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Some further extensions of absolute Cesàro summability for double series

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Abstract
In a recent paper [Savaş and Rhoades in Appl. Math. Lett. 22:1462-1466, 2009], the authors extended the result of Flett [Proc. Lond. Math. Soc. 7:113-141, 1957] to double summability. In this paper, we consider some further extensions of absolute Cesàro summability for double series.

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Let $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}$ be an infinite double series with real or complex numbers, with partial sums

$$s_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}.$$

For any double sequence (x_{mn}) we shall define

$$\Delta_{11}x_{mn} = x_{mn} - x_{m+1,n} - x_{m,n+1} + x_{m+1,n+1}.$$

Denote by $\mathcal{A}_k^2$ the sequence space defined by

$$\mathcal{A}_k^2 = \left\{ (s_{mn})_{m,n=0}^{\infty} : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|a_{mn}|^k < \infty; a_{mn} = \Delta_{11}s_{m-1,n-1} \right\}$$

for $k \geq 1$.

A four-dimensional matrix $T = (t_{mnij} : m,n,i,j = 0,1,\ldots)$ is said to be absolutely kth power conservative for $k \geq 1$, if $T \in B(\mathcal{A}_k^2)$, *i.e.*, if

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|\Delta_{11}s_{m-1,n-1}|^k < \infty,$$

then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|\Delta_{11}t_{m-1,n-1}|^k < \infty,$$

where

$$t_{mn} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} t_{mnij}s_{ij} \quad (m,n = 0,1,\ldots).$$

A double infinite Cesáro matrix $C(\alpha,\beta)$ is a double infinite Hausdorff matrix with entries

$$h_{mnij} = \frac{E_{m-i}^{\alpha-1}E_{n-j}^{\beta-1}}{E_m^{\alpha}E_n^{\beta}},$$

where

$$E_m^{\alpha} = \binom{m+\alpha}{m} = \frac{\Gamma(\alpha+m+1)}{\Gamma(m+1)\Gamma(\alpha+1)} \approx \frac{m^{\alpha}}{\Gamma(\alpha+1)}.$$

The series $\sum\sum a_{mn}$ is said to be summable $|C(\alpha,\beta)|_k$, $k \geq 1$, $\alpha,\beta > -1$, if (see [1])

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left|\Delta_{11}\sigma_{m-1,n-1}^{\alpha\beta}\right|^{k} < \infty, \tag{1}$$

where $\sigma_{mn}^{\alpha\beta}$ denotes the mn-term of the $C(\alpha,\beta)$ transform of a sequence (s_{mn}); *i.e.*,

$$\sigma_{mn}^{\alpha\beta} = \frac{1}{E_m^{\alpha}E_n^{\beta}}\sum_{i=0}^{m}\sum_{j=0}^{n}E_{m-i}^{\alpha-1}E_{n-j}^{\beta-1}s_{ij}. \tag{2}$$

Quite recently, Savaş and Rhoades [2] extended the result of Flett [3] to double summability. Their theorem is as follows.

Theorem 1 *Let $\alpha \geq \gamma > -1$, $\beta \geq \delta > -1$, and $\sum_m\sum_n a_{mn}$ be a double series with partial sums s_{mn}. If $\sum_m\sum_n a_{mn}$is $|C(\gamma,\delta)|_k$-summable, then it is also $|C(\alpha,\beta)|_k$-summable, $k \geq 1$.*

It then follows that if one sets γ=δ = 0, then $C(\alpha,\beta) \in B(\mathcal{A}_k^2)$ for each $\alpha,\beta \geq 0$. In this paper, we consider some further extensions of absolute Cesàro summability for double series.

We shall use the following lemmas.

Lemma 1 *If $\theta > -1$, $\phi > -1$, $\theta - \varphi > 0$ and $\phi - \psi > 0$, then*

$$\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\frac{E_{m-i}^{\varphi}E_{n-j}^{\psi}}{mnE_m^{\theta}E_n^{\phi}} = \frac{1}{ijE_i^{\theta-\varphi-1}E_j^{\phi-\psi-1}}. \tag{3}$$

Proof For $\alpha > -1$, $n \geq 1$ since

$$\frac{1}{E_n^{\alpha}} = \int_0^1 (1-x)^{\alpha}x^{n-1}\,dx$$

and

$$(1-x)^{-\alpha} = \sum_{n=0}^{\infty}E_n^{\alpha-1}x^n,$$

we obtain

$$\begin{aligned}
\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\frac{E_{m-i}^{\varphi}E_{n-j}^{\psi}}{mnE_m^{\theta}E_n^{\phi}} &= \sum_{m=i}^{\infty}\frac{E_{m-i}^{\varphi}}{mE_m^{\theta}}\sum_{n=0}^{\infty}\frac{E_n^{\psi}}{(n+j)E_{n+j}^{\phi}} \\
&= \sum_{m=i}^{\infty}\frac{E_{m-i}^{\varphi}}{mE_m^{\theta}}\sum_{n=0}^{\infty}E_n^{\psi}\int_0^1(1-x)^{\phi}x^{n+j-1}\,dx \\
&= \sum_{m=i}^{\infty}\frac{E_{m-i}^{\varphi}}{mE_m^{\theta}}\int_0^1(1-x)^{\phi}x^{j-1}\left(\sum_{n=0}^{\infty}E_n^{\psi}x^n\right)dx \\
&= \int_0^1(1-x)^{\phi-\psi-1}x^{j-1}dx\sum_{m=0}^{\infty}\frac{E_m^{\varphi}}{(m+i)E_{m+i}^{\theta}} \\
&= \frac{1}{jE_j^{\phi-\psi-1}}\sum_{m=0}^{\infty}E_m^{\varphi}\int_0^1(1-x)^{\theta}x^{m+i-1}\,dx \\
&= \frac{1}{jE_j^{\phi-\psi-1}}\int_0^1(1-x)^{\theta-\varphi-1}x^{i-1}\,dx \\
&= \frac{1}{ijE_i^{\theta-\varphi-1}E_j^{\phi-\psi-1}}.
\end{aligned}$$

□

For single series, Lemma 1 due to Chow [4].

Lemma 2 *Let* $1 \le k \le r < \infty$ *and* $\alpha, \beta > -1$. *For* $i, j \ge 1$, *let*

$$A_{ij} = A_{ij}(\alpha,\beta) = \sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\frac{|E_{m-i}^{\alpha-1}|^{r/k}|E_{n-j}^{\beta-1}|^{r/k}}{mn(E_m^{\alpha})^{r/k}(E_n^{\beta})^{r/k}}. \tag{4}$$

Then, if $k = r$,

$$A_{ij} = \begin{cases} O(i^{-\alpha-1}j^{-\beta-1}), & \text{if } \alpha,\beta < 0, \\ O(i^{-\alpha-1}j^{-1}), & \text{if } \alpha < 0, \beta \ge 0, \\ O(i^{-1}j^{-\beta-1}), & \text{if } \alpha \ge 0, \beta < 0, \\ O(i^{-1}j^{-1}), & \text{if } \alpha,\beta \ge 0. \end{cases} \tag{5}$$

If $k < r < \infty$, *then*

$$A_{ij} = \begin{cases} O(i^{-\frac{\alpha r}{k}-1}j^{-\frac{\beta r}{k}-1}), & \text{if } \alpha < 1-k/r, \beta < 1-k/r, \\ O(i^{-\frac{\alpha r}{k}-1}j^{-\frac{r}{k}}\log j), & \text{if } \alpha < 1-k/r, \beta = 1-k/r, \\ O(i^{-\frac{\alpha r}{k}-1}j^{-\frac{r}{k}}), & \text{if } \alpha < 1-k/r, \beta > 1-k/r, \\ O(i^{-\frac{\alpha r}{k}-1}j^{-\frac{\beta r}{k}-1}\log i), & \text{if } \alpha = 1-k/r, \beta < 1-k/r, \\ O(i^{-\frac{r}{k}}j^{-\frac{r}{k}}\log i\log j), & \text{if } \alpha = 1-k/r, \beta = 1-k/r, \\ O(i^{-\frac{\alpha r}{k}-1}j^{-\frac{r}{k}}\log i), & \text{if } \alpha = 1-k/r, \beta > 1-k/r, \\ O(i^{-\frac{r}{k}}j^{-\frac{\beta r}{k}-1}), & \text{if } \alpha > 1-k/r, \beta < 1-k/r, \\ O(i^{-\frac{r}{k}}j^{-\frac{\beta r}{k}-1}\log j), & \text{if } \alpha > 1-k/r, \beta = 1-k/r, \\ O(i^{-\frac{r}{k}}j^{-\frac{r}{k}}), & \text{if } \alpha > 1-k/r, \beta > 1-k/r. \end{cases} \tag{6}$$

Proof Since $|E_n^\alpha| \le K(\alpha)n^\alpha$ for all α, and $E_n^\alpha \ge M(\alpha)n^\alpha$ for $\alpha > -1$, where $K(\alpha)$ and $M(\alpha)$ are positive constants depending only on α, if $k < r < \infty$, then

$$\begin{aligned}
A_{ij} &= O(1)\sum_{m=i}^{\infty}\sum_{n=j}^{\infty} m^{-\frac{\alpha r}{k}-1} n^{-\frac{\beta r}{k}-1}(m-i+1)^{\frac{(\alpha-1)r}{k}}(n-j+1)^{\frac{(\beta-1)r}{k}} \\
&= O(1)\left(\sum_{m=i}^{2i-1} m^{-\frac{\alpha r}{k}-1}(m-i+1)^{\frac{(\alpha-1)r}{k}} + \sum_{m=2i}^{\infty} m^{-\frac{\alpha r}{k}-1}(m-i+1)^{\frac{(\alpha-1)r}{k}}\right) \\
&\quad \times \left(\sum_{n=j}^{2j-1} n^{-\frac{\beta r}{k}-1}(n-j+1)^{\frac{(\beta-1)r}{k}} + \sum_{n=2j}^{\infty} n^{-\frac{\beta r}{k}-1}(n-j+1)^{\frac{(\beta-1)r}{k}}\right) \\
&= O(1)\left(i^{-\frac{\alpha r}{k}-1}\sum_{m=i}^{2i-1}(m-i+1)^{\frac{(\alpha-1)r}{k}} + \sum_{m=2i}^{\infty} m^{-\frac{r}{k}-1}\right) \\
&\quad \times \left(j^{-\frac{\beta r}{k}-1}\sum_{n=j}^{2j-1}(n-j+1)^{\frac{(\beta-1)r}{k}} + \sum_{n=2j}^{\infty} n^{-\frac{r}{k}-1}\right) \\
&= O(1)i^{-\frac{\alpha r}{k}-1}j^{-\frac{\beta r}{k}-1}\sum_{m=1}^{i}\sum_{n=1}^{j} m^{\frac{(\alpha-1)r}{k}} n^{\frac{(\beta-1)r}{k}} + O(1)i^{-\frac{\alpha r}{k}-1}j^{-r/k}\sum_{m=1}^{i} m^{\frac{(\alpha-1)r}{k}} \\
&\quad + O(1)i^{-r/k}j^{-\frac{\beta r}{k}-1}\sum_{n=1}^{j} n^{\frac{(\beta-1)r}{k}} + O(1)(ij)^{-r/k}.
\end{aligned}$$

According as $\frac{(\alpha-1)r}{k}$ and $\frac{(\beta-1)r}{k} = -1$, < -1 or > -1, we have (6). The case $k = r$ is proved similarly. □

For single series, Lemma 2 due to Mehdi [5].

We now prove the following theorem.

Theorem 2 *Let $r \ge k \ge 1$.*

(i) *It holds that $C(\alpha,\beta) \in (\mathcal{A}_k^2, \mathcal{A}_r^2)$ for each $\alpha, \beta > 1 - k/r$.*

(ii) *If $\alpha, \beta = 1 - k/r$ and the condition $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\log m \log n|a_{mn}|^k = O(1)$ is satisfied then $C(\alpha,\beta) \in (\mathcal{A}_k^2, \mathcal{A}_r^2)$.*

(iii) *If the condition $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{k+(1-\alpha)\frac{r}{k}-2} n^{k+(1-\beta)\frac{r}{k}-2}|a_{mn}|^k = O(1)$ is satisfied then $C(\alpha,\beta) \in (\mathcal{A}_k^2, \mathcal{A}_r^2)$ for each $-k/r < \alpha, \beta < 1 - k/r$.*

(iv) *If $\beta = 1 - k/r$ and the condition $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{k+(1-\alpha)\frac{r}{k}-2} n^{k-1}\log n|a_{mn}|^k = O(1)$ is satisfied then $C(\alpha,\beta) \in (\mathcal{A}_k^2, \mathcal{A}_r^2)$ for each $-k/r < \alpha < 1 - k/r$.*

(v) *If the condition $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{k+(1-\alpha)\frac{r}{k}-2} n^{k-1}|a_{mn}|^k = O(1)$ is satisfied then $C(\alpha,\beta) \in (\mathcal{A}_k^2, \mathcal{A}_r^2)$ for each $-k/r < \alpha < 1 - k/r$ and $\beta > 1 - k/r$.*

(vi) *If $\alpha = 1 - k/r$ and the condition $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{k-1} n^{k+(1-\beta)\frac{r}{k}-2}\log m|a_{mn}|^k = O(1)$ is satisfied then $C(\alpha,\beta) \in (\mathcal{A}_k^2, \mathcal{A}_r^2)$ for each $-k/r < \beta < 1 - k/r$.*

(vii) *If the condition $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{k-1} n^{k-1}\log m|a_{mn}|^k = O(1)$ is satisfied then $C(\alpha,\beta) \in (\mathcal{A}_k^2, \mathcal{A}_r^2)$ for each $\alpha > 1 - k/r$, $\beta < 1 - k/r$.*

(viii) *If the condition $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{k-1} n^{k+(1-\beta)\frac{r}{k}-2}|a_{mn}|^k = O(1)$ is satisfied then $C(\alpha,\beta) \in (\mathcal{A}_k^2, \mathcal{A}_r^2)$ for each $\alpha > 1 - k/r$ and $-k/r < \beta < 1 - k/r$.*

(ix) *If $\beta = 1 - k/r$ and the condition $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{k-1} n^{k-1}\log n|a_{mn}|^k = O(1)$ is satisfied then $C(\alpha,\beta) \in (\mathcal{A}_k^2, \mathcal{A}_r^2)$ for each $\alpha > 1 - k/r$.*

Proof We shall show that $(\sigma_{mn}^{\alpha\beta}) \in \mathcal{A}_r^2$, *i.e.*,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{r-1}\left|\Delta_{11}\sigma_{m-1,n-1}^{\alpha\beta}\right|^r < \infty.$$

Let $\tau_{mn}^{\alpha\beta}$ denote the mn-term of the $C(\alpha,\beta)$-transform in terms of (mna_{mn}), *i.e.*,

$$\tau_{mn}^{\alpha\beta} = \frac{1}{E_m^{\alpha}E_n^{\beta}}\sum_{i=1}^{m}\sum_{j=1}^{n}E_{m-i}^{\alpha-1}E_{n-j}^{\beta-1}ija_{ij}.$$

For $\alpha,\beta > -1$, since

$$\tau_{mn}^{\alpha\beta} = mn\left(\sigma_{mn}^{\alpha\beta} - \sigma_{m,n-1}^{\alpha\beta} - \sigma_{m-1,n}^{\alpha\beta} + \sigma_{m-1,n-1}^{\alpha\beta}\right),$$

to prove the theorem, it will be sufficient to show that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{-1}\left|\tau_{mn}^{\alpha\beta}\right|^r < \infty. \tag{7}$$

Using Hölder's inequality, we have

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\left|\tau_{mn}^{\alpha\beta}\right|^r &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\left|\frac{1}{E_m^{\alpha}E_n^{\beta}}\sum_{i=1}^{m}\sum_{j=1}^{n}E_{m-i}^{\alpha-1}E_{n-j}^{\beta-1}ija_{ij}\right|^r \\
&\leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn(E_m^{\alpha})^r(E_n^{\beta})^r}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}\left|E_{m-i}^{\alpha-1}\right|\left|E_{n-j}^{\beta-1}\right|i^kj^k|a_{ij}|^k\right\}^{r/k} \\
&\quad \times \left\{\sum_{i=1}^{m}\sum_{j=1}^{n}\left|E_{m-i}^{\alpha-1}\right|\left|E_{n-j}^{\beta-1}\right|\right\}^{(k-1)r/k}.
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{i=1}^{m}\sum_{j=1}^{n}\left|E_{m-i}^{\alpha-1}\right|\left|E_{n-j}^{\beta-1}\right| &= \left(\left|E_0^{\alpha-1}\right| + \sum_{i=1}^{m-1}\left|E_{m-i}^{\alpha-1}\right|\right)\left(\left|E_0^{\beta-1}\right| + \sum_{j=1}^{n-1}\left|E_{n-j}^{\beta-1}\right|\right) \\
&= \left(\left|E_0^{\alpha-1}\right| + \left|\sum_{i=1}^{m-1}E_{m-i}^{\alpha-1}\right|\right)\left(\left|E_0^{\beta-1}\right| + \left|\sum_{j=1}^{n-1}E_{n-j}^{\beta-1}\right|\right) \\
&= \left(\left|E_0^{\alpha-1}\right| + \left|\sum_{i=0}^{m}E_{m-i}^{\alpha-1} - E_m^{\alpha-1} - E_0^{\alpha-1}\right|\right) \\
&\quad \times \left(\left|E_0^{\beta-1}\right| + \left|\sum_{j=0}^{n}E_{n-j}^{\beta-1} - E_n^{\beta-1} - E_0^{\beta-1}\right|\right) \\
&= \left(\left|E_0^{\alpha-1}\right| + \left|E_{m-1}^{\alpha} - E_0^{\alpha-1}\right|\right)\left(\left|E_0^{\beta-1}\right| + \left|E_{n-1}^{\beta} - E_0^{\beta-1}\right|\right),
\end{aligned}$$

and using the fact that

$$\left|\frac{E_{m-1}^{\alpha}}{E_m^{\alpha}}\right| = O(1) \quad \text{and} \quad \left|\frac{E_{n-1}^{\beta}}{E_n^{\beta}}\right| = O(1),$$

we obtain

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\left|\tau_{mn}^{\alpha\beta}\right|^{r} \\
&\quad\le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(E_m^{\alpha})^{(k-1)r/k}(E_n^{\beta})^{(k-1)r/k}}{mn(E_m^{\alpha})^{r}(E_n^{\beta})^{r}} \\
&\qquad\times\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}\left|E_{m-i}^{\alpha-1}\right|\left|E_{n-j}^{\beta-1}\right|i^k j^k|a_{ij}|^k\right\}^{r/k} \\
&\quad\le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(E_m^{\alpha})^{-r/k}(E_n^{\beta})^{-r/k}}{mn} \\
&\qquad\times\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}\left|E_{m-i}^{\alpha-1}\right|\left|E_{n-j}^{\beta-1}\right|(ij)^{1-k/r+k^2/r}|a_{ij}|^{k^2/r}(ij)^{-(r-k)+k(r-k)/r}|a_{ij}|^{k(r-k)/r}\right\}^{r/k}.
\end{aligned}$$

Applying Hölder's inequality with indices r/k, $r/(r-k)$, we deduce that

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\left|\tau_{mn}^{\alpha\beta}\right|^{r} &\le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(E_m^{\alpha})^{-r/k}(E_n^{\beta})^{-r/k}}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left|E_{m-i}^{\alpha-1}\right|^{r/k}\left|E_{n-j}^{\beta-1}\right|^{r/k}(ij)^{k-1+r/k}|a_{ij}|^k \\
&\quad\times\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}(ij)^{k-1}|a_{ij}|^k\right\}^{(r-k)/k}.
\end{aligned}$$

Since $(s_{mn})\in\mathcal{A}_k^2$, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\left|\tau_{mn}^{\alpha\beta}\right|^{r} = O(1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(ij)^{k-1}|a_{ij}|^k(ij)^{r/k}\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\frac{|E_{m-i}^{\alpha-1}|^{r/k}|E_{n-j}^{\beta-1}|^{r/k}}{mn(E_m^{\alpha})^{r/k}(E_n^{\beta})^{r/k}}.$$

(i) From Lemma 1, if $\alpha,\beta > 1-k/r$, then

$$\begin{aligned}
\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\frac{(E_{m-i}^{\alpha-1})^{r/k}(E_{n-j}^{\beta-1})^{r/k}}{mn(E_m^{\alpha})^{r/k}(E_n^{\beta})^{r/k}} &= (ij)^{-1}\frac{1}{E_i^{\frac{r}{k}-1}E_j^{\frac{r}{k}-1}} \\
&= O\big((ij)^{-r/k}\big).
\end{aligned}$$

Therefore, for the case $\alpha,\beta > 1-k/r$, we have

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\left|\tau_{mn}^{\alpha\beta}\right|^{r} &= O(1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(ij)^{k-1}|a_{ij}|^k(ij)^{r/k}(ij)^{-r/k} \\
&= O(1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(ij)^{k-1}|a_{ij}|^k = O(1),
\end{aligned}$$

since $(s_{mn})\in\mathcal{A}_k^2$.

(ii) If $\alpha, \beta = 1 - k/r$, from Lemma 2, then

$$\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\frac{|E_{m-i}^{\alpha-1}|^{r/k}|E_{n-j}^{\beta-1}|^{r/k}}{mn(E_m^{\alpha})^{r/k}(E_n^{\beta})^{r/k}} = O\bigl(i^{-\frac{r}{k}}j^{-\frac{r}{k}}\log i\log j\bigr).$$

Hence,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\bigl|\tau_{mn}^{\alpha\beta}\bigr|^{r} = O(1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(ij)^{k-1}\log i\log j|a_{ij}|^{k} = O(1).$$

(iii) If $-k/r < \alpha, \beta < 1 - k/r$, from Lemma 2, then

$$\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\frac{|E_{m-i}^{\alpha-1}|^{r/k}|E_{n-j}^{\beta-1}|^{r/k}}{mn(E_m^{\alpha})^{r/k}(E_n^{\beta})^{r/k}} = O\bigl(i^{-\frac{\alpha r}{k}-1}j^{-\frac{\beta r}{k}-1}\bigr),$$

and then

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\bigl|\tau_{mn}^{\alpha\beta}\bigr|^{r} &= O(1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(ij)^{k+r/k-1}i^{-\frac{\alpha r}{k}-1}j^{-\frac{\beta r}{k}-1}|a_{ij}|^{k}\\ &= O(1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}i^{k+(1-\alpha)\frac{r}{k}-2}j^{k+(1-\beta)\frac{r}{k}-2}|a_{ij}|^{k} = O(1).\end{aligned}$$

(iv) If $\beta = 1 - k/r$ and $-k/r < \alpha < 1 - k/r$, from Lemma 2, then

$$\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\frac{|E_{m-i}^{\alpha-1}|^{r/k}|E_{n-j}^{\beta-1}|^{r/k}}{mn(E_m^{\alpha})^{r/k}(E_n^{\beta})^{r/k}} = O\bigl(i^{-\frac{\alpha r}{k}-1}j^{-\frac{r}{k}}\log j\bigr),$$

therefore, we have

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\bigl|\tau_{mn}^{\alpha\beta}\bigr|^{r} &= O(1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(ij)^{k-1}|a_{ij}|^{k}(ij)^{r/k}i^{-\frac{\alpha r}{k}-1}j^{-\frac{r}{k}}\log j\\ &= O(1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}i^{k+(1-\alpha)\frac{r}{k}-2}j^{k-1}\log j|a_{ij}|^{k} = O(1).\end{aligned}$$

(v) If $-k/r < \alpha < 1 - k/r$ and $\beta > 1 - k/r$, then

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\bigl|\tau_{mn}^{\alpha\beta}\bigr|^{r} &= O(1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(ij)^{k-1}|a_{ij}|^{k}(ij)^{r/k}i^{-\frac{\alpha r}{k}-1}j^{-\frac{r}{k}}\\ &= O(1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}i^{k+(1-\alpha)\frac{r}{k}-2}j^{k-1}|a_{ij}|^{k} = O(1),\end{aligned}$$

by using Lemma 2.

(vi) If $\alpha = 1 - k/r$ and $-k/r < \beta < 1 - k/r$, then

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\left|\tau_{mn}^{\alpha\beta}\right|^{r} &= O(1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(ij)^{k-1}|a_{ij}|^{k}(ij)^{r/k}i^{-\frac{r}{k}}j^{-\frac{\beta r}{k}-1}\log i\\ &= O(1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}i^{k-1}j^{k+(1-\beta)\frac{r}{k}-2}\log i|a_{ij}|^{k} = O(1),\end{aligned}$$

by using Lemma 2.

(vii) If $\alpha = 1 - k/r$ and $\beta > 1 - k/r$, then

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\left|\tau_{mn}^{\alpha\beta}\right|^{r} &= O(1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(ij)^{k-1}|a_{ij}|^{k}(ij)^{r/k}i^{-\frac{r}{k}}j^{-\frac{r}{k}}\log i\\ &= O(1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}i^{k-1}j^{k-1}\log i|a_{ij}|^{k} = O(1),\end{aligned}$$

by using Lemma 2.

(viii) If $\alpha > 1 - k/r$ and $-k/r < \beta < 1 - k/r$, then

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\left|\tau_{mn}^{\alpha\beta}\right|^{r} &= O(1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(ij)^{k-1}|a_{ij}|^{k}(ij)^{r/k}i^{-\frac{r}{k}}j^{-\frac{\beta r}{k}-1}\\ &= O(1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}i^{k-1}j^{k+(1-\beta)\frac{r}{k}-2}|a_{ij}|^{k} = O(1),\end{aligned}$$

by using Lemma 2.

(ix) If $\alpha > 1 - k/r$ and $\beta = 1 - k/r$, then

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\left|\tau_{mn}^{\alpha\beta}\right|^{r} &= O(1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(ij)^{k-1}|a_{ij}|^{k}(ij)^{r/k}i^{-\frac{r}{k}}j^{-\frac{\beta r}{k}-1}\log j\\ &= O(1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}i^{k-1}j^{k-1}\log j|a_{ij}|^{k} = O(1),\end{aligned}$$

by using Lemma 2. □

The one-dimensional version of Theorem 2 appears in [6]. By (5), Theorem 2 includes the following theorem with the special case $r = k$.

Theorem 3 *Let* $k \geq 1$.

(i) *It holds that* $C(\alpha,\beta) \in B(\mathcal{A}_k^2)$ *for each* $\alpha, \beta \geq 0$.

(ii) *If the condition* $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{k-\alpha-1}n^{k-\beta-1}|a_{mn}|^k = O(1)$ *is satisfied then* $C(\alpha,\beta) \in B(\mathcal{A}_k^2)$ *for each* $-1 < \alpha < 0$ *and* $-1 < \beta < 0$.

(iii) *If the condition* $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{k-\alpha-1}n^{k-1}|a_{mn}|^k = O(1)$ *is satisfied then* $C(\alpha,\beta) \in B(\mathcal{A}_k^2)$ *for each* $-1 < \alpha < 0$ *and* $\beta \geq 0$.

(iv) *If the condition* $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{k-1}n^{k-\beta-1}|a_{mn}|^k = O(1)$ *is satisfied then* $C(\alpha,\beta) \in B(\mathcal{A}_k^2)$ *for each* $\alpha \geq 0$ *and* $-1 < \beta < 0$.

Remark 1 Theorem 3 moderates Theorem 1 of [7]. Since Holder's inequality is valid if each of the terms is nonnegative, it should be added the absolute values of the binomial coefficients in the proof of Theorem 1 of [7], when $-1 < \alpha < 0$ and/or $-1 < \beta < 0$. Therefore, if we replace the binomial coefficients with their absolute values, then the inequality (15) of [7] will be true. So, we should add the conditions, given above in (ii), (iii) and (iv) of Theorem 3 in the statement of Theorem 1 of [7], for the cases $-1 < \alpha < 0$ and/or $-1 < \beta < 0$.

Corollary 1 *Let* $\theta^{\alpha}_{mn} = \frac{1}{E^{\alpha}_m} \sum_{i=0}^{m} E^{\alpha-1}_{m-i} s_{in} = C(\alpha, 0)(s_{mn})$.

(i) *It holds that* $C(\alpha, 0) \in B(\mathcal{A}^2_k)$ *for each* $\alpha \geq 0$.

(ii) *If the condition* $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{k-\alpha-1} n^{k-1} |a_{mn}|^k = O(1)$ *is satisfied then* $C(\alpha, 0) \in B(\mathcal{A}^2_k)$ *for each* $-1 < \alpha < 0$.

Corollary 2 *Let* $\theta^{\beta}_{mn} = \frac{1}{E^{\beta}_n} \sum_{j=0}^{n} E^{\beta-1}_{n-j} s_{mj} = (C, 0, \beta)(s_{mn})$.

(i) *It holds that* $C(0, \beta) \in B(\mathcal{A}^2_k)$ *for each* $\beta \geq 0$.

(ii) *If the condition* $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{k-1} n^{k-\beta-1} |a_{mn}|^k = O(1)$ *is satisfied then* $C(0, \beta) \in B(\mathcal{A}^2_k)$ *for each* $-1 < \beta < 0$.

Corollary 3 *Let* $\sigma_{mn} = \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{j=0}^{n} s_{ij} = (C, 1, 1)(s_{mn})$. *Then* $C(1, 1) \in B(\mathcal{A}^2_k)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally and significantly in writing this paper. Both the authors read and approved the final manuscript.

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