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Approximate pexiderized gamma-beta type functions

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Abstract

We show that every unbounded approximate pexiderized gamma-beta type function has a gamma-beta type. That is, we obtain the superstability of the pexiderized gamma-beta type functional equation

$$\beta(x,y)f(x+y) = g(x)h(y)$$

and also investigate the superstability as the following form:

$$\left|\frac{\beta(x,y)f(x+y)}{g(x)h(y)} - 1\right| \le \varphi(x,y).$$

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1 Introduction

In 1940, Ulam gave a wide ranging talk in the Mathematical Club of the University of Wisconsin in which he discussed a number of important unsolved problems (ref. [1]). Among those there was a question concerning the stability of homomorphisms: *Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?* In the next year, Hyers [2] answered the question of Ulam for the case where G_1 and G_2 are Banach spaces. Furthermore, the result of Hyers was generalized by Rassias [3]. Since then, the stability problems of various functional equations have been investigated by many authors (ref. [4]).

Baker, Lawrence and Zorzitto [5] proved the Hyers-Ulam stability of the Cauchy exponential equation $f(x+y) = f(x)f(y)$. That is, if the Cauchy difference $f(x+y) - f(x)f(y)$ of a real-valued function f defined on a real vector space is bounded for all x, y, then f is either bounded or exponential. Their result was generalized by Baker [6]: *Let S be a semi-group and let $f : S \to E$ be a mapping where E is a normed algebra in which the norm is multiplicative. If f satisfies the functional inequality $\|f(xy) - f(x)f(y)\| \le \delta$ for all $x, y \in S$, then f is either bounded or multiplicative.* That is, every unbounded approximate multiplica-

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tive function is multiplicative. Such a phenomenon for functional equations is called the superstability.

The author [7] proved superstability of the pexiderized multiplicative functional equation

$$f(x+y) = g(x)h(x),$$

and the author and Kim [8] also obtained superstability of the gamma-beta type functional equation

$$\beta(x,y)f(x+y) = f(x)f(y),$$

where $\beta(x,y)$ is a beta-type function.

In this paper, we generalize it to the pexiderized gamma-beta type functional equation

$$\beta(x,y)f(x+y) = g(x)h(y). \tag{1.1}$$

And then we prove the superstability of this equation and obtain the superstability in the sense of Ger [9].

2 Definitions and solutions

Throughout this paper, we denote by D an additive subset (that is, $x+y \in D$ for all $x,y \in D$) of R containing all positive integers Z^+.

Definition 1 Let a function $\beta : D \times D \to R - \{0\}$ satisfy the following conditions (a)∼(e):

(a) $\beta(x,y) = \beta(y,x)$ $(x,y \in D)$,
(b) $|\beta(n,m)| \leq 1$ $(n,m \in Z^+)$,
(c) $\frac{\beta(x,y)\beta(z,x+y)}{\beta(x,y+z)\beta(y,z)} = 1$ $(x,y \in D)$,
(d) $\lim_{n\to\infty} \prod_{i=1}^{n-1} |\beta(im,m)| = 0$ $(m \in Z^+)$,
(e) $|\beta(x,n)| < \infty$ ($n \in Z^+$ and fixed $x \in D$).

Then we call β *a beta-type function.*

Definition 2 Let a function $\varphi : D \to [0,\infty)$ and a beta-type function $\beta : D \times D \to R - \{0\}$ be given. If a function $f : D \to R$ satisfies that

$$\left|\beta(x,y)f(x+y) - f(x)f(y)\right| \leq \varphi(x,y)$$

for all $(x,y) \in D \times D$, then we call f *a* $\{\varphi,\beta\}$*-approximate gamma-beta type function.* In the case of $\varphi = 0$, we call f a gamma-beta type function.

Definition 3 Let a function $\varphi : D \to [0,\infty)$ and a beta-type function $\beta : D \times D \to R - \{0\}$ be given. If a function $f : D \to R$ satisfies that

$$\left|\beta(x,y)f(x+y) - g(x)h(y)\right| \leq \varphi(x,y)$$

for all $(x,y) \in D \times D$ and for some functions $g,h : D \to R$, then we call f a $\{\varphi,\beta\}$-*approximate pexiderized gamma-beta type function*. In the case of $\varphi = 0$, we call f a pexiderized gamma-beta type function.

Examples and solutions

If $f, g, h : R^+ \to R^+$ are functions satisfying equation (1.1) and $\beta(x,y) = \frac{1}{a^{xy}}$ $(a > 1)$, then β is a beta-type function and $f(x) = a^{\frac{x^2}{2}+3}$, $g(x) = a^{\frac{x^2}{2}+2}$, $h(x) = a^{\frac{x^2}{2}+1}$ are solutions of it.

Now, we consider the gamma and the beta functions. Note that the beta function $B(x,y)$ is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \quad (x > 0, y > 0)$$

and the gamma function is defined by

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt \quad (x > 0).$$

It is well known that B and Γ satisfy the gamma-beta functional equation

$$B(x,y)\Gamma(x+y) = \Gamma(x)\Gamma(y) \tag{2.1}$$

for all $x, y \in (0,\infty)$. Also, $B(x,y) = B(y,x)$ and

$$B(n,m) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)} = \frac{(n-1)!(m-1)!}{(n+m-1)!} < 1$$

for all $x, y \in (0,\infty)$ and nonnegative integers n, m. By (2.1), we have

$$\begin{aligned}\prod_{i=1}^{n-1} B(im,m) &= \frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} \cdot \frac{\Gamma(2m)\Gamma(m)}{\Gamma(3m)} \cdot \cdots \cdot \frac{\Gamma((n-1)m)\Gamma(m)}{\Gamma(nm)} \\ &= \frac{\Gamma(m)^n}{\Gamma(nm)} \\ &= \frac{[(m-1)!]^n}{(nm-1)!} \to 0\end{aligned}$$

as $n \to \infty$ and

$$\frac{B(x,y)B(z,x+y)}{B(x,y+z)B(y,z)} = 1$$

for all $x, y, z \in (0,\infty)$. Also, for all $x \in (0,\infty)$ and $n \in Z^+$,

$$B(x,n) = \frac{\Gamma(n)\Gamma(x)}{\Gamma(x+n)} = \frac{(n-1)!}{(x+n-1)(x+n-2)\cdots(x+1)x} < \frac{1}{x}.$$

Thus, $B : (0,\infty) \times (0,\infty) \to (0,\infty)$ is a beta-type function and Γ is a gamma-beta type function.

If $\beta(x,y)$ is the beta function and

$$f(x) = 6a^{x+1}\Gamma(x), \qquad g(x) = 3a^x\Gamma(x), \qquad h(x) = 2a^{x+1}\Gamma(x),$$

then f, g, h are the solutions of equation (1.1).

3 Superstability of a gamma-beta type functional equation

The following Theorem 1 with $\phi(x) = \delta$ states that every unbounded approximate pexiderized gamma-beta type function is a gamma-beta type function.

Theorem 1 *Let a function $\phi : D \to [0,\infty)$ be given and let $\varphi(x,y) = \min\{\phi(x),\phi(y)\}$. Suppose that $\beta : D \times D \to R - \{0\}$ is a beta-type function and $f,g,h : D \to R$ are functions such that f is a $\{\varphi,\beta\}$-approximate gamma-beta type function, $f(s) = g(s)$ for some $s \in D$ and*

$$\left|\beta(x,y)f(x+y) - g(x)h(y)\right| \le \varphi(x,y) \tag{3.1}$$

for all $(x,y) \in D \times D$.

(a) *If h is unbounded, then f and g are unbounded gamma-beta type functions.*
(b) *If $|g(m)| \ge \max\{2,(8\phi(s)+4\phi(m))/|h(m)||g(s)|\}$ for some positive integer m, then f and g are unbounded gamma-beta type functions.*

Proof (a) Suppose that h is unbounded. Since f is a $\{\varphi,\beta\}$-approximate gamma-beta type function,

$$\begin{aligned}\left|h(x) - f(x)\right| &= \frac{1}{|g(s)|}\left|g(s)h(x) - f(s)f(x)\right| \\ &\le \frac{|\beta(x,s)f(x+s) - f(x)f(s)| + |\beta(x,s)f(x+s) - h(x)g(s)|}{|g(s)|} \\ &\le \frac{2\phi(s)}{|g(s)|}\end{aligned} \tag{3.2}$$

for all $x \in D$. Also, since

$$\begin{aligned}\left|g(x)h(y) - g(y)h(x)\right| &\le \left|g(x)h(y) - \beta(x,y)f(x+y)\right| + \left|\beta(y,x)f(x+y) - g(y)h(x)\right| \\ &\le 2\phi(y),\end{aligned}$$

we have

$$\left|g(x)\right| \le \left|h(x)\right|\left|\frac{g(y)+2\phi(y)}{h(y)}\right|$$

and

$$\left|h(x)\right| \le \left|g(x)\right|\left|\frac{h(y)+2\phi(y)}{g(y)}\right|$$

for all $x \in D$ and for fixed $y \in D$. Thus, f and g are unbounded. By the unboundedness of h, we can choose a sequence $\{y_n\}$ in Z^+ such that $|h(y_n)| \to \infty$ as $n \to \infty$. By the conditions (a), (c) and (e) of the beta-type function β and (3.2), we have

$$\begin{aligned}&\left|h(y_n)\right|\left|\beta(x,y)g(x+y) - g(x)g(y)\right| \\ &\quad\le \left|\beta(x,y)\right|\left|h(y_n)g(x+y) - \beta(x+y,y_n)f(x+y+y_n)\right| \\ &\qquad+ \left|\beta(y,y_n)\right|\left|\beta(x,y+y_n)f(x+y+y_n) - g(x)h(y+y_n)\right|\end{aligned}$$

$$
\begin{aligned}
&\quad + \left|\beta(y, y_n)\right|\left|h(y + y_n) - f(y + y_n)\right|\left|g(x)\right| \\
&\quad + \left|g(x)\right|\left|\beta(y, y_n)f(y + y_n) - g(y)h(y_n)\right| \\
&\leq \left|\beta(x, y)\right|\phi(x + y) + \left|\beta(y, y_n)\right|\phi(x) \\
&\quad + \left|g(x)\right|\left|\beta(y, y_n)\right|\frac{2\phi(s)}{|g(s)|} + \left|g(x)\right|\phi(y) < \infty
\end{aligned}
\tag{3.3}
$$

for all sufficiently large y_n and $(x, y) \in D \times D$. It follows from (3.3) by dividing $|h(y_n)|$ that

$$\beta(x, y)g(x, y) = g(x)g(y)$$

for all $(x, y) \in D \times D$. Also, by letting $f = g = h$ in (3.3) and using the property of an approximate gamma-beta type function, we have

$$\beta(x, y)f(x, y) = f(x)f(y)$$

for all $(x, y) \in D \times D$.

(b) If we replace x by m and also y by m in (3.1), respectively, we get

$$\left|\beta(m, m)f(2m) - g(m)h(m)\right| \leq \phi(m).$$

Note that $|f(x) - h(x)| \leq 2\phi(s)/|g(s)|$ from the proof of (a). An induction argument implies that for all $n \geq 2$,

$$
\begin{aligned}
&\left|f(nm)\prod_{i=1}^{n-1}\beta(im, m) - g(m)^{n-1}h(m)\right| \\
&\quad \leq \phi(m)\prod_{i=1}^{n-2}\left|\beta(im, m)\right| \\
&\qquad + \left|g(m)\right|\left(\frac{2\phi(s)}{|g(s)|}\prod_{i=1}^{n-2}\left|\beta(im, m)\right| + \phi(m)\prod_{i=1}^{n-3}\left|\beta(im, m)\right|\right) \\
&\qquad + \left|g(m)\right|^2\left(\frac{2\phi(s)}{|g(s)|}\prod_{i=1}^{n-3}\left|\beta(im, m)\right| + \phi(m)\prod_{i=1}^{n-4}\left|\beta(im, m)\right|\right) \\
&\qquad + \cdots + \left|g(m)\right|^{n-2}\left(\frac{2\phi(s)}{|g(s)|}\left|\beta(m, m)\right| + \phi(m)\right).
\end{aligned}
\tag{3.4}
$$

To prove the inequality (3.4) by the induction, suppose that the inequality (3.4) holds for $k = n \geq 2$. Let $k = n + 1$. Then we have

$$
\begin{aligned}
&\left|f\big((n + 1)m\big)\prod_{i=1}^{n}\beta(im, m) - g(m)^{n}h(m)\right| \\
&\quad \leq \left|\beta(nm, m)f\big((n + 1)m\big) - g(m)h(nm)\right|\prod_{i=1}^{n-1}\left|\beta(im, m)\right| \\
&\qquad + \left|g(m)\right|\left|h(nm) - f(nm)\right|\prod_{i=1}^{n-1}\left|\beta(im, m)\right|
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \left|g(m)\right| \left| f(nm) \prod_{i=1}^{n-1} \beta(im, m) - g(m)^{n-1} h(m) \right| \\
&\le \phi(m) \prod_{i=1}^{n-1} \left|\beta(im, m)\right| + \left|g(m)\right| \frac{2\phi(s)}{|g(s)|} \prod_{i=1}^{n-1} \left|\beta(im, m)\right| \\
&\quad + \left|g(m)\right| \left| f(nm) \prod_{i=1}^{n-1} \beta(im, m) - g(m)^{n-1} h(m) \right|
\end{aligned}
$$

for all $n \ge 2$. And thus we get

$$
\begin{aligned}
&\left| f\bigl((n+1)m\bigr) \prod_{i=1}^{n} \beta(im, m) - g(m)^n h(m) \right| \\
&\le \phi(m) \prod_{i=1}^{n-1} \left|\beta(im, m)\right| \\
&\quad + \left|g(m)\right| \left(\frac{2\phi(s)}{|g(s)|} \prod_{i=1}^{n-1} \left|\beta(im, m)\right| + \phi(m) \prod_{i=1}^{n-2} \left|\beta(im, m)\right| \right) \\
&\quad + \left|g(m)\right|^2 \left(\frac{2\phi(s)}{|g(s)|} \prod_{i=1}^{n-2} \left|\beta(im, m)\right| + \phi(m) \prod_{i=1}^{n-3} \left|\beta(im, m)\right| \right) \\
&\quad + \cdots + \left|g(m)\right|^{n-1} \left(\frac{2\phi(s)}{|g(s)|} \left|\beta(m, m)\right| + \phi(m) \right)
\end{aligned}
$$

for all $n \ge 2$. By the induction, the inequality (3.4) holds for all $n \in Z+$. Note that

$$
\prod_{i=1}^{n-1} \beta(im, m) < \infty, \quad \text{and} \quad \left|g(m)^{n-1}\right| \to \infty \quad \text{as } n \to \infty.
$$

By dividing $g(m)^{n-1}h(m)$ by (3.4), we get

$$
\begin{aligned}
&\left| \frac{f(nm) \prod_{i=1}^{n-1} \beta(im, m)}{g(m)^{n-1} h(m)} - 1 \right| \\
&\le \frac{1}{g(m)^{n-1} h(m)} \left(\phi(m) + \left(\frac{2\phi(s)}{|g(s)|} + \phi(m) \right) \left|g(m)\right| \right. \\
&\quad \left. + \left|g(m)\right|^2 \left(\frac{2\phi(s)}{|g(s)|} + \phi(m) \right) + \cdots + \left|g(m)\right|^{n-2} \left(\frac{2\phi(s)}{|g(s)|} + \phi(m) \right) \right) \\
&\le \frac{2\phi(s) + \phi(m)}{|h(m)||g(m)||g(s)|} \left(\frac{1}{|g(m)|^{m-2}} + \frac{1}{|g(m)|^{m-3}} + \cdots + \frac{1}{|g(m)|} + 1 \right) \\
&\le \frac{2\phi(s) + \phi(m)}{|h(m)||g(m)||g(s)|} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} \cdots \right) \\
&\le 2 \frac{2\phi(s) + \phi(m)}{|h(m)||g(m)||g(s)|} \\
&\le \frac{1}{2}
\end{aligned}
$$

for all positive integer n. Thus, we can easily show that

$$\left|f(nm)\right| \to \infty \quad \text{as } n \to \infty.$$

Thus, f is unbounded and so h is unbounded. By (a), we complete the proof. □

Corollary 1 *Let $\delta > 0$ be given and $\beta(x,y)$ be a beta-type function on $(0,\infty)$. Suppose that f is a function from $(0,\infty)$ into $(0,\infty)$ with $f(m) \geq \max\{2,(12\delta)^{1/3}\}$ for some positive integer m such that*

$$\left|\beta(x,y)f(x+y) - f(x)f(y)\right| \leq \delta$$

for all $x, y \in (0,\infty)$. Then

$$\beta(x,y)f(x+y) = f(x)f(y)$$

for all $x, y \in (0,\infty)$.

Proof By Theorem 1 with $\phi(x) = \delta$ and $s = m$, we complete the proof. □

Corollary 2 *Let $\delta > 0$ be given. Suppose that $h, g : (0,\infty) \to (0,\infty)$ are functions with $g(1) = 1$, h is unbounded, $|g(m)| \geq \max(2, \sqrt{12\delta})$ for some positive integer m and*

$$\left|B(x,y)\Gamma(x+y) - g(x)h(y)\right| \leq \delta$$

for all $x, y \in (0,\infty)$, where $B(x,y)$ is the beta function and $\Gamma(x)$ is the gamma function. Then

$$B(x,y)g(x+y) = g(x)g(y)$$

for all $x, y \in (0,\infty)$.

Corollary 3 *Let $\delta > 0$ and $a > 1$ be given. Suppose that $f : (0,\infty) \to (0,\infty)$ is a function with $|f(m)| \geq \max\{2,(12\delta)^{1/3}\}$ for some positive integer m and*

$$\left|\frac{1}{a^{xy}}f(x+y) - f(x)f(y)\right| \leq \delta$$

for all $x, y \in (0,\infty)$. Then

$$f(x+y) = a^{xy}f(x)f(y)$$

for all $x, y \in (0,\infty)$.

Proof Let $\beta(x,y) = \frac{1}{a^{xy}}$ for all $x, y \in (0,\infty)$. Then $\beta(x,y) = \beta(y,x)$ and $0 < \beta(x,y) < 1$. Also,

$$\frac{\beta(x,y)\beta(z,x+y)}{\beta(x,y+z)\beta(y,z)} = \frac{a^{x(y+z)}a^{yz}}{a^{xy}a^{z(x+y)}} = 1$$

for all $x, y, z \in (0, \infty)$ and

$$\prod_{i=1}^{n-1} \beta(im, m) = \prod_{i=1}^{n-1} \frac{1}{a^{im^2}} \to 0$$

as $n \to \infty$. Also, $\beta(x, n) < \infty$ for fixed x. Thus, $\beta(x, y)$ is a beta-type function. By Theorem 1 with $\phi(x) = \delta$ and $s = m$, we complete the proof. □

Corollary 4 *Let $\delta > 0$ and $k > 1$ be given. Suppose that $f : R \to R$ is a function with $|f(m)| \geq \max\{2, (12\delta)^{1/3}\}$ for some positive integer m such that*

$$\left| \frac{1}{k} f(x + y) - f(x)f(y) \right| \leq \delta$$

for all $x, y \in R$. Then

$$f(x + y) = kf(x)f(y)$$

for all $x, y \in R$.

Proof By Theorem 1 with $\beta(x, y) = \frac{1}{k}, s = m$ and $\phi(x) = \delta$, we complete the proof. □

4 Superstability of a gamma-beta type functional equation in the sense of Ger

Ger [9] suggested a new type of stability for the exponential equation of the following form:

$$\left| \frac{f(x + y)}{f(x)f(y)} - 1 \right| \leq \delta.$$

In this section, the superstability problem in the sense of Ger for a gamma-beta type functional equation will be investigated.

Theorem 2 *Let $\varphi : D \times D \to (0, 1)$ be a function such that $\varphi(x, y_n) \to 0$ as $y_n \to \infty$ and let a beta-type function $\beta : D \times D \to (0, \infty)$ be given.*

(a) *Suppose that a function $f : D \to (0, \infty)$ satisfies*

$$\left| \frac{\beta(x, y)f(x + y)}{f(x)f(y)} - 1 \right| \leq \varphi(x, y) \tag{4.1}$$

for all $(x, y) \in D \times D$. Then

$$\beta(x, y)f(x + y) = f(x)f(y)$$

for all $(x, y) \in D \times D$.

(b) *Suppose that the inequality* (4.1) *holds and functions $f, g, h : D \to (0, \infty)$ satisfy*

$$\left| \frac{\beta(x, y)f(x + y)}{g(x)h(y)} - 1 \right| \leq \varphi(x, y) \tag{4.2}$$

for all $(x, y) \in D \times D$. *If* $f(s) = g(s)$ *for some* $s \in D$, *then*

$$\beta(x, y)g(x + y) = g(x)g(y)$$

for all $(x, y) \in D \times D$.

(c) *Suppose that the inequalities* (4.1) *and* (4.2) *hold. If* $f(s) = h(s)$ *for some* $s \in D$ *and* $\varphi(x, y) = \varphi(y, x)$, *then*

$$\beta(x, y)h(x + y) = h(x)h(y)$$

for all $(x, y) \in D \times D$.

Proof (a) Choose a sequence $\{y_n\}$ in D such that $y_n \to \infty$. For all $x, y, y_n \in D$, we have

$$\begin{aligned}\frac{\beta(x, y)f(x + y)}{f(x)f(y)} &= \frac{f(x + y)f(y_n)}{\beta(x + y, y_n)f(x + y + y_n)} \cdot \frac{\beta(x, y + y_n)f(x + y + y_n)}{f(x)f(y + y_n)} \\ &\quad \cdot \frac{\beta(y, y_n)f(y + y_n)}{f(y)f(y_n)} \cdot \frac{\beta(x, y)\beta(x + y, y_n)}{\beta(x, y + y_n)\beta(y, y_n)}.\end{aligned}$$

By the condition (c) of a beta-type function β and (4.1), we have

$$\begin{aligned}&\lim_{y_n \to \infty} \frac{1}{1 + \varphi(x + y, y_n)} \big(1 - \varphi(x, y + y_n)\big)\big(1 - \varphi(y, y_n)\big) \\ &\quad = 1 \\ &\quad \le \frac{\beta(x, y)f(x + y)}{f(x)f(y)} \\ &\quad \le \lim_{y_n \to \infty} \frac{1}{1 - \varphi(x + y, y_n)} \big(1 + \varphi(x, y + y_n)\big)\big(1 + \varphi(y, y_n)\big) \\ &\quad = 1\end{aligned}$$

for all $x, y \in D$. Thus, we complete the proof of (a).

(b) Choose a sequence $\{y_n\}$ in D such that $y_n \to \infty$. For all $y, y_n \in D$, we have

$$\frac{h(y + y_n)}{f(y + y_n)} = \frac{\beta(s, y + y_n)f(s + y + yn)}{f(y + y_n)f(s)} \cdot \frac{g(s)h(y + y_n)}{\beta(s, y + y_n)f(s + y + y_n)}$$

and for all $x, y, y_n \in D$, we get

$$\begin{aligned}\frac{\beta(x, y)g(x + y)}{g(x)g(y)} &= \frac{g(x + y)h(y_n)}{\beta(x + y, y_n)f(x + y + y_n)} \cdot \frac{\beta(x, y + y_n)f(x + y + y_n)}{g(x)h(y + y_n)} \\ &\quad \cdot \frac{h(y + y_n)}{f(y + y_n)} \cdot \frac{\beta(y, y_n)f(y + y_n)}{g(y)h(y_n)} \cdot \frac{\beta(x, y)\beta(x + y, y_n)}{\beta(x, y + y_n)\beta(y, y_n)}.\end{aligned}$$

By the condition (c) of a beta-type function β and (4.2), we have

$$\begin{aligned}&\lim_{y_n \to \infty} \frac{1}{1 + \varphi(x + y, y_n)} \big(1 - \varphi(x, y + y_n)\big) \frac{1}{1 + \varphi(s, y + y_n)} \big(1 - \varphi(y, y_n)\big) \\ &\quad = 1\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\beta(x,y)g(x+y)}{g(x)g(y)} \\
&\le \lim_{y_n\to\infty} \frac{1}{1-\varphi(x+y,y_n)}\bigl(1+\varphi(x,y+y_n)\bigr)\frac{1}{1-\varphi(s,y+y_n)}\bigl(1+\varphi(y,y_n)\bigr) \\
&= 1
\end{aligned}
$$

for all $x, y \in D$. Thus, we complete the proof of (b). Similarly, we obtain (c) from (b). □

Remark 1 Consider the following inequalities: For all $(x,y) \in D \times D$,

$$\frac{1}{1+\varphi(x,y)} \le \frac{\beta(x,y)f(x+y)}{f(x)f(y)} \le 1+\varphi(x,y) \tag{4.3}$$

and

$$\frac{1}{1+\varphi(x,y)} \le \frac{\beta(x,y)f(x+y)}{g(x)h(y)} \le 1+\varphi(x,y), \tag{4.4}$$

where $\varphi : D \times D \to (0,\infty)$ is a function such that $\varphi(x,y_n) \to 0$ as $y_n \to \infty$. If we replace the inequality (4.1) by (4.3) and (4.2) by (4.3) respectively, then we have the same result as Theorem 2.

Corollary 5 *Let $\varphi : (0,\infty) \times (0,\infty) \to (0,1)$ be a function such that $\varphi(x,y_n) \to 0$ as $y_n \to \infty$ and let the beta function B and the gamma function Γ be given. If functions $g, h : (0,\infty) \to (0,\infty)$ satisfy*

$$\left|\frac{B(x,y)\Gamma(x+y)}{g(x)h(y)} - 1\right| \le \varphi(x,y)$$

for all $(x,y) \in (0,\infty) \times (0,\infty)$. If $g(1) = 1$, then

$$\beta(x,y)g(x+y) = g(x)g(y)$$

for all $(x,y) \in (0,\infty) \times (0,\infty)$.

Corollary 6 *Let $\varphi : (0,\infty) \times (0,1) \to (0,\infty)$ be a function such that $\varphi(x,y_n) \to 0$ as $y_n \to \infty$ and let $a > 1$ be given. If a function $f : (0,\infty) \to (0,\infty)$ satisfies*

$$\left|\frac{\frac{1}{a^{xy}}f(x+y)}{f(x)f(y)} - 1\right| \le \varphi(x,y)$$

for all $(x,y) \in (0,\infty) \times (0,\infty)$, then

$$\frac{1}{a^{xy}}f(x+y) = f(x)f(y)$$

for all $(x,y) \in (0,\infty) \times (0,\infty)$.

Corollary 7 *Let $\varphi : (0,\infty) \times (0,\infty) \to (0,1)$ be a function such that $\varphi(x,y_n) \to 0$ as $y_n \to \infty$ and let $k > 1$ be given. If a function $f : (0,\infty) \to (0,\infty)$ satisfies*

$$\left|\frac{\frac{1}{k}f(x+y)}{f(x)f(y)} - 1\right| \le \varphi(x,y)$$

for all $(x, y) \in (0, \infty) \times (0, \infty)$, *then*

$$\frac{1}{k} f(x + y) = f(x) f(y)$$

for all $(x, y) \in (0, \infty) \times (0, \infty)$.

Competing interests
The author did not provide this information.

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References

1. Ulam, SM: Problems in Modern Mathematics. Proc. Chap. VI. Wiley, New York (1964)
2. Hyers, DH: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. USA **27**, 222-224 (1941)
3. Rassias, ThM: The problem of S.M. Ulam for approximately multiplication mappings. J. Math. Anal. Appl. **246**, 352-378 (2000)
4. Forti, GL: Hyers-Ulam stability of functional equations in several variables. Aequ. Math. **50**, 146-190 (1995)
5. Baker, J, Lawrence, J, Zorzitto, F: The stability of the equation $f(x + y) = f(x) + f(y)$. Proc. Am. Math. Soc. **74**, 242-246 (1979)
6. Baker, J: The stability of the cosine equations. Proc. Am. Math. Soc. **80**, 411-416 (1980)
7. Lee, YW: Superstability and stability of the Pexiderized multiplicative functional equation. Hindawi Pub. Corp. J. Inequal. Appl. **2010**, 486325 (2010). doi:1155/2010/486325
8. Lee, YW, Kim, GH: Approximate gamma-beta type functions. Nonlinear Anal., Theory Methods Appl. **71**, e1567-e1574 (2009)
9. Ger, R: Superstability is not natural. Rocznik Naukowo-Dydaktyczny WSP Krakkowie, Prace Mat. **159**, 109-123 (1993)

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