(\alpha+2)}\bigl[|a|^{\alpha-1} + |b|^{\alpha-1}\bigr]. \tag{5.1}$$

Proof The assertion follows by taking $m = 1$ and from inequality (2.2) applied to the mapping $f(x) = x^{\alpha}$, $x \in [a,b]$ with $n \in N$.

Moreover, by setting $\alpha = 1$ in inequality (5.1), we get

$$\bigl|A(a,b) - L(a,b)\bigr| \le \frac{5}{36}(b-a). \qquad \square$$

Proposition 5.2 *Let $f : [a,b] \to \mathrm{R}$, $0 < a < b$ and $n \in N$. Then we have*

$$\left|\frac{1}{3}A\bigl(a^{\alpha}, b^{\alpha}\bigr) + \frac{2}{3}A^{\alpha}(a,b) - L_{\alpha}^{\alpha}(a,b)\right| \le \alpha(b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{1/p}\frac{\alpha}{(\alpha+1)^{1/q}} \times \Bigl[\bigl(\bigl|a^{\alpha-1}\bigr|^{q} + \bigl|A^{\alpha-1}(a,b)\bigr|^{q}\bigr)^{1/q} + \bigl(\bigl|A^{\alpha-1}(a,b)\bigr|^{q} + \bigl|b^{\alpha-1}\bigr|^{q}\bigr)^{1/q}\Bigr]. \tag{5.2}$$

Proof The assertion follows by taking $m = 1$ and from inequality (2.6) applied to the mapping $f(x) = x^{\alpha}$, $x \in [a,b]$ and $n \in N$.

Moreover, by setting $\alpha = 1$ in inequality (5.2), we get

$$\left|A(a,b) - L(a,b)\right| \leq 2(b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{1/p}. \qquad \square$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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