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The general iterative methods for equilibrium problems and fixed point problems of a countable family of nonexpansive mappings in Hilbert spaces

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Abstract
In this paper, the researcher introduces the general iterative scheme for finding a common element of the set of equilibrium problems and fixed point problems of a countable family of nonexpansive mappings in Hilbert spaces. The results presented in this paper improve and extend the corresponding results announced by many others.
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1 Introduction

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let C be a nonempty closed and convex subset of H, and let $T : C \to C$ be a nonlinear mapping. In this paper, we use $F(T)$ to denote the fixed point set of T.

Recall the following definitions.

(1) The mapping T is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

Further, let F be a bifunction from $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The so-called equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $y \in C$ such that

$$F(y, u) \ge 0, \quad \forall u \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $EP(F)$. Given a mapping $A : C \to H$, let $F(y, u) = \langle Ay, u - y\rangle$ for all $y, u \in C$. Then $z \in EP(F)$ if and only if $\langle Az, u - z\rangle \ge 0$ for all $u \in C$. Numerous problems in physics, optimization and economics reduce to finding a solution of (1.2).

(2) The mappings $\{T_n\}_{n\in\mathbb{N}}$ are said to be a family of nonexpansive mappings from H into itself if

$$\|T_n x - T_n y\| \le \|x - y\|, \quad \forall x, y \in H, \tag{1.3}$$

and denoted by $F(T_n) = \{x \in H : T_n x = x\}$ is the fixed point set of T_n. Finding an optimal point in $\bigcap_{n\in\mathbb{N}} F(T_n)$ of the fixed point sets of each mapping is a matter of interest in various branches of science.

Recently, many authors considered the iterative methods for finding a common element of the set of solutions to problem (1.2) and of the set of fixed points of nonexpansive mappings; see, for example, [1, 2] and the references therein.

Next, let $A : C \to H$ be a nonlinear mapping. We recall the following definitions.

(3) A is said to be *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

(4) A is said to be *strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C.$$

In such a case, A is said to be α-*strongly monotone.*

(5) A is said to be inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

In such a case, A is said to be α-*inverse-strongly monotone.*

The classical variational inequality is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in C. \tag{1.4}$$

In this paper, we use $VI(C,A)$ to denote the set of solutions to problem (1.4). One can easily see that the variational inequality problem is equivalent to a fixed point problem. $u \in C$ is a solution to problem (1.4) if and only if u is a fixed point of the mapping $P_C(I - \lambda)T$, where $\lambda > 0$ is a constant.

The variational inequality has been widely studied in the literature; see, for example, the work of Plubtieng and Punpaeng [3] and the references therein.

Recently, Ceng *et al.* [4] considered an iterative method for the system of variational inequalities (1.4). They got a strongly convergence theorem for problem (1.4) and a fixed point problem for a single nonexpansive mapping; see [4] for more details.

On the other hand, Moudafi [5] introduced the viscosity approximation method for nonexpansive mappings (see [6] for further developments in both Hilbert and Banach spaces).

A mapping $f : C \to C$ is called α-contractive if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in C. \tag{1.5}$$

Let f be a contraction on C. Starting with an arbitrary initial $x_1 \in C$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \sigma_n)Tx_n + \sigma_n f(x_n), \quad n \geq 0, \tag{1.6}$$

where $\{\sigma_n\}$ is a sequence in $(0,1)$. It is proved [5, 6] that under certain appropriate conditions imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (1.6) strongly converges to the unique solution q in C of the variational inequality

$$\langle (I-f)q, p-q\rangle \geq 0, \quad p \in C.$$

Let A be a strongly positive linear bounded operator on a Hilbert space H with a constant $\bar{\gamma}$; that is, there exists $\bar{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.7}$$

Recently, Marino and Xu [7] introduced the following general iterative method:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{1.8}$$

where A is a strongly positive bounded linear operator on H. They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.8) converges strongly to the unique solution of the variational inequality

$$\langle (A-\gamma f)x^*, x - x^*\rangle \geq 0, \quad x \in C, \tag{1.9}$$

which is the optimality condition for the minimization problem

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (*i.e.*, $h'(x) = \gamma f(x)$ for $x \in H$).

In 2007, Takahashi and Takahashi [2] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions (1.2) and the set of fixed points of a nonexpansive mapping in Hilbert spaces. Let $S : C \to H$ be a nonexpansive mapping. Starting with arbitrary initial $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ recursively by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (1-\alpha_n)Su_n, & \forall n \in \mathbb{N}. \end{cases} \tag{1.10}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \cap EP(F)$, where $z = P_{F(S)\cap EP(F)} f(z)$.

Next, Plubtieng and Punpaeng, [3] introduced an iterative scheme by the general iterative method for finding a common element of the set of solutions (1.2) and the set of fixed points of nonexpansive mappings in Hilbert spaces.

Let $S : H \to H$ be a nonexpansive mapping. Starting with an arbitrary $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I-\alpha_n A)Su_n, & \forall n \in \mathbb{N}. \end{cases} \tag{1.11}$$

They proved that if the sequences $\{\alpha_n\}$ and $\{r_n\}$ of parameters satisfy appropriate conditions, then the sequence $\{x_n\}$ generated by (1.11) converges strongly to the unique solution of the variational inequality

$$\big\langle (A-\gamma f)z, x-z\big\rangle \geq 0, \quad \forall x \in F(S)\cap EP(F), \tag{1.12}$$

which is the optimality condition for the minimization problem

$$\min_{x\in F(S)\cap EP(F)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (*i.e.*, $h'(x) = \gamma f(x)$ for $x \in H$).

Let $T_1, T_2, \ldots$ be an infinite sequence of mappings of C into itself, and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i \in \mathbb{N}$. Then for any $n \in \mathbb{N}$, Takahashi [8] (see [9]) defined a mapping W_n of C into itself as follows:

$$\begin{aligned}
U_{n,n+1} &= I,\\
U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I,\\
U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I,\\
&\vdots\\
U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I,\\
U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I,\\
&\vdots\\
U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I,\\
W_n = U_{n,1} &= \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I.
\end{aligned} \tag{1.13}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$.

Recently, using process (1.13), Yao *et al.* [10] proved the following result.

Theorem 1.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F: C\times C \to \mathbb{R}$ be an equilibrium bifunction satisfying the conditions*:

(1) *F is monotone, that is, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$*;

(2) *for each $x, y, z \in C$, $\lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y)$*;

(3) *for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.*

Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into C such that $\bigcap_{i=1}^{\infty} F(T_i) \cap EP(F) \neq \emptyset$. Suppose $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$ and $\{r_n\} \subset (0,\infty)$. Suppose the following conditions are satisfied:

(1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(3) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$.

Let f be a contraction of H into itself, and let $x_0 \in H$ be given arbitrarily. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated iteratively by

$$\begin{cases} F(y_n, x) + \frac{1}{r_n}\langle x - y_n, y_n - x_n\rangle \geq 0, & \forall x \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n y_n, \end{cases}$$

converge strongly to $x^* \in \bigcap_{i=1}^{\infty} F(T_i) \cap EP(F)$, the unique solution of the minimization problem

$$\min_{x \in \bigcap_{i=1}^{\infty} F(T_i) \cap EP(F)} \frac{1}{2}\|x\|^2 - h(x),$$

where h is a potential function for f.

Very recently, using process (1.13), Chen [11] proved the following result.

Theorem 1.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings from C to C such that the common fixed point set $\Omega = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $f : C \to H$ be an α-contraction, and let $A : H \to H$ be a self-adjoint, strongly positive bounded linear operator with a coefficient $\bar{\gamma} > 0$. Let σ be a constant such that $0 < \gamma\alpha < \bar{\gamma}$. For an arbitrary initial point x_0 belonging to C, one defines a sequence $\{x_n\}_{n\geq 0}$ iteratively*

$$x_{n+1} = P_C\big[\alpha_n \gamma f(x_n) + (I - \alpha_n A) W_n x_n\big], \quad \forall n \geq 0, \tag{1.14}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$. Assume the sequence $\{\alpha_n\}$ satisfies the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ generated by (1.14) *converges in norm to the unique solution x^*, which solves the following variational inequality*:

$$x^* \in \Omega \quad \textit{such that } \big\langle (A - \gamma f)x^*, x^* - \hat{x}\big\rangle \geq 0, \forall \hat{x} \in \Omega. \tag{1.15}$$

Motivated by this result, we introduce the following explicit general iterative scheme:

$$\begin{cases} x_1 \in H, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in H, \\ x_{n+1} = P_C[\alpha_n \gamma f(x_n) + (I - \alpha_n A) W_n u_n], & \forall n \in \mathbb{N}, \end{cases} \tag{1.16}$$

where $\{T_n\}_{n\in\mathbb{N}}$ is a family of nonexpansive mappings from H into itself such that $\bigcap_{n\in\mathbb{N}} F(T_n)$ is nonempty, $F : C \times C \to \mathbb{R}$ is an equilibrium bifunction, A is a strongly positive operator on H, f is a contraction of H into itself with $\alpha \in (0,1)$, $\{\alpha_n\}$, $\{r_n\}$, $\{\lambda_n\}$ suitable sequences in $\mathbb{R}$ and $\{W_n\}$ is the sequence of a W-mapping generated by $\{T_n\}_{n\in\mathbb{N}}$ and $\{\lambda_n\}$. Let U be defined by $Ux = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$ for every $x \in C$ using process (1.13). We shall prove under mild conditions that $\{x_n\}$ and $\{u_n\}$ strongly converge to a point $x^* \in \bigcap_{i=1}^{\infty} F(T_i) \cap EP(F)$, which is the unique solution of the variational inequality

$$\big\langle (A - \gamma f)x^*, x^* - \hat{x} \big\rangle \geq 0, \quad \forall \hat{x} \in \bigcap_{i=1}^{\infty} F(T_i) \cap EP(F), \tag{1.17}$$

or, equivalently, the unique solution of the minimization problem

$$\min_{x \in \bigcap_{i=1}^{\infty} F(T_i) \cap EP(F)} \frac{1}{2} \langle A\hat{x}, \hat{x} \rangle - h(\hat{x}),$$

where h is a potential function for γf.

2 Preliminaries

Let H be a real Hilbert space with the norm $\|\cdot\|$ and the inner product $\langle \cdot, \cdot \rangle$, and let C be a closed convex subset of H. We call $f : C \to H$ an α-contraction if there exists a constant $\alpha \in [0, 1)$ such that

$$\big\| f(x) - f(y) \big\| \leq \alpha \|x - y\|, \quad \forall x, y \in C.$$

Let A be a strongly positive linear bounded operator on a Hilbert space H with a constant $\bar{\gamma}$; that is, there exists $\bar{\gamma} > 0$ such that

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2, \quad \forall x \in H.$$

Next, we denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\to$, respectively. A space X is said to satisfy Opial's condition [12] if for each sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$, we have

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|, \quad \forall y \in X, y \neq x.$$

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called the (nearest point or metric) projection of H onto C. In addition, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x \rangle \geq 0, \tag{2.1}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H, y \in C. \tag{2.2}$$

Recall that a mapping $T : H \to H$ is said to be firmly nonexpansive mapping if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle, \quad \forall x, y \in H.$$

It is well known that P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y \rangle, \quad \forall x, y \in H. \tag{2.3}$$

If A is an α-inverse-strongly monotone mapping of C into H, then it is obvious that A is $\frac{1}{\alpha}$-*Lipschitz* continuous. We also have that for all $x, y \in C$ and $\lambda > 0$,

$$\begin{aligned}\|(I-\lambda A)x-(I-\lambda A)y\|^2 &= \|x-y-\lambda(Ax-Ay)\|^2 \\ &= \|x-y\|^2 - 2\lambda\langle Ax-Ay, x-y\rangle + \lambda^2\|Ax-Ay\|^2 \\ &\leq \|x-y\|^2 + \lambda(\lambda-2\alpha)\|Ax-Ay\|^2. \end{aligned} \tag{2.4}$$

So, if $\lambda \leq 2\alpha$, then $I-\lambda A$ is a nonexpansive mapping of C into H.

The following lemmas will be useful for proving the convergence result of this paper.

Lemma 2.1 *Let H be a real Hilbert space. Then for all $x, y \in H$,*

(1) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$;
(2) $\|x+y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$.

Lemma 2.2 ([13]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X, and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \leq 0$. Then $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

Lemma 2.3 ([14]) *Assume that $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions*:

(A1) *$F(x,x) = 0$ for all $x \in C$;*
(A2) *F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;*
(A3) *for each $x, y, z \in C$, $\lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y)$;*
(A4) *for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.*

Lemma 2.4 ([14]) *Assume that $F : C \times C \to \mathbb{R}$ satisfies* (A1)-(A4). *For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows*:

$$T_r(x) = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \forall y \in C \right\}$$

for all $z \in H$. Then the following hold:

1. *T_r is single-valued;*
2. *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x-y\rangle;$$

3. *$F(T_r) = EP(F)$;*
4. *$EP(F)$ is closed and convex.*

Lemma 2.5 ([12]) *Let H be a Hilbert space, C be a closed convex subset of H, and $S : C \to C$ be a nonexpansive mapping with $F(S) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to $x \in C$ and if $\{(I-S)x_n\}$ converges strongly to y, then $(I-S)x = y$.*

Lemma 2.6 ([6]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7 ([7]) *Let* H *be a Hilbert space,* C *be a nonempty closed convex subset of* H*, and* $f: H \to H$ *be a contraction with a coefficient* $0 < \alpha < 1$*, and let* A *be a strongly positive linear bounded operator with a coefficient* $\bar{\gamma} > 0$*. Then, for* $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$,

$$\langle x - y, (A - \gamma f)x - (A - \gamma f)y \rangle \ge (\bar{\gamma} - \gamma\alpha)\|x - y\|^2, \quad x, y \in H.$$

That is, $A - \gamma f$ *is strongly monotone with a coefficient* $\bar{\gamma} - \gamma\alpha$.

Lemma 2.8 ([7]) *Assume* A *is a strongly positive linear bounded operator on a Hilbert space* H *with a coefficient* $\bar{\gamma} > 0$ *and* $0 < \rho \le \|A\|^{-1}$*. Then* $\|I - \rho A\| \le 1 - \rho\bar{\gamma}$.

Lemma 2.9 ([9] and [15]) *Let* C *be a nonempty closed convex subset of a Banach space* E*. Let* $\{T_i\}_{i=1}^{\infty}$ *be a sequence of nonexpansive mappings of* C *into itself with* $\bigcap_{i=1}^{\infty} F(T_i) \ne \emptyset$*, and let* $\{\lambda_i\}_{i=1}^{\infty}$ *be a real sequence such that* $0 < \lambda_i \le b < 1$, $\forall i \ge 1$*. Then*:

(1) W_n *is nonexpansive and* $F(W_n) = \bigcap_{i=1}^{\infty} F(T_i)$ *for each* $n \ge 1$;

(2) *for each* $x \in C$ *and for each positive integer* k*, the* $\lim_{n\to\infty} U_{n,k}x$ *exists*;

(3) *the mapping* $U: C \to C$ *defined by*

$$Ux = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad x \in C$$

is a nonexpansive mapping satisfying $F(U) = \bigcap_{n=1}^{\infty} F(T_i)$ *and it is called the* W*-mapping generated by* $T_1, T_2, \dots$ *and* $\lambda_1, \lambda_2, \dots$;

(4) $\lim_{m,n\to\infty} \sup_{x\in K} \|W_m x - W_n x\| = 0$ *for any bounded subset* K *of* E.

3 Main results

In this section, we introduce our algorithm and prove its strong convergence.

Theorem 3.1 *Let* C *be a closed convex subset of a real Hilbert space* H*. Let* F *be a bifunction from* $H \times H$ *into* $\mathbb{R}$ *satisfying* (A1)-(A4)*. Let* f *be a contraction of* H *into itself with* $\alpha \in (0,1)$*, and let* T_n *be a sequence of nonexpansive mappings of* C *into itself such that* $\Omega = \bigcap_{n=1}^{\infty} F(T_n) \cap EP(F) \ne \emptyset$*. Let* $A: H \to H$ *be a strongly positive bounded linear operator with a coefficient* $\bar{\gamma} > 0$ *with* $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$*. Let* $\lambda_1, \lambda_2, \dots$ *be a sequence of real numbers such that* $0 < \lambda_n \le b < 1$ *for every* $n = 1, 2, \dots$*. Let* W_n *be a* W*-mapping of* C *into itself generated by* $T_n, T_{n-1}, \dots, T_1$ *and* $\lambda_n, \lambda_{n-1}, \dots, \lambda_1$*. Let* U *be defined by* $Ux = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$ *for every* $x \in C$*. Let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by* $x_1 \in H$ *and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \ge 0, & \forall y \in H, \\ x_{n+1} = P_C[\alpha_n \gamma f(x_n) + (I - \alpha_n A)W_n u_n], & \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *and* $\{r_n\}$ *is a sequence in* $[0,\infty)$*. Suppose that* $\{\alpha_n\}$ *and* $\{r_n\}$ *satisfy the following conditions*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $\lim_{n\to\infty} r_n = r > 0$.

Then both $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $x^* \in \Omega$, *which is the unique solution of the variational inequality*

$$\langle (A - \gamma f)x^*, x^* - \hat{x}\rangle \geq 0, \quad \forall \hat{x} \in \Omega. \tag{3.2}$$

Equivalently, one has $x^* = P_\Omega(I - A + \gamma f)(x^*)$.

Proof We observe that $P_\Omega(\gamma f + (I - A))$ is a contraction. Indeed, for all $x, y \in H$, we have

$$\begin{aligned} \|P_\Omega(\gamma f + (I-A))(x) - P_\Omega(\gamma f + (I-A))(y)\| &\leq \|(\gamma f + (I-A))(x) - (\gamma f + (I-A))(y)\| \\ &\leq \gamma\|f(x) - f(y)\| + \|I - A\|\|x - y\| \\ &\leq \gamma\alpha\|x - y\| + (1 - \overline{\gamma})\|x - y\| \\ &< (1 - (\overline{\gamma} - \gamma\alpha))\|x - y\|. \end{aligned}$$

Banach's contraction mapping principle guarantees that $P_\Omega(\gamma f + (I - A))$ has a unique fixed point, say $x^* \in H$. That is, $x^* = P_\Omega(\gamma f + (I - A))(x^*)$. Note that by Lemma 2.4, we can write

$$x_{n+1} = P_C[\alpha_n\gamma f(x_n) + (I - \alpha_n A)W_n T_{r_n} x_n],$$

where

$$T_{r_n}(x) = \left\{ z \in H : F(z, y) + \frac{1}{r_n}\langle y - z, z - x\rangle \geq 0, \forall y \in H \right\}.$$

Moreover, since $\alpha_n \to 0$ as $n \to \infty$ by condition (C1), we assume that $\alpha_n \leq \|A\|^{-1}$ for all $n \in \mathbb{N}$. From Lemma 2.8, we know that if $0 < \rho < \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$. We divide the proof into seven steps as follows.

Step 1. Show that the sequences $\{x_n\}$ and $\{u_n\}$ are bounded.

Let $\hat{x} \in \Omega$. Then $\hat{x} \in EP(F)$. From Lemma 2.4, we have

$$\|u_n - \hat{x}\| = \|T_{r_n} x_n - T_{r_n}\hat{x}\| \leq \|x_n - \hat{x}\|.$$

Thus, we have

$$\begin{aligned} \|x_{n+1} - \hat{x}\| &= \|P_C[\alpha_n\gamma f(x_n) + (I - \alpha_n A)W_n u_n] - \hat{x}\| \\ &\leq \|\alpha_n\gamma f(x_n) + (I - \alpha_n A)W_n u_n - \hat{x}\| \\ &\leq \alpha_n\gamma\|f(x_n) - f(\hat{x})\| + \|I - \alpha_n A\|\|W_n u_n - \hat{x}\| + \alpha_n\|\gamma f(\hat{x}) - A\hat{x}\| \\ &\leq \alpha_n\gamma\alpha\|x_n - \hat{x}\| + (1 - \alpha_n\bar{\gamma})\|u_n - \hat{x}\| + \alpha_n\|\gamma f(\hat{x}) - A\hat{x}\| \\ &= (1 - \alpha_n(\bar{\gamma} - \gamma\alpha))\|x_n - \hat{x}\| + \alpha_n(\bar{\gamma} - \gamma\alpha)\frac{\|\gamma f(\hat{x}) - A\hat{x}\|}{(\bar{\gamma} - \gamma\alpha)}. \end{aligned}$$

By induction, we have

$$\|x_n - \hat{x}\| \le \max\left\{\|x_1 - \hat{x}\|, \frac{\|\gamma f(\hat{x}) - A\hat{x}\|}{\bar{\gamma} - \gamma\alpha}\right\}, \quad \forall n \ge 0.$$

This shows that the sequence $\{x_n\}$ is bounded, so are $\{u_n\}$, $\{f(x_n)\}$ and $\{W_n u_n\}$.

Step 2. Show that $\|W_{n+1}u_n - W_n u_n\| \to 0$ as $n \to \infty$.

Let $\hat{x} \in \Omega$. Since T_i and $U_{n,i}$ are nonexpansive and $T_i\hat{x} = \hat{x} = U_{n,i}\hat{x}$ for every $n \in \mathbb{N}$ and $i \le n+1$, it follows that

$$\begin{aligned}
\|W_{n+1}u_n - W_n u_n\| &= \|\lambda_1 T_1 U_{n+1,2}u_n - \lambda_1 T_1 U_{n,2}u_n\| \\
&\le \lambda_1 \|U_{n+1,2}u_n - U_{n,2}u_n\| \\
&= \lambda_1 \|\lambda_2 T_2 U_{n+1,3}u_n - \lambda_2 T_2 U_{n,3}u_n\| \\
&\le \lambda_1\lambda_2 \|U_{n+1,3}u_n - U_{n,3}u_n\| \\
&\vdots \\
&\le \left(\prod_{i=1}^{n} \lambda_i\right) \|U_{n+1,n+1}u_n - \hat{x} + \hat{x} - U_{n,n+1}u_n\| \\
&\le \left(\prod_{i=1}^{n} \lambda_i\right) \big(\|U_{n+1,n+1}u_n - \hat{x} + \hat{x} - U_{n,n+1}u_n\|\big) \\
&\le 2\left(\prod_{i=1}^{n} \lambda_i\right) \|u_n - \hat{x}\|.
\end{aligned}$$

Since $\{u_n\}$ is bounded and $0 < \lambda_n \le b < 1$ for any $n \in \mathbb{N}$, the following holds:

$$\lim_{n\to\infty} \|W_{n+1}u_n - W_n u_n\| = 0.$$

Step 3. Show that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

Setting $S = 2P_C - I$, we have S is nonexpansive. Note that $W_n = (1-\lambda_1)I + \lambda_1 T_1 U_{n,2}$. Then we can write

$$\begin{aligned}
x_{n+1} &= \frac{I+S}{2}\big[\alpha_n \gamma f(x_n) + (I - \alpha_n A)W_n u_n\big] \\
&= \frac{1-\alpha_n}{2} W_n u_n + \frac{\alpha_n}{2}\big(\gamma f(x_n) - AW_n u_n + W_n u_n\big) \\
&\quad + \frac{1}{2} S\big[\alpha_n \gamma f(x_n) + (I - \alpha_n A)W_n u_n\big] \\
&= \frac{1-\alpha_n}{2}\big[(1-\lambda_1)I + \lambda_1 T_1 U_{n,2}\big]u_n + \frac{\alpha_n}{2}\big(\gamma f(x_n) - AW_n u_n + W_n u_n\big) \\
&\quad + \frac{1}{2} S\big[\alpha_n \gamma f(x_n) + (I - \alpha_n A)W_n u_n\big] \\
&= \frac{(1-\lambda_1)(1-\alpha_n)}{2} u_n + \frac{\lambda_1(1-\alpha_n)}{2} T_1 U_{n,2}u_n + \frac{\alpha_n}{2}\big(\gamma f(x_n) - AW_n u_n + W_n u_n\big) \\
&\quad + \frac{1}{2} S\big[\alpha_n \gamma f(x_n) + (I - \alpha_n A)W_n u_n\big].
\end{aligned} \tag{3.3}$$

Note that

$$0 < \lim_{n\to\infty} \frac{(1-\lambda_1)(1-\alpha_n)}{2} = \frac{1-\lambda_1}{2} < 1,$$

and

$$\frac{\lambda_1(1-\alpha_n)}{2} + \frac{1}{2} = \frac{1+\lambda_1}{2} - \frac{\lambda_1}{2}\alpha_n.$$

From (3.3), we have

$$\begin{aligned} x_{n+1} &= \left[1 - \left(\frac{1+\lambda_1}{2} + \frac{1-\lambda_1}{2}\alpha_n\right)\right]u_n + \left(\frac{1+\lambda_1}{2} + \frac{1-\lambda_1}{2}\alpha_n\right) \\ &\quad \times \left(\frac{\lambda_1(1-\alpha_n)}{2} T_1 U_{n,2}u_n + \frac{\alpha_n}{2}\big(\gamma f(x_n) - AW_n u_n + W_n u_n\big)\right. \\ &\quad \left. + \frac{1}{2} S\big[\alpha_n \gamma f(x_n) + (I-\alpha_n A)W_n u_n\big]\right) \Big/ \left(\frac{1+\lambda_1}{2} + \frac{1-\lambda_1}{2}\alpha_n\right) \\ &= \left[1 - \left(\frac{1+\lambda_1}{2} + \frac{1-\lambda_1}{2}\alpha_n\right)\right]u_n + \left(\frac{1+\lambda_1}{2} + \frac{1-\lambda_1}{2}\alpha_n\right) y_n, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} y_n &= \left(\frac{\lambda_1(1-\alpha_n)}{2} T_1 U_{n,2}u_n + \frac{\alpha_n}{2}\big(\gamma f(x_n) - AW_n u_n + W_n u_n\big)\right. \\ &\quad \left. + \frac{1}{2} S\big[\alpha_n \gamma f(x_n) + (I-\alpha_n A)W_n u_n\big]\right) \Big/ \left(\frac{1+\lambda_1}{2} + \frac{1-\lambda_1}{2}\alpha_n\right) \\ &= \big(\lambda_1(1-\alpha_n)T_1 U_{n,2}u_n + \alpha_n\big(\gamma f(x_n) - AW_n u_n + W_n u_n\big) \\ &\quad + S\big[\alpha_n \gamma f(x_n) + (I-\alpha_n A)W_n u_n\big]\big) / \big(1 + \lambda_1 + (1-\lambda_1)\alpha_n\big). \end{aligned}$$

Set $e_n = \gamma f(x_n) - AW_n u_n + W_n u_n$ and $d_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A)W_n u_n$ for all n. Then

$$y_n = \frac{\lambda_1(1-\alpha_n)T_1 U_{n,2}u_n + \alpha_n e_n + Sd_n}{1+\lambda_1+(1-\lambda_1)\alpha_n}, \quad \forall n \geq 0.$$

It follows that

$$\begin{aligned} y_{n+1} - y_n &= \frac{\lambda_1(1-\alpha_{n+1})T_1 U_{n+1,2}u_{n+1} + \alpha_{n+1}e_{n+1} + Sd_{n+1}}{1+\lambda_1+(1-\lambda_1)\alpha_{n+1}} \\ &\quad - \frac{\lambda_1(1-\alpha_n)T_1 U_{n,2}u_n + \alpha_n e_n + Sd_n}{1+\lambda_1+(1-\lambda_1)\alpha_n} \\ &= \frac{\lambda_1(1-\alpha_{n+1})}{1+\lambda_1+(1-\lambda_1)\alpha_{n+1}}(T_1 U_{n+1,2}u_{n+1} - T_1 U_{n,2}u_n) \\ &\quad + \left(\frac{\lambda_1(1-\alpha_{n+1})}{1+\lambda_1+(1-\lambda_1)\alpha_{n+1}} - \frac{\lambda_1(1-\alpha_n)}{1+\lambda_1+(1-\lambda_1)\alpha_n}\right) T_1 U_{n,2}u_n \\ &\quad + \frac{\alpha_{n+1}e_{n+1}}{1+\lambda_1+(1-\lambda_1)\alpha_{n+1}} - \frac{\alpha_n e_n}{1+\lambda_1+(1-\lambda_1)\alpha_n} \\ &\quad + \frac{Sd_{n+1} - Sd_n}{1+\lambda_1+(1-\lambda_1)\alpha_{n+1}} + \left(\frac{1}{1+\lambda_1+(1-\lambda_1)\alpha_{n+1}} - \frac{1}{1+\lambda_1+(1-\lambda_1)\alpha_n}\right) Sd_n. \end{aligned}$$

Thus,

$$\begin{aligned}
\|y_{n+1}-y_n\| &\le \frac{\lambda_1(1-\alpha_{n+1})}{1+\lambda_1+(1-\lambda_1)\alpha_{n+1}}\|T_1U_{n+1,2}u_{n+1}-T_1U_{n,2}u_n\| \\
&\quad + \left|\frac{\lambda_1(1-\alpha_{n+1})}{1+\lambda_1+(1-\lambda_1)\alpha_{n+1}} - \frac{\lambda_1(1-\alpha_n)}{1+\lambda_1+(1-\lambda_1)\alpha_n}\right|\|T_1U_{n,2}u_n\| \\
&\quad + \frac{\alpha_{n+1}}{1+\lambda_1+(1-\lambda_1)\alpha_{n+1}}\|e_{n+1}\| + \frac{\alpha_n}{1+\lambda_1+(1-\lambda_1)\alpha_n}\|e_n\| \\
&\quad + \frac{1}{1+\lambda_1+(1-\lambda_1)\alpha_{n+1}}\|Sd_{n+1}-Sd_n\| \\
&\quad + \left|\frac{1}{1+\lambda_1+(1-\lambda_1)\alpha_{n+1}} - \frac{1}{1+\lambda_1+(1-\lambda_1)\alpha_n}\right|\|Sd_n\|.
\end{aligned}$$

Since S is nonexpansive, we obtain that

$$\begin{aligned}
\|Sd_{n+1}-Sd_n\| &\le \|d_{n+1}-d_n\| \\
&= \big\|\alpha_{n+1}\gamma f(x_{n+1}) + (I-\alpha_{n+1}A)W_{n+1}u_{n+1} - \big(\alpha_n\gamma f(x_n) + (I-\alpha_nA)W_nu_n\big)\big\| \\
&\le \alpha_{n+1}\big\|\gamma f(x_{n+1}) - AW_{n+1}u_{n+1}\big\| + \alpha_n\big\|\gamma f(x_n) - AW_nu_n)\big\| \\
&\quad + \|W_{n+1}u_{n+1} - W_nu_n\| \\
&\le \alpha_{n+1}\big\|\gamma f(x_{n+1}) - AW_{n+1}u_{n+1}\big\| + \alpha_n\big\|\gamma f(x_n) - AW_nu_n)\big\| \\
&\quad + \|W_{n+1}u_{n+1} - W_{n+1}u_n\| + \|W_{n+1}u_n - W_nu_n\| \\
&\le \alpha_{n+1}\big\|\gamma f(x_{n+1}) - AW_{n+1}u_{n+1}\big\| + \alpha_n\big\|\gamma f(x_n) - AW_nu_n)\big\| + \|u_{n+1}-u_n\|. \\
&\quad + \|W_{n+1}u_n - W_nu_n\|.
\end{aligned}$$

Since T_i and $U_{n,i}$ are nonexpansive, we have

$$\begin{aligned}
\|T_1U_{n+1,2}u_n - T_1U_{n,2}u_n\| &\le \|U_{n+1,2}u_n - U_{n,2}u_n\| \\
&= \|\lambda_2T_2U_{n+1,3}u_n - \lambda_2T_2U_{n,3}u_n\| \\
&\le \lambda_2\|U_{n+1,3}u_n - U_{n,3}u_n\| \\
&\le \cdots \\
&\le \lambda_2\cdots\lambda_n\|U_{n+1,n+1}u_n - U_{n,n+1}u_n\| \\
&\le M\prod_{i=2}^{n}\lambda_i,
\end{aligned}$$

where $M > 0$ is a constant such that $\|U_{n+1,n+1}u_n - U_{n,n+1}u_n\| \le M$ for all $n \ge 0$. So,

$$\begin{aligned}
\|T_1U_{n+1,2}u_{n+1} - T_1U_{n,2}u_n\| &\le \|T_1U_{n+1,2}u_{n+1} - T_1U_{n+1,2}u_n\| + \|T_1U_{n+1,2}u_n - T_1U_{n,2}u_n\| \\
&\le \|u_{n+1}-u_n\| + M\prod_{i=2}^{n}\lambda_i.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|y_{n+1}-y_n\| &\le \frac{\lambda_1(1-\alpha_{n+1})}{1+\lambda_1+(1-\lambda_1)\alpha_{n+1}}\|u_{n+1}-u_n\| + M\prod_{i=2}^{n}\lambda_i \\
&\quad + \left|\frac{\lambda_1(1-\alpha_{n+1})}{1+\lambda_1+(1-\lambda_1)\alpha_{n+1}} - \frac{\lambda_1(1-\alpha_n)}{1+\lambda_1+(1-\lambda_1)\alpha_n}\right|\|T_1U_{n,2}u_n\| \\
&\quad + \frac{\alpha_{n+1}}{1+\lambda_1+(1-\lambda_1)\alpha_{n+1}}\|e_{n+1}\| + \frac{\alpha_n}{1+\lambda_1+(1-\lambda_1)\alpha_n}\|e_n\| \\
&\quad + \frac{1}{1+\lambda_1+(1-\lambda_1)\alpha_{n+1}}\big(\alpha_{n+1}\big\|\gamma f(x_{n+1})-AW_{n+1}u_{n+1}\big\| \\
&\quad + \alpha_n\big\|\gamma f(x_n)-AW_nu_n\big\| + \|u_{n+1}-u_n\|\big) + \|W_{n+1}u_n - W_nu_n\| \\
&\quad + \left|\frac{1}{1+\lambda_1+(1-\lambda_1)\alpha_{n+1}} - \frac{1}{1+\lambda_1+(1-\lambda_1)\alpha_n}\right|\|Sd_n\| \\
&= \frac{\lambda_1(1-\alpha_{n+1})}{1+\lambda_1+(1-\lambda_1)\alpha_{n+1}}\|u_{n+1}-u_n\| \\
&\quad + \left|\frac{\lambda_1(1-\alpha_{n+1})}{1+\lambda_1+(1-\lambda_1)\alpha_{n+1}} - \frac{\lambda_1(1-\alpha_n)}{1+\lambda_1+(1-\lambda_1)\alpha_n}\right|\|T_1U_{n,2}u_n\| \\
&\quad + \frac{\alpha_{n+1}}{1+\lambda_1+(1-\lambda_1)\alpha_{n+1}}\|e_{n+1}\| + \frac{\alpha_n}{1+\lambda_1+(1-\lambda_1)\alpha_n}\|e_n\| \\
&\quad + \frac{1}{1+\lambda_1+(1-\lambda_1)\alpha_{n+1}}\big(\alpha_{n+1}\big\|\gamma f(x_{n+1})-AW_{n+1}u_{n+1}\big\| \\
&\quad + \alpha_n\big\|\gamma f(x_n)-AW_nu_n\big\| + \|u_{n+1}-u_n\|\big) + \|W_{n+1}u_n - W_nu_n\| \\
&\quad + \left|\frac{1}{1+\lambda_1+(1-\lambda_1)\alpha_{n+1}} - \frac{1}{1+\lambda_1+(1-\lambda_1)\alpha_n}\right|\|Sd_n\|.
\end{aligned}$$

Note that:

(1) By condition (C1), we have

$$\frac{\lambda_1(1-\alpha_{n+1})}{1+\lambda_1+(1-\lambda_1)\alpha_{n+1}} - \frac{\lambda_1(1-\alpha_n)}{1+\lambda_1+(1-\lambda_1)\alpha_n} \to 0$$

and

$$\frac{1}{1+\lambda_1+(1-\lambda_1)\alpha_{n+1}} - \frac{1}{1+\lambda_1+(1-\lambda_1)\alpha_n} \to 0.$$

(2) $\|W_{n+1}u_n - W_nu_n\| \to 0$ as $n \to \infty$ because of Step 2.

Therefore,

$$\limsup_{n\to\infty}\big(\|y_{n+1}-y_n\| - \|u_{n+1}-u_n\|\big) \le 0.$$

By Lemma 2.2, we get

$$\lim_{n\to\infty}\|y_n-u_n\| = 0.$$

Hence, from (3.4), we deduce

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}\left(\frac{1+\lambda_1}{2}+\frac{1-\lambda_1}{2}\alpha_n\right)\|y_n-u_n\| = 0. \tag{3.5}$$

Step 4. Show that $\|x_n - W_nu_n\| \to 0$ as $n\to\infty$. Indeed, we have

$$\begin{aligned}
\|x_n - W_nu_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - W_nu_n\| \\
&= \|x_n - x_{n+1}\| + \big\|P_C\big[\alpha_n\gamma f(x_n) + (I-\alpha_nA)W_nu_n\big] - W_nu_n\big\| \\
&\le \|x_n - x_{n+1}\| + \big\|\alpha_n\gamma f(x_n) + (I-\alpha_nA)W_nu_n - W_nu_n\big\| \\
&= \|x_n - x_{n+1}\| + \big\|-\alpha_nAW_nu_n + \alpha_n\gamma f(x_n)\big\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\big(\|A\|\|W_nu_n\| + \gamma\big\|f(x_n)\big\|\big).
\end{aligned}$$

Then

$$\lim_{n\to\infty}\|x_n - W_nu_n\| \le \lim_{n\to\infty}\|x_n - x_{n+1}\| + \alpha_n\big(\|A\|\|W_nu_n\| + \gamma\big\|f(x_n)\big\|\big) = 0. \tag{3.6}$$

Thus, from (3.6), we obtain

$$\lim_{n\to\infty}\|x_n - W_nu_n\| = 0.$$

Step 5. Show that $\|x_n - u_n\| \to 0$ as $n\to\infty$.

Let $\hat{x}\in\Omega$. Since T_{r_n} is firmly nonexpansive, it follows

$$\begin{aligned}
\|\hat{x}-u_n\|^2 &= \|T_{r_n}\hat{x} - T_{r_n}x_n\|^2 \\
&\le \langle T_{r_n}x_n - T_{r_n}\hat{x}, x_n-\hat{x}\rangle \\
&\le \langle T_{r_n}x_n - \hat{x}, x_n-\hat{x}\rangle \\
&= \langle u_n - \hat{x}, x_n-\hat{x}\rangle \\
&= \frac{1}{2}\big(\|u_n-\hat{x}\|^2 + \|x_n-\hat{x}\|^2 - \|x_n-u_n\|^2\big).
\end{aligned}$$

Then

$$\|u_n-\hat{x}\|^2 \le \|x_n-\hat{x}\|^2 - \|x_n-u_n\|^2.$$

Since we have

$$\begin{aligned}
\|x_{n+1}-\hat{x}\|^2 &= \big\|P_C\big[\alpha_n\gamma f(x_n) + (I-\alpha_nA)W_nu_n\big] - P_C\hat{x}\big\|^2 \\
&= \big\|\alpha_n\gamma f(x_n) + (I-\alpha_nA)W_nu_n - \hat{x}\big\|^2 \\
&= \big\|(I-\alpha_nA)(W_nu_n-\hat{x}) + \alpha_n\big(\gamma f(x_n) - A\hat{x}\big)\big\|^2 \\
&\le (1-\alpha_n\bar{\gamma})^2\|W_nu_n-\hat{x}\|^2 + 2\alpha_n\big\langle\gamma f(x_n) - A\hat{x}, x_{n+1}-\hat{x}\big\rangle \\
&\le (1-\alpha_n\bar{\gamma})^2\|u_n-\hat{x}\|^2 + 2\alpha_n\gamma\big\langle f(x_n) - f(\hat{x}), x_{n+1}-\hat{x}\big\rangle
\end{aligned}$$

$$\begin{aligned}
&+ 2\alpha_n\langle \gamma f(\hat{x}) - A\hat{x}, x_{n+1} - \hat{x}\rangle \\
&\le (1-\alpha_n\bar{\gamma})^2\big(\|x_n - \hat{x}\|^2 - \|x_n - u_n\|^2\big) + 2\alpha_n\gamma\alpha\|x_n - \hat{x}\|\|x_{n+1} - \hat{x}\| \\
&\quad + 2\alpha_n\big\|\gamma f(\hat{x}) - A\hat{x}\big\|\|x_{n+1} - \hat{x}\| \\
&= \big(1 - 2\alpha_n\bar{\gamma} + (\alpha_n\bar{\gamma})^2\big)\|x_n - \hat{x}\|^2 - (1-\alpha_n\bar{\gamma})^2\|x_n - u_n\|^2 \\
&\quad + 2\alpha_n\gamma\alpha\|x_n - \hat{x}\|\|x_{n+1} - \hat{x}\| + 2\alpha_n\big\|\gamma f(\hat{x}) - A\hat{x}\big\|\|x_{n+1} - \hat{x}\| \\
&\le \|x_n - \hat{x}\|^2 + \alpha_n\bar{\gamma}^2\|x_n - \hat{x}\|^2 - (1-\alpha_n\bar{\gamma})^2\|x_n - u_n\|^2 \\
&\quad + 2\alpha_n\gamma\alpha\|x_n - \hat{x}\|\|x_{n+1} - \hat{x}\| + 2\alpha_n\big\|\gamma f(\hat{x}) - A\hat{x}\big\|\|x_{n+1} - \hat{x}\|,
\end{aligned}$$

and hence

$$\begin{aligned}
(1-\alpha_n\bar{\gamma})^2\|x_n - u_n\|^2 &\le \|x_n - \hat{x}\|^2 - \|x_{n+1} - \hat{x}\|^2 + \alpha_n\bar{\gamma}^2\|x_n - \hat{x}\|^2 \\
&\quad + 2\alpha_n\gamma\alpha\|x_n - \hat{x}\|\|x_{n+1} - \hat{x}\| + 2\alpha_n\big\|\gamma f(\hat{x}) - A\hat{x}\big\|\|x_{n+1} - \hat{x}\| \\
&\le \|x_n - x_{n+1}\|\big(\|x_n - \hat{x}\|\|x_{n+1} - \hat{x}\|\big) + \alpha_n\bar{\gamma}^2\|x_n - \hat{x}\|^2 \\
&\quad + 2\alpha_n\gamma\alpha\|x_n - \hat{x}\|\|x_{n+1} - \hat{x}\| + 2\alpha_n\big\|\gamma f(\hat{x}) - A\hat{x}\big\|\|x_{n+1} - \hat{x}\|.
\end{aligned}$$

Therefore, we have $\|x_n - u_n\| \to 0$ as $n \to \infty$.

Step 6. Show that $\limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, x_n - x^*\rangle \le 0$.

We can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty}\big\langle \gamma f\big(x^*\big) - Ax^*, x_{n_i} - x^*\big\rangle = \limsup_{n\to\infty}\big\langle \gamma f\big(x^*\big) - Ax^*, x_n - x^*\big\rangle.$$

Let

$$A(x_{n_i}) = \Big\{x \in H : \limsup_{i\to\infty}\|x_{n_i} - x\| = \inf_{y\in H}\limsup_{i\to\infty}\|x_{n_i} - y\|\Big\}$$

be the asymptotic center of $\{x_{n_i}\}$. Since $\{x_{n_i}\}$ is bounded and H is a Hilbert space, it is well known that $A(x_{n_i})$ is a singleton; say $A(x_{n_i}) = \{\tilde{x}\}$. Set

$$L = \sup_{i\in\mathbb{N}}\big\|\gamma f(x_{n_i}) - AW_{n_i}u_{n_i}\big\|$$

and for every $x \in H$ define

$$Wx = \lim_{i\to\infty} W_{n_i}x \tag{3.7}$$

and

$$T_r(x) = \Big\{z \in H : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in H\Big\}.$$

Note that

$$\begin{aligned}
\|x_{n_i} - W\tilde{x}\| &\le \|x_{n_i+1} - x_{n_i}\| + \|x_{n_i+1} - W\tilde{x}\| \\
&= \|x_{n_i+1} - x_{n_i}\| + \big\|P_C\big[\alpha_{n_i}\gamma f(x_{n_i}) + (I - \alpha_{n_i}A)W_{n_i}u_{n_i}\big] - W\tilde{x}\big\|
\end{aligned}$$

$$\begin{aligned}
&\le \|x_{n_i+1} - x_{n_i}\| + \left\|\alpha_{n_i}\gamma f(x_{n_i}) + (I - \alpha_{n_i}A)W_{n_i}u_{n_i} - W\tilde{x}\right\| \\
&= \|x_{n_i+1} - x_{n_i}\| + \left\|W_{n_i}u_{n_i} - W\tilde{x} + \alpha_{n_i}\big(\gamma f(x_{n_i}) - AW_{n_i}u_{n_i}\big)\right\| \\
&\le \|x_{n_i+1} - x_{n_i}\| + \left\|W_{n_i}u_{n_i} - W_{n_i}x_{n_i}\right\| + \|W_{n_i}x_{n_i} - W_{n_i}\tilde{x}\| \\
&\quad + \|W_{n_i}\tilde{x} - W\tilde{x}\| + \alpha_{n_i}L \\
&\le \|x_{n_i+1} - x_{n_i}\| + \|u_{n_i} - x_{n_i}\| + \|x_{n_i} - \tilde{x}\| + \|W_{n_i}\tilde{x} - W\tilde{x}\| + \alpha_{n_i}L.
\end{aligned}$$

By Steps 1-5, condition (C1) and (3.7), we derive

$$\begin{aligned}
\limsup_{i\to\infty} \|x_{n_i} - W\tilde{x}\| &\le \limsup_{i\to\infty} \|u_{n_i} - x_{n_i}\| + \|x_{n_i} - \tilde{x}\| + \|W_{n_i}\tilde{x} - W\tilde{x}\| \\
&\le \limsup_{i\to\infty} \|x_{n_i} - \tilde{x}\|.
\end{aligned}$$

That is, $W\tilde{x} \in A(x_{n_i})$. Therefore $W\tilde{x} = \tilde{x}$. Next, we show that $\tilde{x} = T_r\tilde{x}$.

Note that for any $x \in H$ and $a, b > 0$, we have

$$F(T_a x, y) + \frac{1}{a}\langle y - T_a x, T_a x - x\rangle \ge 0, \quad \forall y \in H$$

and

$$F(T_b x, y) + \frac{1}{b}\langle y - T_b x, T_b x - x\rangle \ge 0, \quad \forall y \in H,$$

then

$$F(T_a x, T_b x) + \frac{1}{a}\langle T_b x - T_a x, T_a x - x\rangle \ge 0$$

and

$$F(T_b x, T_a x) + \frac{1}{b}\langle T_a x - T_b x, T_a x - x\rangle \ge 0.$$

Summing up the last inequalities and using (A2), we obtain

$$\left\langle T_a x - T_b x, \frac{T_b x - x}{b} - \frac{T_a x - x}{a}\right\rangle \ge 0.$$

Hence we have

$$\begin{aligned}
0 &\le \left\langle T_a x - T_b x, T_b x - x - \frac{b}{a}(T_a x - x)\right\rangle \\
&= \left\langle T_a x - T_b x, T_b x - T_a x + T_a x - x - \frac{b}{a}(T_a x - x)\right\rangle \\
&= \left\langle T_a x - T_b x, (T_b x - T_a x) + \left(1 - \frac{b}{a}\right)(T_a x - x)\right\rangle \\
&\le -\|T_a x - T_b x\|^2 + \left|1 - \frac{b}{a}\right| \|T_a x - T_b x\| \big(\|T_a x\| + \|x\|\big).
\end{aligned}$$

We derive then

$$\|T_a x - T_b x\| \le \frac{|b-a|}{a}\bigl(\|T_a x\| + \|x\|\bigr).$$

It follows that

$$\begin{aligned}
\|x_{n_i} - T_r\tilde{x}\| &\le \|x_{n_i+1} - x_{n_i}\| + \|x_{n_i+1} - T_r\tilde{x}\| \\
&= \|x_{n_i+1} - x_{n_i}\| + \bigl\|P_C\bigl[\alpha_{n_i}\gamma f(x_{n_i}) + (I - \alpha_{n_i}A)W_{n_i}u_{n_i}\bigr] - T_r\tilde{x}\bigr\| \\
&\le \|x_{n_i+1} - x_{n_i}\| + \bigl\|\alpha_{n_i}\gamma f(x_{n_i}) + (I - \alpha_{n_i}A)W_{n_i}u_{n_i} - T_r\tilde{x}\bigr\| \\
&= \|x_{n_i+1} - x_{n_i}\| + \bigl\|W_{n_i}u_{n_i} - T_r\tilde{x} + \alpha_{n_i}\bigl(\gamma f(x_{n_i}) - AW_{n_i}u_{n_i}\bigr)\bigr\| \\
&\le \|x_{n_i+1} - x_{n_i}\| + \|W_{n_i}u_{n_i} - x_{n_i}\| + \|T_{r_{n_i}}x_{n_i} - T_{r_{n_i}}\tilde{x}\| + \|x_{n_i} - T_{r_{n_i}}x_{n_i}\| \\
&\quad + \|T_{r_{n_i}}\tilde{x} - T_r\tilde{x}\| + \alpha_{n_i}L \\
&\le \|x_{n_i+1} - x_{n_i}\| + \|u_{n_i} - x_{n_i}\| + \|x_{n_i} - \tilde{x}\| + \|x_{n_i} - u_{n_i}\| \\
&\quad + \|T_{r_{n_i}}\tilde{x} - T_r\tilde{x}\| + \alpha_{n_i}L \\
&\le \|x_{n_i+1} - x_{n_i}\| + \|u_{n_i} - x_{n_i}\| + \|x_{n_i} - \tilde{x}\| + \|x_{n_i} - u_{n_i}\| \\
&\quad + \frac{|r_{n_i} - r|}{r}\bigl(\|T_r\tilde{x}\| + \|\tilde{x}\|\bigr) + \alpha_{n_i}L.
\end{aligned}$$

By Steps 2-5, conditions (C1) and (C3), we obtain

$$\limsup_{i\to\infty}\|x_{n_i} - T_r\tilde{x}\| \le \limsup_{i\to\infty}\|x_{n_i} - \tilde{x}\|$$

and $\tilde{x} = T_r\tilde{x}$. Thus $\tilde{x} \in F(W) \cap F(T_r) = \Omega$ by Lemma 2.3 and 2.9. Fix $t \in (0,1)$, $x \in H$ and set $y = \tilde{x} + tx$. Then

$$\|x_{n_i} - \tilde{x} - tx\|^2 \le \|x_{n_i} - \tilde{x}\|^2 + 2t\langle x, \tilde{x} + tx - x_{n_i}\rangle.$$

By the minimizing property of $\tilde{x}$ and since $\|\cdot\|^2$ is continuous and increasing in $[0,\infty)$, we have

$$\begin{aligned}
\limsup_{i\to\infty}\|x_{n_i} - \tilde{x}\|^2 &\le \limsup_{i\to\infty}\|x_{n_i} - \tilde{x} - tx\|^2 \\
&\le \limsup_{i\to\infty}\|x_{n_i} - \tilde{x}\|^2 + 2t\limsup_{i\to\infty}\langle x, \tilde{x} + tx - x_{n_i}\rangle.
\end{aligned}$$

Thus,

$$\limsup_{i\to\infty}\langle x, \tilde{x} + tx - x_{n_i}\rangle \ge 0.$$

On the other hand,

$$\langle x, \tilde{x} - x_{n_i}\rangle = \langle x, \tilde{x} + tx - x_{n_i}\rangle - t\|x\|^2.$$

Hence we obtain

$$\limsup_{i\to\infty}\langle x, \tilde{x} - x_{n_i}\rangle = \lim_{t\to 0}\Bigl(\limsup_{i\to\infty}\langle x, \tilde{x} + tx - x_{n_i}\rangle - t\|x\|^2\Bigr) \geq 0.$$

Set $x = \gamma f(x^*) - Ax^*$. Since $\tilde{x} \in \Omega$, we obtain

$$\begin{aligned}
0 &\leq \limsup_{i\to\infty}\bigl\langle \gamma f\bigl(x^*\bigr) - Ax^*, \tilde{x} - x_{n_i}\bigr\rangle \\
&\leq \bigl\langle \gamma f\bigl(x^*\bigr) - Ax^*, \tilde{x} - x^*\bigr\rangle + \lim_{i\to\infty}\bigl\langle \gamma f\bigl(x^*\bigr) - Ax^*, x^* - x_{n_i}\bigr\rangle \\
&\leq \lim_{i\to\infty}\bigl\langle \gamma f\bigl(x^*\bigr) - Ax^*, x^* - x_{n_i}\bigr\rangle.
\end{aligned}$$

So that

$$\limsup_{n\to\infty}\bigl\langle \gamma f\bigl(x^*\bigr) - Ax^*, x_n - x^*\bigr\rangle = -\lim_{i\to\infty}\bigl\langle \gamma f\bigl(x^*\bigr) - Ax^*, x_{n_i} - x^*\bigr\rangle \leq 0.$$

Step 7. Show that both $\{x_n\}$ and $\{u_n\}$ strongly converge to $x^* \in \Omega$, which is the unique solution of the variational inequality (3.2). Indeed, we note that

$$\bigl\|x_{n+1} - x^*\bigr\|^2 = \bigl\langle x_{n+1} - d_n, x_{n+1} - x^*\bigr\rangle + \bigl\langle d_n - x^*, x_{n+1} - x^*\bigr\rangle.$$

Since $\langle x_{n+1} - d_n, x_{n+1} - x^*\rangle \leq 0$, we get

$$\begin{aligned}
\bigl\|x_{n+1} - x^*\bigr\|^2 &\leq \bigl\langle d_n - x^*, x_{n+1} - x^*\bigr\rangle \\
&= \bigl\langle \alpha_n\gamma f(x_n) + (I - \alpha_n A)W_n u_n - x^*, x_{n+1} - x^*\bigr\rangle \\
&= \bigl\langle \alpha_n\gamma f(x_n) - \alpha_n\gamma f\bigl(x^*\bigr) + W_n u_n - \alpha_n A W_n u_n - x^* \\
&\quad + \alpha_n A x^* + \alpha_n\gamma f\bigl(x^*\bigr) - \alpha_n A x^*, x_{n+1} - x^*\bigr\rangle \\
&= \bigl\langle \alpha_n\gamma\bigl(f(x_n) - f\bigl(x^*\bigr)\bigr) + (I - \alpha_n A)\bigl(W_n u_n - x^*\bigr), x_{n+1} - x^*\bigr\rangle \\
&\quad + \alpha_n\bigl\langle \gamma f\bigl(x^*\bigr) - Ax^*, x_{n+1} - x^*\bigr\rangle \\
&\leq \bigl(\alpha_n\gamma\bigl\|f(x_n) - f\bigl(x^*\bigr)\bigr\| + \|I - \alpha_n A\|\bigl\|W_n u_n - x^*\bigr\|\bigr)\bigl\|x_{n+1} - x^*\bigr\| \\
&\quad + \alpha_n\bigl\langle \gamma f\bigl(x^*\bigr) - Ax^*, x_{n+1} - x^*\bigr\rangle \\
&\leq \bigl(1 - \alpha_n(\bar{\gamma} - \gamma\alpha)\bigr)\bigl\|x_n - x^*\bigr\|\bigl\|x_{n+1} - x^*\bigr\| + \alpha_n\bigl\langle \gamma f\bigl(x^*\bigr) - Ax^*, x_{n+1} - x^*\bigr\rangle \\
&\leq \biggl(\frac{[1 - \alpha_n(\bar{\gamma} - \gamma\alpha)]^2}{2}\biggr)\bigl\|x_n - x^*\bigr\|^2 + \frac{1}{2}\bigl\|x_{n+1} - x^*\bigr\|^2 \\
&\quad + \alpha_n\bigl\langle \gamma f\bigl(x^*\bigr) - Ax^*, x_{n+1} - x^*\bigr\rangle.
\end{aligned}$$

It then follows that

$$\bigl\|x_{n+1} - x^*\bigr\|^2 \leq \bigl[1 - \alpha_n(\bar{\gamma} - \gamma\alpha)\bigr]\bigl\|x_n - x^*\bigr\|^2 + 2\alpha_n\bigl\langle \gamma f\bigl(x^*\bigr) - Ax^*, x_{n+1} - x^*\bigr\rangle. \tag{3.8}$$

Let $a_n = \|x_n - x^*\|^2$, $\gamma_n = \alpha_n(\bar{\gamma} - \gamma\alpha)$ and $\delta_n = 2\alpha_n\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle$.

Then, we can write the last inequality as

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n.$$

Note that in virtue of condition (C2), $\sum_{n=1}^{\infty} \gamma_n = \infty$. Moreover,

$$\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} = \frac{1}{\bar{\gamma} - \gamma\alpha} \limsup_{n\to\infty} 2\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle.$$

By Step 5, we obtain

$$\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0. \tag{3.9}$$

Now, applying Lemma 2.6 to (3.8), we conclude that $x_n \to x^*$ as $n \to \infty$. Furthermore, since $\|u_n - x^*\| = \|T_{r_n}x_n - T_{r_n}x^*\| \leq \|x_n - x^*\|$, we then have that $u_n \to x^*$ as $n \to \infty$. The proof is now complete. □

Setting $A \equiv I$ and $\gamma = 1$ in Theorem 3.1, we have the following result.

Corollary 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be an equilibrium bifunction satisfying the conditions*:

(1) *F is monotone, that is, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;*
(2) *for each $x, y, z \in C$, $\lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y)$;*
(3) *for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.*

Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into C such that $\bigcap_{i=1}^{\infty} F(T_i) \cap EP(F) \neq \emptyset$. Suppose $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfy the following conditions:

(1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(2) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$.

Let f be a contraction of C into itself, and let $x_0 \in H$ be given arbitrarily. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated iteratively by

$$\begin{cases} F(y_n, x) + \frac{1}{r_n}\langle x - y_n, y_n - x_n \rangle \geq 0, \quad \forall x \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) W_n y_n, \end{cases}$$

converge strongly to $x^* \in \bigcap_{i=1}^{\infty} F(T_i) \cap EP(F)$, the unique solution of the minimization problem

$$\min_{x \in \bigcap_{i=1}^{\infty} F(T_i) \cap EP(F)} \frac{1}{2}\|x\|^2 - h(x),$$

where h is a potential function for f.

Setting $F = 0$ in Theorem 3.1, we have the following result.

Corollary 3.3 ([11]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings from C to C such that the common* *fixed point set* $\Omega = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. *Let* $f : C \to H$ *be an* α*-contraction and* $A : H \to H$ *be a strongly positive bounded linear operator with a coefficient* $\bar{\gamma} > 0$. *Let* γ *be a constant such that* $0 < \gamma\alpha < \bar{\gamma}$. *For an arbitrary initial point* x_0 *belonging to* C, *one defines a sequence* $\{x_n\}_{n\geq 0}$ *iteratively*

$$x_{n+1} = P_C\bigl[\alpha_n \gamma f(x_n) + (I - \alpha_n A) W_n x_n\bigr], \quad \forall n \geq 0, \tag{3.10}$$

where $\{\alpha_n\}$ *is a real sequence in* $[0,1]$. *Assume that the sequence* $\{\alpha_n\}$ *satisfies the following conditions*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ *generated by* (3.10) *converges in norm to the unique solution* x^*, *which solves the following variational inequality*:

$$x^* \in \Omega \quad \text{such that } \bigl\langle (A - \gamma f)x^*, x^* - \hat{x}\bigr\rangle \leq 0, \forall \hat{x} \in \Omega. \tag{3.11}$$

Competing interests

The author declares that they have no competing interests.

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