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Univalent functions in the Banach algebra of continuous functions

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Abstract
In this paper, we investigate several interesting properties of a composition operator defined on the open unit ball B_0 of the Banach algebra $C(T)$. We also consider the Noshiro-Warschawski theorem in the Banach algebra of continuous functions.

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1 Introduction and definitions

Throughout this paper, $C(T)$ denotes the Banach algebra, with sup norm, of continuous complex-valued functions defined on a compact metric space T. Let $B(f:r)$ be an open ball in $C(T)$ centered at $f \in C(T)$ with radius r. In particular, for the sake of brevity, we use the simplified notation B_0 instead of $B(0:1)$.

Let $\mathcal{A}$ denote the class of functions $\varphi(z)$ of the form

$$\varphi(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Also, let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are *univalent* in the unit disk $\mathcal{U}$. A function $\varphi(z)$ belonging to the class $\mathcal{S}$ is said to be *convex* in $\mathcal{U}$ if and only if

$$\Re\left\{1 + \frac{z\varphi''(z)}{\varphi'(z)}\right\} > 0 \quad (z \in \mathcal{U}).$$

We denote by $\mathcal{K}$ the class of all functions in $\mathcal{S}$ which are convex in $\mathcal{U}$.

Corresponding to the function $\varphi \in \mathcal{A}$, we define a composition operator $F_\varphi : B_0 \to C(T)$ by

$$F_\varphi(f) = \varphi \circ f = f + \sum_{n=2}^{\infty} a_n f^n. \tag{1.2}$$

We denote by S_C the class of all functions F_φ which are injective in the open unit ball B_0. We note that Nikić ([1], Definition 2) defined a similar class S_C without using the function φ. In this case, we cannot ensure the convergence of the series

$$f + \sum_{n=2}^{\infty} a_n f^n.$$

Now we let G be an open nonempty subset of $C(T)$. A function $F : G \to C(T)$ is said to be *L-differentiable* at a point $f \in G$ if there exists $\lambda \in C(T)$ and a map η defined in a ball $B(0 : r)$ with values in $C(T)$ such that

$$\lim_{h \to 0} \frac{\eta(h)}{\|h\|} = 0$$

and such that

$$F(f + h) - F(f) = \lambda h + \eta(h)$$

for all $h \in B(0 : r)$. We call λ the *L-derivative* of F at f and denote it by $F'(f)$. From [1], we see that

$$F'_\varphi(f) = \varphi' \circ f, \tag{1.3}$$

where φ' is a derivative of φ.

In the present paper, we investigate several geometric properties of the class S_C associated with the theory of univalent functions.

2 Geometric properties of the composition operator F_φ

We begin by proving the following theorem.

Theorem 1 $F_\varphi \in S_C$ *if and only if* $\varphi \in \mathcal{S}$.

Proof ($\Leftarrow$) Suppose that $F_\varphi(f) = F_\varphi(g)$ for the functions f and g in B_0. Then it means that

$$\varphi\bigl(f(t)\bigr) = \varphi\bigl(g(t)\bigr)$$

for all $t \in T$. Since φ is univalent, $f(t) = g(t)$ for all $t \in T$.

($\Rightarrow$) Let $\varphi(z_1) = \varphi(z_2)$ for z_1 and z_2 in $\mathcal{U}$. If we take the constant functions f and g such that $f = z_1$ and $g = z_2$, then it is obvious that

$$f \in B_0 \quad \text{and} \quad g \in B_0.$$

Furthermore, from (1.2) it is easy to see that

$$F_\varphi(f) = F_\varphi(g).$$

Since F_φ is injective, we have $f = g$. Hence we get $z_1 = z_2$. This completes the proof of Theorem 1. □

By using Brange's theorem [2], we obtain the following.

Corollary 1 *If*

$$F_\varphi(f) = f + \sum_{n=2}^{\infty} a_n f^n \in S_C,$$

then

$$|a_n| \leq n.$$

Now we prove the Noshiro-Warschawski theorem ([3], Theorem 2.16) in the Banach algebra $C(T)$.

Theorem 2 *If the L-derivative $F'_\varphi(f)$ has a positive real part for all $f \in B_0$, then*

$$F_\varphi \in S_C.$$

Proof If $f_1 \in B_0, f_2 \in B_0$ and $f_1 \neq f_2$, then there exists $t \in T$ such that

$$f_1(t) \neq f_2(t). \tag{2.1}$$

By the hypothesis,

$$\Re\{F'_\varphi(f)\} > 0 \tag{2.2}$$

for all $f \in B_0$. It follows from (1.3) that

$$\Re\{\varphi'(f(t))\} > 0 \quad (f \in B_0 : t \in T). \tag{2.3}$$

Since

$$\varphi(f_2(t)) - \varphi(f_1(t)) = \int_{f_1(t)}^{f_2(t)} \varphi'(x)\,dx = (f_2(t) - f_1(t)) \int_0^1 \varphi'(\lambda f_2(t) + (1-\lambda) f_1(t))\,d\lambda$$

and

$$\lambda f_2(t) + (1-\lambda) f_1(t) \in B_0,$$

equations (2.1) and (2.3) imply that

$$\varphi(f_2(t)) \neq \varphi(f_1(t)).$$

Hence

$$F_\varphi(f_1(t)) \neq F_\varphi(f_2(t))$$

at $t \in T$, which shows that F_φ is injective. □

Remark Since T is compact, $\{f(t): t \in T\}$ is a closed proper subset of $\mathcal{U}$. Hence the condition (2.2) does not imply

$$\Re\{\varphi'(z)\} > 0 \quad (z \in \mathcal{U}).$$

Next we obtain the following.

Theorem 3 *Let*

$$\varphi(z) = \frac{z}{1-z}.$$

Then

$$\{F_\varphi(f): f \in B_0\}$$

is a convex subset in $C(T)$.

Proof Assume that

$$\alpha > 0, \qquad \beta > 0 \quad \text{and} \quad \alpha + \beta = 1.$$

For the functions f and g in B_0, we let

$$u(t) \equiv \alpha F_\varphi(f(t)) + \beta F_\varphi(g(t))$$

and

$$v(t) \equiv \frac{u(t)}{1+u(t)}.$$

Then we have

$$u(t) = \frac{v(t)}{1-v(t)} = F_\varphi(v(t)).$$

Since

$$\begin{aligned} 1 - |v(t)|^2 &= 1 - v(t)\overline{v(t)} \\ &= 1 - \frac{u(t)}{1+u(t)} \frac{\overline{u(t)}}{1+\overline{u(t)}} \\ &= \frac{1}{1+\overline{u(t)}}\big(1 + u(t) + \overline{u(t)}\big)\frac{1}{1+u(t)} \\ &= \frac{1 + 2\Re\{u(t)\}}{1+|u(t)|^2} > 0, \end{aligned}$$

the function v belongs to B_0. Thus we have

$$u = F_\varphi(v) \in \{F_\varphi(f): f \in B_0\}.$$

This completes the proof of Theorem 3. □

We now recall that the function

$$\varphi_\eta(z) = \frac{z}{1-\eta z} \quad \bigl(\eta \in \mathbb{C}, |\eta| = 1\bigr)$$

is the well-known extremal function (see [3]) for the class $\mathcal{K}$ of convex functions. If we let

$$\varphi(z) = \frac{z}{1-z},$$

then we note that

$$\varphi_\eta(z) = \eta^{-1}\varphi(\eta z). \tag{2.4}$$

Making use of Theorem 3 and (2.4), we can derive the following.

Corollary 2 *If φ is an extreme point of $\mathcal{K}$, then*

$$\bigl\{F_\varphi(f) : f \in B_0\bigr\}$$

is a convex subset in $C(T)$.

It is well known that the sharp inequality

$$\bigl|f^{(n)}(z)\bigr| \leq \frac{n!(n+|z|)}{(1-|z|)^{n+2}} \quad (n = 1, 2, 3, \ldots) \tag{2.5}$$

holds for every $f \in \mathcal{S}$ (see [3, p.70, Exercise 6]).

In view of the inequality (2.5), we have a generalization of [1, Theorem 2] as follows.

Theorem 4 *If $f \in B_0$ and $\varphi \in \mathcal{S}$, then the nth L-derivative of F_φ at f satisfies*

$$\bigl\|F^{(n)}(f)\bigr\| \leq \frac{n!(n+\|f\|)}{(1-\|f\|)^{n+2}}.$$

Remark The proof would run parallel to that of [1, Theorem 2] because there are many similarities. But, as we have seen in equation (1.2), we find it to be different from the definition of the class S_C, which was given by Nikić [1]. So, we include the proof of Theorem 4.

Proof Applying (1.2) and (1.3), it is not difficult to show that

$$F_\varphi^{(n)}(f) = \varphi^{(n)} \circ f \quad (n = 1, 2, 3, \ldots),$$

where $\varphi^{(n)}$ is the nth derivative of φ. Since

$$F_\varphi^{(n)}(f) \in C(T)$$

and T is a compact metric space, there exists a point $\xi \in T$ such that

$$\bigl\|F_\varphi^{(n)}(f)\bigr\| = \bigl|F_\varphi^{(n)}\bigl(f(\xi)\bigr)\bigr| = \bigl|\varphi^{(n)}\bigl(f(\xi)\bigr)\bigr|. \tag{2.6}$$

Since $\varphi \in \mathcal{S}$, from (2.4) we have

$$\left|\varphi^{(n)}\bigl(f(\xi)\bigr)\right| \leq \frac{n!(n+|f(\xi)|)}{(1-|f(\xi)|)^{n+2}} \leq \frac{n!(n+\|f\|)}{(1-\|f\|)^{n+2}}. \tag{2.7}$$

Combining (2.6) and (2.7), we obtain the desired result. □

3 Examples

Example 1 Let the function φ be defined by (1.1). For a fixed radius $0 < r < 1$, we let $T = \{z \in \mathbb{C} : |z| \leq r\}$. If we define a continuous function $f : T \to \mathbb{C}$ by $f(z) = z$, then

$$F_\varphi(f) = \varphi$$

on T.

Example 2 Setting $\varphi(z) = z$ in (1.2), we have

$$F_\varphi(f) = f.$$

Example 3 If $\varphi \in \mathcal{A}$ satisfies

$$\Re\bigl\{\varphi'(z)\bigr\} > 0 \quad (z \in \mathcal{U}),$$

then the Noshiro-Warschawski theorem implies that φ is univalent. Hence, by Theorem 1, we obtain

$$F_\varphi \in S_C.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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