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A note on the Analogue of Lebesgue-Radon-Nikodym theorem with respect to weighted p-adic q-measure on $\mathbb{Z}_p$

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Abstract

We give the analogue of the Lebesgue-Radon-Nikodym theorem with respect to a weighted p-adic q-measure on $\mathbb{Z}_p$. In a special case, when the weight q^x is 1, we can derive the same result as Kim *et al.* (Abstr. Appl. Anal. 2011:637634, 2011). And if $q = 1$, we have the same result as Kim (Russ. J. Math. Phys. 19:193-196, 2012).

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Keywords: p-adic q-measure; Lebesgue-Radon-Nikodym

1 Introduction

Let p be a fixed odd prime number. Throughout this paper, the symbols $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers, and the p-adic completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p| = p^{-\nu_p(p)} = \frac{1}{p}$ and $\nu_p(0) = \infty$.

When one speaks of q-extension, q can be regarded as an indeterminate, a complex $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. In this paper, we assume that $q \in \mathbb{C}_p$ with $|1 - q| < 1$, and we use the notations of q-numbers as follows:

$$[x]_q = [x : q] = \frac{1 - q^x}{1 - q} \quad \text{and} \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}. \tag{1.1}$$

For any positive integer N, let

$$a + p^N\mathbb{Z}_p = \{x \in \mathbb{Z}_p | x \equiv a \pmod{p^N}\}, \tag{1.2}$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < p^N$ (see [1–8]).

It is known that the fermionic p-adic q-measure on $\mathbb{Z}_p$ is given by Kim as follows:

$$\mu_{-q}(a + p^N\mathbb{Z}_p) = \frac{(-q)^a}{[p^N]_{-q}} = \frac{1 + q}{1 + q^{p^N}}(-q)^a, \quad \text{(see [1, 6, 9–12])}. \tag{1.3}$$

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. From (1.3), the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \tag{1.4}$$

where $f \in C(\mathbb{Z}_p)$ (see [1, 6, 9–12]).

Let us assume $q \in \mathbb{C}_p$ with $|q-1| < 1$. By (1.4), we get

$$\int_{\mathbb{Z}_p} q^{-x} e^{[x]_q t}\,d\mu_{-q}(x) = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}, \tag{1.5}$$

(see [7, 8, 13]) where $E_{n,q}$ are q-Euler numbers. The q-Euler polynomials, $E_{n,q}(x)$, are also defined by

$$\int_{\mathbb{Z}_p} q^{-y} e^{[x+y]_q t}\,d\mu_{-y}(t) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \tag{1.6}$$

By (1.5) and (1.6), we get

$$E_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} E_{l,q} = (x + E_q)^n,$$

with the usual convention of replacing $(E_q)^n$ by $E_{n,q}$ (see [1, 2, 7, 8, 13]),

$$E_{n,q} = \int_{\mathbb{Z}_p} q^{-x} [x]_q^n\,d\mu_{-q}(x) = \frac{[\iota]_q}{(1-q)^n} \sum_{l=0}^{n} (-1)^l \binom{n}{l} \frac{1}{[l]_q}.$$

We will give the analogue of the Lebesgue-Radon-Nikodym theorem with respect to a weighted p-adic q-measure on $\mathbb{Z}_p$. In a special case, when the weight q^x is 1, we can derive the same result as Kim *et al.* [9]. And if $q = 1$, we have the same result as Kim [4].

2 Lebesgue-Radon-Nikody-type theorem with respect to a weighted *p*-adic *q*-measure on $\mathbb{Z}_p$

For any positive integer a and n, with $a < p^n$ and $f \in C(\mathbb{Z}_p)$, let us define

$$\tilde{\mu}_{f,-q}\big(a + p^n \mathbb{Z}_p\big) = \int_{a+p^n\mathbb{Z}_p} q^{-x} f(x)\,d\mu_{-q}(x), \tag{2.1}$$

where the integral is the fermionic p-adic q-integral on $\mathbb{Z}_p$.

From (1.3), (1.4), and (2.1), we note that

$$\begin{aligned}&\tilde{\mu}_{f,-q}\big(a + p^n \mathbb{Z}_p\big)\\ &\quad= \lim_{m\to\infty} \frac{1}{[p^{m+n}]_{-q}} \sum_{x=0}^{p^m-1} q^{-(a+p^n x)} f\big(a + p^n x\big)(-q)^{a+p^n x}\end{aligned}$$

$$= \lim_{m\to\infty} \frac{(-1)^a}{[p^m]_{-q}} \sum_{x=0}^{p^{m-n}-1} f(a+p^n x)(-q)^{-p^n x} q^{p^n x}(-1)^x$$

$$= \frac{[2]_q}{[2]_{q^{p^n}}}(-1)^a \lim_{m\to\infty} \frac{1}{[p^{m-n}]_{-q^{p^n}}} \sum_{x=0}^{p^{m-n}-1} f(a+p^n x)\left(-q^{p^n}\right)^x$$

$$= \frac{[2]_q}{[2]_{q^{p^n}}}(-1)^a \int_{\mathbb{Z}_p} q^{p^n x} f(a+p^n x)\, d\mu_{-q^{p^n}}(x). \tag{2.2}$$

By (2.2), we get

$$\tilde{\mu}_{f,-q}(a+p^n\mathbb{Z}_p) = \frac{[2]_q}{[2]_{q^{p^n}}}(-1)^a \int_{\mathbb{Z}_p} q^{p^n x} f(a+p^n x)\, d\mu_{-q^{p^n}}(x). \tag{2.3}$$

Therefore, by (2.3), we obtain the following theorem.

Theorem 1 *For $f, g \in C(\mathbb{Z}_p)$, we have*

$$\tilde{\mu}_{\alpha f+\beta g,-q} = \alpha\tilde{\mu}_{f,-q} + \beta\tilde{\mu}_{g,-q}, \tag{2.4}$$

where α, β are constants.

From (2.2) and (2.4), we note that

$$\left|\tilde{\mu}_{f,-q}(a+p^n\mathbb{Z}_p)\right| \le M\|f_q\|_\infty, \tag{2.5}$$

where $\|f_q\|_\infty = \sup_{x\in\mathbb{Z}_p} |q^{-x}f(x)|$ and M is some positive constant.

Now, we recall the definition of the strongly fermionic p-adic q-measure on $\mathbb{Z}_p$. If μ_{-q} satisfies the following equation:

$$\left|\mu_{-q}(a+p^n\mathbb{Z}_p) - \mu_{-q}(a+p^{n+1}\mathbb{Z}_p)\right| \le \delta_{n,q}, \tag{2.6}$$

where $\delta_{n,q} \to 0$ and $n \to \infty$ and $\delta_{n,q}$ is independent of a, then μ_{-q} is called a weakly fermionic p-adic q-measure on $\mathbb{Z}_p$.

If $\delta_{n,q}$ is replaced by $Cp^{-\nu_p(1-q^n)}$ (C is some constant), then μ_{-q} is called a strongly fermionic p-adic q-measure on $\mathbb{Z}_p$.

Let $P(x) \in \mathbb{C}_p[[x]_q]$ be an arbitrary q-polynomial with $\sum a_i[x]_q^i$. Then we see that $\mu_{P,-q}$ is a strongly fermionic p-adic q-measure on $\mathbb{Z}_p$. Without loss of generality, it is enough to prove the statement for $P(x) = [x]_q^k$.

Let a be an integer with $0 \le a < p^n$. Then we get

$$\tilde{\mu}_{P,-q}(a+p^n\mathbb{Z}_p) = \frac{[2]_q}{[2]_{q^{p^n}}}(-1)^a \lim_{m\to\infty} \frac{1}{[p^{m-n}]_{-q^{p^n}}} \sum_{i=0}^{p^{m-n}-1} \left[a+ip^n\right]_q^k (-1)^i q^{p^n i}, \tag{2.7}$$

and

$$q^{p^n i} = \sum_{l=0}^{i} \binom{i}{l} [p^n]_q^l (q-1)^l.$$

By (2.7), we easily get

$$\begin{aligned}\tilde{\mu}_{P,-q}\left(a+p^n\mathbb{Z}_p\right) &\equiv \frac{[2]_q}{[2]_{q^{p^n}}}(-1)^a[a]_q^k \quad \left(\bmod\left[p^n\right]_q\right)\\ &\equiv \frac{[2]_q}{[2]_{q^{p^n}}}(-1)^a P(a) \quad \left(\bmod\left[p^n\right]_q\right).\end{aligned} \tag{2.8}$$

Let x be arbitrary in $\mathbb{Z}_p$ with $x \equiv x_n (\bmod p^n)$ and $x \equiv x_{n+1} (\bmod p^{n+1})$, where x_n and x_{n+1} are positive integers such that $0 \le x_n < p^n$ and $0 \le x_{n+1} < p^{n+1}$. Thus, by (2.8), we have

$$\left|\tilde{\mu}_{P,-q}\left(a+p^n\mathbb{Z}_p\right) - \tilde{\mu}_{P,-q}\left(a+p^{n+1}\mathbb{Z}_p\right)\right| \le Cp^{-\nu_p(1-q^{p^n})}, \tag{2.9}$$

where C is some positive constant and $n \gg 0$.

Let

$$f_{\tilde{\mu}_{P,-q}}(a) = \lim_{n\to\infty} \tilde{\mu}_{P,-q}\left(a+p^n\mathbb{Z}_p\right). \tag{2.10}$$

Then by (2.5), (2.7), and (2.8), we get

$$f_{\tilde{\mu}_{P,-q}}(a) = \frac{[2]_q}{2}(-1)^a[a]_q^k = \frac{[2]_q}{2}(-1)^a P(a). \tag{2.11}$$

Since $f_{\tilde{\mu}_{P,-q}}(x)$ is continuous on $\mathbb{Z}_p$, it follows, for all $x \in \mathbb{Z}_p$,

$$f_{\tilde{\mu}_{P,-q}}(x) = \frac{[2]_q}{2}(-1)^x P(x). \tag{2.12}$$

Let $g \in C(\mathbb{Z}_p)$. By (2.10), (2.11), and (2.12), we get

$$\begin{aligned}\int_{\mathbb{Z}_p} g(x)\,d\tilde{\mu}_{P,-q}(x) &= \lim_{n\to\infty}\sum_{i=0}^{p^n-1} g(i)\tilde{\mu}_{P,-q}\left(i+p^n\mathbb{Z}_p\right)\\ &= \frac{[2]_q}{2}\lim_{n\to\infty}\sum_{i=0}^{p^n-1} g(i)(-q)^i[i]_q^k\\ &= \int_{\mathbb{Z}_p} q^{-x}g(x)[x]_q^k\,d\mu_{-q}(x).\end{aligned} \tag{2.13}$$

Therefore, by (2.13), we obtain the following theorem.

Theorem 2 *Let $P(x) \in \mathbb{C}_p[[x]_q]$ be an arbitrary q-polynomial with $\sum a_i[x]_q^i$. Then $\tilde{\mu}_{P,-q}$ is a strongly fermionic weighted p-adic q-measure on $\mathbb{Z}_p$, and for all $x \in \mathbb{Z}_p$,*

$$f_{\tilde{\mu}_{P,-q}} = (-1)^x\frac{[2]_q}{2}P(x). \tag{2.14}$$

Furthermore, for any $g \in C(\mathbb{Z}_p)$, we have

$$\int_{\mathbb{Z}_p} g(x)\,d\tilde{\mu}_{P,-q}(x) = \int_{\mathbb{Z}_p} q^{-x}g(x)P(x)\,d\mu_{-q}(x), \tag{2.15}$$

where the second integral is a fermionic p-adic q-integral on $\mathbb{Z}_p$.

Let $f(x)=\sum_{n=0}^{\infty}a_{n,q}\binom{x}{n}_q$ be the q-Mahler expansion of a continuous function on $\mathbb{Z}_p$, where

$$\binom{x}{n}_q=\frac{[x]_q[x-1]_q\cdots[x-n+1]_q}{[n]_q!}\quad\bigl(\text{see [4]}\bigr). \tag{2.16}$$

Then we note that $\lim_{n\to\infty}|a_{n,q}|=0$.

Let

$$f_m(x)=\sum_{i=0}^{m}a_{i,q}\binom{x}{i}_q\in\mathbb{C}_p\bigl[[x]_q\bigr]. \tag{2.17}$$

Then

$$\|f-f_m\|_\infty\le\sup_{m\le n}|a_{n,q}|. \tag{2.18}$$

Writing $f=f_m+f-f_m$, we easily get

$$\begin{aligned}&\bigl|\tilde{\mu}_{f,-q}\bigl(a+p^n\mathbb{Z}_p\bigr)-\tilde{\mu}_{f,-q}\bigl(a+p^{n+1}\mathbb{Z}_p\bigr)\bigr|\\&\quad\le\max\bigl\{\bigl|\tilde{\mu}_{f_m,-q}\bigl(a+p^n\mathbb{Z}_p\bigr)-\tilde{\mu}_{f_m,-q}\bigl(a+p^{n+1}\mathbb{Z}_p\bigr)\bigr|,\\&\qquad\bigl|\tilde{\mu}_{f-f_m,-q}\bigl(a+p^n\mathbb{Z}_p\bigr)-\tilde{\mu}_{f-f_m,-q}\bigl(a+p^{n+1}\mathbb{Z}_p\bigr)\bigr|\bigr\}.\end{aligned} \tag{2.19}$$

From Theorem 2, we note that

$$\bigl|\tilde{\mu}_{f-f_m,-q}\bigl(a+p^n\mathbb{Z}_p\bigr)\bigr|\le\|f-f_m\|_\infty\le C_1p^{-\nu_p(1-q^{p^n})}, \tag{2.20}$$

where C_1 is some positive constant.

For $m\gg 0$, we have $\|f\|_\infty=\|f_m\|_\infty$.

So,

$$\bigl|\tilde{\mu}_{f_m,-q}\bigl(a+p^n\mathbb{Z}_p\bigr)-\tilde{\mu}_{f_m,-q}\bigl(a+p^{n+1}\mathbb{Z}_p\bigr)\bigr|\le C_2p^{-\nu_p(1-q^{p^n})}, \tag{2.21}$$

where C_2 is also some positive constant.

By (2.20) and (2.21), we see that

$$\begin{aligned}&\bigl|f(a)-\tilde{\mu}_{f,-q}\bigl(a+p^n\mathbb{Z}_p\bigr)\bigr|\\&\quad\le\max\bigl\{\bigl|f(a)-f_m(a)\bigr|,\bigl|f_m(a)-\tilde{\mu}_{f_m,-q}\bigl(a+p^n\mathbb{Z}_p\bigr)\bigr|,\bigl|\tilde{\mu}_{f-f_m,-q}\bigl(a+p^n\mathbb{Z}_p\bigr)\bigr|\bigr\}\\&\quad\le\max\bigl\{\bigl|f(a)-f_m(a)\bigr|,\bigl|f_m(a)-\tilde{\mu}_{f_m,-q}\bigl(a+p^n\mathbb{Z}_p\bigr)\bigr|,\|f-f_m\|_\infty\bigr\}.\end{aligned} \tag{2.22}$$

If we fix $\epsilon>0$ and fix m such that $\|f-f_m\|\le\epsilon$, then for $n\gg 0$, we have

$$\bigl|f(a)-\tilde{\mu}_{f,-q}\bigl(a+p^n\mathbb{Z}_p\bigr)\bigr|\le\epsilon. \tag{2.23}$$

Hence, we have

$$f_{\mu_{f,-q}}(a)=\lim_{n\to\infty}\tilde{\mu}_{f,-q}\bigl(a+p^n\mathbb{Z}_p\bigr)=\frac{[2]_q}{2}(-1)^af(a). \tag{2.24}$$

Let m be a sufficiently large number such that $\|f - f_m\|_\infty \leq p^{-n}$.

Then we get

$$\begin{aligned}\tilde{\mu}_{f,-q}\bigl(a + p^n\mathbb{Z}_p\bigr) &= \tilde{\mu}_{f_m,-q}\bigl(a + p^n\mathbb{Z}_p\bigr) + \tilde{\mu}_{f-f_m,-q}\bigl(a + p^n\mathbb{Z}_p\bigr)\\ &= \tilde{\mu}_{f_m,-q}\bigl(a + p^n\mathbb{Z}_p\bigr)\\ &= (-1)^a\frac{[2]_q}{[2]_{q^{p^n}}}f(a) \quad \bigl(\mathrm{mod}\bigl[p^n\bigr]_q\bigr).\end{aligned} \tag{2.25}$$

For any $g \in C(\mathbb{Z}_p)$, we have

$$\int_{\mathbb{Z}_p} g(x)\,d\tilde{\mu}_{f,-q}(x) = \int_{\mathbb{Z}_p} q^{-x}f(x)g(x)\,d\mu_{-q}(x). \tag{2.26}$$

Assume that f is the function from $C(\mathbb{Z}_p, \mathbb{C}_p)$ to $\mathrm{Lip}(\mathbb{Z}_p, \mathbb{C}_p)$. By the definition of $\tilde{\mu}_{-q}$, we easily see that $\tilde{\mu}_{-q}$ is a strongly p-adic q-measure on $\mathbb{Z}_p$, and for $n \gg 0$,

$$\bigl|f_{\tilde{\mu}_{-q}}(a) - \tilde{\mu}_{-q}\bigl(a + p^n\mathbb{Z}_p\bigr)\bigr| \leq C_3 p^{-\nu_p(1-q^{p^n})}, \tag{2.27}$$

where C_3 is some positive constant.

If $\tilde{\mu}_{1,-q}$ is associated with strongly fermionic weighted p-adic q-measure on $\mathbb{Z}_p$, then we have

$$\bigl|\tilde{\mu}_{1,-q}\bigl(a + p^n\mathbb{Z}_p\bigr) - f_{\tilde{\mu}_{-q}}(a)\bigr| \leq C_4 p^{-\nu_p(1-q^{p^n})}, \tag{2.28}$$

where $n \gg 0$ and C_4 is some positive constant.

From (2.28), we get

$$\begin{aligned}&\bigl|\tilde{\mu}_{-q}\bigl(a + p^n\mathbb{Z}_p\bigr) - \tilde{\mu}_{1,-q}\bigl(a + p^n\mathbb{Z}_p\bigr)\bigr|\\ &\quad\leq \bigl|\tilde{\mu}_{-q}\bigl(a + p^n\mathbb{Z}_p\bigr) - f_{\tilde{\mu}_{-q}}(a)\bigr| + \bigl|f_{\tilde{\mu}_{-q}}(a) - \tilde{\mu}_{1,-q}\bigl(a + p^n\mathbb{Z}_p\bigr)\bigr| \leq K,\end{aligned} \tag{2.29}$$

where K is some positive constant.

Therefore, $\tilde{\mu}_{-q} - \tilde{\mu}_{1,-q}$ is a q-measure on $\mathbb{Z}_p$. Hence, we obtain the following theorem.

Theorem 3 *Let $\tilde{\mu}_{-q}$ be a strongly fermionic weighted p-adic q-measure on $\mathbb{Z}_p$, and assume that the fermionic weighted Radon-Nikodym derivative $f_{\tilde{\mu}_{-q}}$ on $\mathbb{Z}_p$ is a continuous function on $\mathbb{Z}_p$. Suppose that $\tilde{\mu}_{1,-q}$ is the strongly fermionic weighted p-adic q-measure associated to $f_{\tilde{\mu}_{-q}}$. Then there exists a q-measure $\tilde{\mu}_{2,-q}$ on $\mathbb{Z}_p$ such that*

$$\tilde{\mu}_{-q} = \tilde{\mu}_{1,-q} + \tilde{\mu}_{2,-q}. \tag{2.30}$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

HP carried out the q-analogue version of similar material studies. SR conceived of the study and participated in its design and coordination. JJ participated in the sequence alignment and drafted the manuscript. All authors read and approved the final manuscript.

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References

1. Bayad, A, Kim, T: Identities involving values of Bernstein, q-Bernoulli, and q-Euler polynomials. Russ. J. Math. Phys. **18**(2), 133-143 (2011)
2. Choi, J, Kim, T, Kim, YH: A note on the q-analogues of Euler numbers and polynomials. Honam Math. J. **33**(4), 529-534 (2011)
3. Kim, T: q-Volkenborn integration. Russ. J. Math. Phys. **9**(3), 288-299 (2002)
4. Kim, T: Lebesgue-Radon-Nikodym theorem with respect to fermionic p-adic invariant measure on $\mathbb{Z}_p$. Russ. J. Math. Phys. **19**(2), 193-196 (2012)
5. Kim, T: Lebesgue-Radon-Nikodym theorem with respect to q-Volkenborn distribution on μ_q. Appl. Math. Comput. **187**(1), 266-271 (2007)
6. Kim, T: A note on q-Bernstein polynomials. Russ. J. Math. Phys. **18**(1), 73-82 (2011)
7. Kim, T: Note on the Euler numbers and polynomials. Adv. Stud. Contemp. Math. **17**(2), 131-136 (2008)
8. Kim, T: Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$. Russ. J. Math. Phys. **16**(4), 484-491 (2009)
9. Kim, T, Dolgy, DV, Lee, SH, Ryoo, CS: Analogue of Lebesgue-Radon-Nikodym theorem with respect to p-adic q-measure on $\mathbb{Z}_p$. Abstr. Appl. Anal. **2011**, Article ID 637634 (2011)
10. Kim, T, Kim, SD, Park, DW: On uniform differentiability and q-Mahler expansion. Adv. Stud. Contemp. Math. **4**(1), 35-41 (2001)
11. Jeong, J, Rim, SH: A note on the Lebesgue-Radon-Nikodym theorem with respect to weighted p-adic invariant integral on $\mathbb{Z}_p$. Abstr. Appl. Anal. **2012**, Article ID 696720 (2012)
12. Kim, T, Choi, J, Kim, H: A note on the weighted Lebesgue-Radon-Nikodym theorem with respect to p-adic invariant integral on $\mathbb{Z}_p$. J. Appl. Math. Inform. **30**(1-2), 211-217 (2012)
13. Kim, T: New approach to q-Euler polynomials of higher order. Russ. J. Math. Phys. **17**(2), 218-225 (2010)

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