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A class of analytic functions involving in the Dziok-Srivastava operator

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Abstract

Let $\mathcal{A}$ be a class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{0.1}$$

which are analytic in the open unit disk $\mathbb{U}$. By means of the Dziok-Srivastava operator, we introduce a new subclass

$$\mathcal{S}_m^l(\alpha_1, \alpha, \mu) \quad \left(l \le m+1, l, m \in \mathbb{N} \cup \{0\}, -\frac{\pi}{2} < \alpha < \frac{\pi}{2}, \mu > -\cos\alpha\right)$$

of $\mathcal{A}$. In particular, $\mathcal{S}_0^1(2,0,0)$ coincides with the class of uniformly convex functions introduced by Goodman. The order of starlikeness and the radius of α-spirallikeness of order β ($\beta < 1$) are computed. Inclusion relations and convolution properties for the class $\mathcal{S}_m^l(\alpha_1, \alpha, \mu)$ are obtained. A special member of $\mathcal{S}_m^l(\alpha_1, \alpha, \mu)$ is also given. The results presented here not only generalize the corresponding known results, but also give rise to several other new results.

MSC: Primary 30C45

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1 Introduction

Let $\mathcal{A}$ be a class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. For $\beta < 1$, a function $f(z) \in \mathcal{A}$ is said to be starlike of order β in $\mathbb{U}$ if

$$\Re \frac{zf'(z)}{f(z)} > \beta \quad (z \in \mathbb{U}). \tag{1.2}$$

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This class is denoted by $\mathcal{S}^*(\beta)$ $(\beta < 1)$. For $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ and $\beta < 1$, a function $f(z) \in \mathcal{A}$ is said to be α-spirallike of order β in $\mathbb{U}$ if

$$\Re\left\{e^{i\alpha}\frac{zf'(z)}{f(z)}\right\} > \beta\cos\alpha \quad (z \in \mathbb{U}). \tag{1.3}$$

When $0 \leq \beta < 1$, it is well known that all the starlike functions of order β and α-spirallike functions of order β are univalent in $\mathbb{U}$. A function $f(z) \in \mathcal{A}$ is said to be convex univalent in $\mathbb{U}$ if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad (z \in \mathbb{U}). \tag{1.4}$$

We denote this class by $\mathcal{K}$. Also, let $\mathcal{UCV}(\subset \mathcal{K})$ be the class of uniformly convex functions in $\mathbb{U}$ introduced by Goodman [1]. It was shown in [2] that $f(z) \in \mathcal{A}$ is in $\mathcal{UCV}$ if and only if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in \mathbb{U}). \tag{1.5}$$

In [2], Rønning investigated the class $\mathcal{S}_p$ defined by

$$\mathcal{S}_p = \left\{f(z) \in \mathcal{S}^*(0) : f(z) = zg'(z), g(z) \in \mathcal{UCV}\right\}. \tag{1.6}$$

The uniformly convex and related functions have been studied by many authors (see, *e.g.*, [1–10] and the references therein).

If

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A} \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A},$$

then the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

For

$$\alpha_j \in \mathbb{C} \quad (j = 1, 2, \ldots, l) \quad \text{and} \quad \beta_j \in \mathbb{C}\backslash\{0, -1, -2, \ldots\} \quad (j = 1, 2, \ldots, m),$$

the generalized hypergeometric function

$${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$$

is defined by the following infinite series:

$${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{z^n}{n!}$$
$$\bigl(l \leq m + 1; l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{U}\bigr),$$

where $(c)_n$ is the Pochhammer symbol defined by

$$(c)_n = \begin{cases} 1 & (n = 0), \\ c(c+1)\cdots(c+n-1) & (n \in \mathbb{N}). \end{cases}$$

Corresponding to the function

$$z \cdot {}_l F_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z),$$

the Dziok-Srivastava operator (see [11])

$$H(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) : \mathcal{A} \to \mathcal{A}$$

is defined by the following Hadamard product:

$$H(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z) = \big(z \cdot {}_l F_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)\big) * f(z)$$
$$(l \le m+1; l, m \in \mathbb{N}_0; z \in \mathbb{U}).$$

If $f(z) \in \mathcal{A}$ is given by (1.1), then we have

$$H(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z) = z + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{a_{n+1}}{n!} z^{n+1} \quad (z \in \mathbb{U}). \tag{1.7}$$

In order to make the notation simple, we write

$$H_m^l(\alpha_1) = H(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) \quad (l \le m+1; l, m \in \mathbb{N}_0). \tag{1.8}$$

It should also be remarked that the Dziok-Srivastava operator $H_m^l(\alpha_1)$ is a generalization of several linear operators considered in earlier investigations (see [12–19], also see [20]).

In this paper we introduce and investigate the following subclass of $\mathcal{A}$.

Definition A function $f(z) \in \mathcal{A}$ is said to be in $\mathcal{S}_m^l(\alpha_1, \alpha, \mu)$ if it satisfies the condition

$$\Re\left\{ e^{i\alpha} \frac{z(H_m^l(\alpha_1)f(z))'}{H_m^l(\alpha_1)f(z)} \right\} + \mu > \left| \frac{z(H_m^l(\alpha_1)f(z))'}{H_m^l(\alpha_1)f(z)} - 1 \right| \quad (z \in \mathbb{U}), \tag{1.9}$$

where

$$l \le m+1, \quad l, m \in \mathbb{N}_0, \quad -\frac{\pi}{2} < \alpha < \frac{\pi}{2} \quad \text{and} \quad \mu > -\cos\alpha. \tag{1.10}$$

Note that $f(z) = z \in \mathcal{S}_m^l(\alpha_1, \alpha, \mu)$ and that

$$\mathcal{S}_0^1(1, \alpha, 0) = \left\{ f(z) \in \mathcal{A} : \Re\left\{ e^{i\alpha} \frac{zf'(z)}{f(z)} \right\} > \left| \frac{zf'(z)}{f(z)} - 1 \right| (z \in \mathbb{U}) \right\}. \tag{1.11}$$

Also,

$$\mathcal{S}_0^1(1,0,0) = \mathcal{S}_p \quad \text{and} \quad \mathcal{S}_0^1(2,0,0) = \mathcal{UCV}. \tag{1.12}$$

Throughout this paper we assume, unless otherwise stated, that l, m, α and μ satisfy (1.10).

2 Subordination theorem

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. We say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and we write $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that

$$\left|w(z)\right| \leq |z| \quad \text{and} \quad f(z) = g\bigl(w(z)\bigr) \quad (z \in \mathbb{U}).$$

If $g(z)$ is univalent in $\mathbb{U}$, then

$$f(z) \prec g(z) \quad \Leftrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Theorem 1 *A function $f(z) \in \mathcal{A}$ is in $\mathcal{S}_m^l(\alpha_1, \alpha, \mu)$ if and only if*

$$e^{i\alpha} \frac{z(H_m^l(\alpha_1)f(z))'}{H_m^l(\alpha_1)f(z)} \prec h(z)\cos\alpha + i\sin\alpha, \tag{2.1}$$

where

$$\begin{aligned} h(z) &= 1 + \frac{2}{\pi^2}\left(1 + \frac{\mu}{\cos\alpha}\right)\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 \\ &= 1 + \frac{8}{\pi^2}\left(1 + \frac{\mu}{\cos\alpha}\right)\left\{z + \frac{2}{3}z^2 + \frac{23}{45}z^3 + \cdots\right\} \quad (z \in \mathbb{U}). \end{aligned} \tag{2.2}$$

Proof Let us define $w(z) = u + iv$ by

$$e^{i\alpha} \frac{z(H_m^l(\alpha_1)f(z))'}{H_m^l(\alpha_1)f(z)} = w(z)\cos\alpha + i\sin\alpha \quad (z \in \mathbb{U}). \tag{2.3}$$

Then $w(0) = 1$ and the inequality (1.9) can be rewritten as

$$u > \frac{\cos\alpha}{2(\cos\alpha + \mu)}v^2 + \frac{1}{2}\left(1 - \frac{\mu}{\cos\alpha}\right). \tag{2.4}$$

Thus

$$w(\mathbb{U}) \subset \Omega = \bigl\{w = u + iv : u \text{ and } v \text{ satisfy (2.4)}\bigr\}.$$

It follows from (2.2) that $h(0) = 1$. In order to prove the theorem, it suffices to show that the function $w = h(z)$ given by (2.2) maps $\mathbb{U}$ conformally onto the parabolic region Ω.

Note that $\frac{1}{2}(1 - \frac{\mu}{\cos\alpha}) < 1$. Consider the transformations

$$w_1 = \sqrt{w-1}, \qquad w_2 = \exp\left(\pi w_1 \sqrt{\frac{2\cos\alpha}{\cos\alpha + \mu}}\right), \qquad t = \frac{1}{2}\left(w_2 + \frac{1}{w_2}\right). \tag{2.5}$$

It is easy to verify that the composite function

$$t = \mathrm{ch}\left(\pi\sqrt{\frac{2\cos\alpha(w-1)}{\cos\alpha+\mu}}\right) = g(w) \quad \text{(say)}$$

maps $\Omega^+ = \Omega \cap \{w = u + iv : v > 0\}$ conformally onto the upper half-plane $\mathrm{Im}(t) > 0$ so that $w = \Re(w) \in [\frac{1}{2}(1 - \frac{\mu}{\cos\alpha}), +\infty)$ corresponds to $t = \Re(t) \in [-1, +\infty)$ and $w = 1$ to $t = 1$. With the help of the symmetry principle, the function $t = g(w)$ maps Ω conformally onto the region $G = \{t : |\arg(t+1)| < \pi\}$. Since

$$t = 2\left(\frac{1+z}{1-z}\right)^2 - 1 \tag{2.6}$$

maps $\mathbb{U}$ onto G, we see that

$$\begin{aligned} w = g^{-1}(t) &= 1 + \frac{1}{2\pi^2}\left(1 + \frac{\mu}{\cos\alpha}\right)\left(\log\left(t + \sqrt{t^2-1}\right)\right)^2 \\ &= 1 + \frac{2}{\pi^2}\left(1 + \frac{\mu}{\cos\alpha}\right)\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 \\ &= h(z) \end{aligned}$$

maps $\mathbb{U}$ conformally onto Ω. The proof of the theorem is now completed. □

Corollary 1 *Let $f(z) \in \mathcal{S}_m^l(\alpha_1, \alpha, \mu)$. Then for $z \in \mathbb{U}$,*

$$\left|\left(\frac{H_m^l(\alpha_1)f(z)}{z}\right)^{\sec\alpha e^{i\alpha}}\right| \le \exp\left\{\frac{2}{\pi^2}\left(1 + \frac{\mu}{\cos\alpha}\right)\int_0^1 \frac{1}{\rho}\left(\log\frac{1+\sqrt{\rho|z|}}{1-\sqrt{\rho|z|}}\right)^2 d\rho\right\} \tag{2.7}$$

and

$$\left|\left(\frac{H_m^l(\alpha_1)f(z)}{z}\right)^{\sec\alpha e^{i\alpha}}\right| \ge \exp\left\{-\frac{8}{\pi^2}\left(1 + \frac{\mu}{\cos\alpha}\right)\int_0^1 \frac{1}{\rho}\left(\arctan\sqrt{\rho|z|}\right)^2 d\rho\right\}. \tag{2.8}$$

The results are sharp.

Proof From Theorem 1 we have

$$\frac{e^{i\alpha}}{\cos\alpha}\left(\frac{z(H_m^l(\alpha_1)f(z))'}{H_m^l(\alpha_1)f(z)} - 1\right) \prec h(z) - 1$$

for $f(z) \in \mathcal{S}_m^l(\alpha_1, \alpha, \mu)$ and $h(z)$ given by (2.2). Since the function $h(z) - 1$ is univalent and starlike (with respect to the origin) in $\mathbb{U}$, using the result of Suffridge [21, Theorem 3], we get

$$\frac{e^{i\alpha}}{\cos\alpha}\int_0^z\left(\frac{(H_m^l(\alpha_1)f(t))'}{H_m^l(\alpha_1)f(t)} - \frac{1}{t}\right)dt \prec \int_0^z \frac{h(t)-1}{t}\,dt.$$

This implies that

$$\frac{e^{i\alpha}}{\cos\alpha}\log\frac{H_m^l(\alpha_1)f(z)}{z}=\int_0^1\frac{h(\rho w(z))-1}{\rho}\,d\rho \quad (z\in\mathbb{U}), \tag{2.9}$$

where $w(z)$ is analytic and $|w(z)|\leq|z|$ in $\mathbb{U}$.

Noting that $h(z)$ maps the disk $|z|<\rho$ $(0<\rho\leq 1)$ onto a region which is convex and symmetric with respect to the real axis, we know that

$$h\bigl(-\rho|z|\bigr)\leq\Re\bigl\{h\bigl(\rho w(z)\bigr)\bigr\}\leq h\bigl(\rho|z|\bigr) \quad (z\in\mathbb{U}). \tag{2.10}$$

Now (2.2), (2.9) and (2.10) lead to

$$\log\left|\left(\frac{H_m^l(\alpha_1)f(z)}{z}\right)^{\sec\alpha e^{i\alpha}}\right|\leq\frac{2}{\pi^2}\left(1+\frac{\mu}{\cos\alpha}\right)\int_0^1\frac{1}{\rho}\left(\log\frac{1+\sqrt{\rho|z|}}{1-\sqrt{\rho|z|}}\right)^2 d\rho$$

and

$$\begin{aligned}\log\left|\left(\frac{H_m^l(\alpha_1)f(z)}{z}\right)^{\sec\alpha e^{i\alpha}}\right|&\geq\frac{2}{\pi^2}\left(1+\frac{\mu}{\cos\alpha}\right)\int_0^1\frac{1}{\rho}\left(\log\frac{1+i\sqrt{\rho|z|}}{1-i\sqrt{\rho|z|}}\right)^2 d\rho\\&=-\frac{8}{\pi^2}\left(1+\frac{\mu}{\cos\alpha}\right)\int_0^1\frac{1}{\rho}\bigl(\arctan\sqrt{\rho|z|}\bigr)^2 d\rho\end{aligned}$$

for $z\in\mathbb{U}$. Hence we have (2.7) and (2.8).

Furthermore, for

$$\alpha_j\in\mathbb{C}\setminus\{0,-1,-2,\ldots\} \quad (j=1,\ldots,l),$$

it is easy to see that the function $f_0(z)$ in $\mathcal{S}_m^l(\alpha_1,\alpha,\mu)$, defined by

$$\begin{aligned}&H_m^l(\alpha_1)f_0(z)\\&\quad=z\exp\left\{\frac{2}{\pi^2}\left(1+\frac{\mu}{\cos\alpha}\right)\cos\alpha e^{-i\alpha}\int_0^z\frac{1}{t}\left(\log\frac{1+\sqrt{t}}{1-\sqrt{t}}\right)^2 dt\right\} \quad (z\in\mathbb{U}),\end{aligned} \tag{2.11}$$

shows that the estimates (2.7) and (2.8) are sharp. □

Corollary 2 *Let $f(z)\in\mathcal{S}_m^l(\alpha_1,\alpha,\mu)$, where*

$$\alpha_j\in\mathbb{C}\backslash\{0,-1,-2,\ldots\} \quad (j=1,2,\ldots,l).$$

Then

$$\begin{aligned}f(z)&=z\exp\left\{\frac{2\cos\alpha e^{-i\alpha}}{\pi^2}\left(1+\frac{\mu}{\cos\alpha}\right)\int_0^1\frac{1}{\rho}\left(\log\frac{1+\sqrt{\rho w(z)}}{1-\sqrt{\rho w(z)}}\right)^2 d\rho\right\}\\&\quad*\left\{z+\sum_{n=1}^{\infty}\frac{n!(\beta_1)_n\cdots(\beta_m)_n}{(\alpha_1)_n\cdots(\alpha_l)_n}z^{n+1}\right\} \quad (z\in\mathbb{U}),\end{aligned} \tag{2.12}$$

where $w(z)$ is analytic in $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1$ $(z\in\mathbb{U})$.

Proof From (2.9) and (2.2), we have

$$\begin{aligned} &H_m^l(\alpha_1)f(z) \\ &\quad = z\exp\left\{\frac{2\cos\alpha e^{-i\alpha}}{\pi^2}\left(1+\frac{\mu}{\cos\alpha}\right)\int_0^1\frac{1}{\rho}\left(\log\frac{1+\sqrt{\rho w(z)}}{1-\sqrt{\rho w(z)}}\right)^2 d\rho\right\} \quad (z\in\mathbb{U}). \end{aligned} \tag{2.13}$$

For

$$\alpha_j \in \mathbb{C}\backslash\{0,-1,-2,\ldots\} \quad (j=1,2,\ldots,l),$$

from (2.13) and (1.7), we obtain (2.12). □

3 Properties of the class $\mathcal{S}_m^l(\alpha_1,\alpha,\mu)$

Theorem 2 *Let $f(z)\in\mathcal{S}_m^l(\alpha_1,\alpha,\mu)$. Then*

$$H_m^l(\alpha_1)f(z)\in\mathcal{S}^*\left(\frac{1}{2}\left(1-\frac{\mu}{\cos\alpha}\right)\right) \tag{3.1}$$

and the order $\frac{1}{2}(1-\frac{\mu}{\cos\alpha})$ is sharp.

Proof Let $h(z)$ be given by (2.2). It follows from the proof of Theorem 1 that

$$\partial h(\mathbb{U}) = \left\{w=u+iv : u=\frac{\cos\alpha}{2(\cos\alpha+\mu)}v^2+\frac{1}{2}\left(1-\frac{\mu}{\cos\alpha}\right)\right\}. \tag{3.2}$$

By using (3.2), we find that

$$\min_{|z|=1 (z\neq 1)}\Re\left\{e^{-i\alpha}\left(h(z)\cos\alpha+i\sin\alpha\right)\right\} = \min_{v\in(-\infty,+\infty)} g(v)\cos\alpha+\sin^2\alpha,$$

where

$$g(v) = \frac{\cos^2\alpha}{2(\cos\alpha+\mu)}v^2+\frac{\cos\alpha-\mu}{2}+v\sin\alpha \quad (-\infty<v<+\infty).$$

Since

$$g'(v) = \frac{\cos^2\alpha}{\cos\alpha+\mu}v+\sin\alpha, \qquad g''(v)>0,$$

the function $g(v)$ attains its minimum value at

$$v_0 = -\frac{(\cos\alpha+\mu)\sin\alpha}{\cos^2\alpha}.$$

Thus

$$\begin{aligned} &\min_{|z|=1 (z\neq 1)}\Re\left\{e^{-i\alpha}\left(h(z)\cos\alpha+i\sin\alpha\right)\right\} \\ &\quad = g(v_0)\cos\alpha+\sin^2\alpha \end{aligned}$$

$$= -\frac{\sin^2\alpha(\cos\alpha+\mu)}{2\cos\alpha} + \frac{\cos\alpha(\cos\alpha-\mu)}{2} + \sin^2\alpha$$
$$= \frac{1}{2}\left(1-\frac{\mu}{\cos\alpha}\right). \tag{3.3}$$

If $f(z) \in \mathcal{S}_m^l(\alpha_1,\alpha,\mu)$, then we deduce from Theorem 1 and (3.3) that

$$\Re\frac{z(H_m^l(\alpha_1)f(z))'}{H_m^l(\alpha_1)f(z)} > \frac{1}{2}\left(1-\frac{\mu}{\cos\alpha}\right) \quad (z \in \mathbb{U})$$

and the order $\frac{1}{2}(1-\frac{\mu}{\cos\alpha})$ in (3.1) is sharp for the function $f_0(z)$ defined by (2.11). □

Theorem 3 *Let $f(z) \in \mathcal{S}_m^l(\alpha_1,\alpha,\mu)$ and $\frac{1}{2}(1-\frac{\mu}{\cos\alpha}) \le \beta < 1$. Then $H_m^l(\alpha_1)f(z)$ is α-spirallike of order β in $|z| < \rho$, where*

$$\rho = \rho(\beta,\alpha,\mu) = \left(\tan\left(\frac{\pi}{4}\sqrt{\frac{2\cos\alpha(1-\beta)}{\cos\alpha+\mu}}\right)\right)^2. \tag{3.4}$$

The result is sharp.

Proof From (3.4) and (2.2) we have

$$0 < \rho \le 1\left(\frac{1}{2}\left(1-\frac{\mu}{\cos\alpha}\right) \le \beta < 1\right)$$

and

$$h(-\rho) = 1 + \frac{2}{\pi^2}\left(1+\frac{\mu}{\cos\alpha}\right)\left(\log\frac{1+i\sqrt{\rho}}{1-i\sqrt{\rho}}\right)^2$$
$$= 1 - \frac{8}{\pi^2}\left(1+\frac{\mu}{\cos\alpha}\right)(\arctan\sqrt{\rho})^2$$
$$= \beta.$$

Hence

$$\inf_{|z|<\rho}\Re h(z) = h(-\rho) = \beta. \tag{3.5}$$

Let $f(z) \in \mathcal{S}_m^l(\alpha_1,\alpha,\mu)$. Then it follows from Theorem 1 and (3.5) that

$$\Re\left\{e^{i\alpha}\frac{z(H_m^l(\alpha_1)f(z))'}{H_m^l(\alpha_1)f(z)}\right\} > \beta\cos\alpha \quad \bigl(|z| < \rho\bigr),$$

that is, $H_m^l(\alpha_1)f(z)$ is α-spirallike of order β in $|z| < \rho$. Also, the result is sharp for the function $f_0(z)$ defined by (2.11). □

Setting $\beta = \frac{1}{2}(1-\frac{\mu}{\cos\alpha})$, Theorem 3 reduces to the following.

Corollary 3 *Let $f(z) \in \mathcal{S}_m^l(\alpha_1,\alpha,\mu)$. Then $H_m^l(\alpha_1)f(z)$ is α-spirallike of order $\frac{1}{2}(1-\frac{\mu}{\cos\alpha})$ in $\mathbb{U}$. The result is sharp.*

For $\beta \leq 1$, a function $f(z) \in A$ is said to be prestarlike of order β in $\mathbb{U}$ if

$$\begin{cases} \frac{z}{(1-z)^{2(1-\beta)}} * f(z) \in \mathcal{S}^*(\beta), & \beta < 1, \\ \Re\frac{f(z)}{z} > \frac{1}{2}, & \beta = 1, \end{cases} \tag{3.6}$$

(see [20]). We denote this class by $\mathcal{R}(\beta)$ ($\beta \leq 1$). The following lemma is due to Ruscheweyh [20, p.54].

Lemma 1 *Let $\beta \leq 1$, $f(z) \in \mathcal{R}(\beta)$ and $g(z) \in \mathcal{S}^*(\beta)$. Then, for any analytic function $F(z)$ in $\mathbb{U}$,*

$$\frac{f * (Fg)}{f * g}(\mathbb{U}) \subset \overline{\mathrm{co}}\big(F(\mathbb{U})\big),$$

where $\overline{\mathrm{co}}(F(\mathbb{U}))$ denotes the convex hull of $F(\mathbb{U})$.

Applying the lemma, we derive Theorems 4 and 5 below.

Theorem 4 *Let*

$$\alpha_1 > 0 \quad \textit{and} \quad \alpha_1' \geq \max\left\{\alpha_1, 1 + \frac{\mu}{\cos\alpha}\right\}. \tag{3.7}$$

Then

$$\mathcal{S}_m^l\big(\alpha_1', \alpha, \mu\big) \subset \mathcal{S}_m^l(\alpha_1, \alpha, \mu). \tag{3.8}$$

Proof Define

$$\phi(z) = z + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n}{(\alpha_1')_n} z^{n+1} \quad (z \in \mathbb{U})$$

for α_1 and α_1' satisfying (3.7). Then $\phi(z) \in \mathcal{A}$ and

$$\frac{z}{(1-z)^{\alpha_1'}} * \phi(z) = \frac{z}{(1-z)^{\alpha_1}} \quad (z \in \mathbb{U}). \tag{3.9}$$

In view of $\alpha_1' \geq \alpha_1 > 0$, it follows from (3.9) that

$$\frac{z}{(1-z)^{\alpha_1'}} * \phi(z) \in \mathcal{S}^*\left(1 - \frac{\alpha_1}{2}\right) \subset \mathcal{S}^*\left(1 - \frac{\alpha_1'}{2}\right),$$

which implies that

$$\phi(z) \in \mathcal{R}\left(1 - \frac{\alpha_1'}{2}\right). \tag{3.10}$$

Also, for $f(z) \in \mathcal{A}$, (3.9) leads to

$$\begin{cases} H_m^l(\alpha_1) f(z) = \phi(z) * H_m^l(\alpha_1') f(z), \\ z(H_m^l(\alpha_1) f(z))' = \phi(z) * (z(H_m^l(\alpha_1') f(z))'). \end{cases} \tag{3.11}$$

Let $f(z) \in \mathcal{S}_m^l(\alpha_1', \alpha, \mu)$. Then, by Theorems 1 and 2, we have

$$\begin{cases} F(z) = \frac{z(H_m^l(\alpha_1')f(z))'}{H_m^l(\alpha_1')f(z)} \prec e^{-i\alpha}(h(z)\cos\alpha + i\sin\alpha), \\ H_m^l(\alpha_1')f(z) \in \mathcal{S}^*(\frac{1}{2}(1-\frac{\mu}{\cos\alpha})) \subset \mathcal{S}^*(1-\frac{\alpha_1'}{2}) \end{cases} \tag{3.12}$$

for $h(z)$ given by (2.2) and $\alpha_1' \geq 1 + \frac{\mu}{\cos\alpha}$. Since the function $e^{-i\alpha}(h(z)\cos\alpha + i\sin\alpha)$ is convex univalent in $\mathbb{U}$, from (3.10), (3.11), (3.12) and the lemma, we deduce that

$$\begin{aligned} \frac{z(H_m^l(\alpha_1)f(z))'}{H_m^l(\alpha_1)f(z)} &= \frac{\phi(z) * (z(H_m^l(\alpha_1')f(z))')}{\phi(z) * H_m^l(\alpha_1')f(z)} \\ &= \frac{\phi(z) * (F(z)H_m^l(\alpha_1')f(z)}{\phi(z) * H_m^l(\alpha_1')f(z)} \\ &\prec e^{-i\alpha}\bigl(h(z)\cos\alpha + i\sin\alpha\bigr). \end{aligned}$$

Therefore, by Theorem 1, $f(z) \in \mathcal{S}_m^l(\alpha_1, \alpha, \mu)$ and (3.8) is proved. □

Theorem 5 *Let $f(z) \in \mathcal{S}_m^l(\alpha_1, \alpha, \mu)$ and $g(z) \in \mathcal{R}(\frac{1}{2}(1-\frac{\mu}{\cos\alpha}))$. Then*

$$(f * g)(z) \in \mathcal{S}_m^l(\alpha_1, \alpha, \mu). \tag{3.13}$$

Proof Let $f(z) \in \mathcal{S}_m^l(\alpha_1, \alpha, \mu)$. According to Theorems 1 and 2, we have

$$F(z) = \frac{z(H_m^l(\alpha_1)f(z))'}{H_m^l(\alpha_1)f(z)} \prec e^{-i\alpha}\bigl(h(z)\cos\alpha + i\sin\alpha\bigr)$$

and

$$H_m^l(\alpha_1)f(z) \in \mathcal{S}^*\left(\frac{1}{2}\left(1-\frac{\mu}{\cos\alpha}\right)\right). \tag{3.14}$$

If we put $\phi(z) = (f * g)(z)$, then

$$\begin{aligned} \frac{z(H_m^l(\alpha_1)\phi(z))'}{H_m^l(\alpha_1)\phi(z)} &= \frac{g(z) * (z(H_m^l(\alpha_1)f(z))')}{g(z) * H_m^l(\alpha_1)f(z)} \\ &= \frac{g(z) * (F(z)H_m^l(\alpha_1)f(z))}{g(z) * H_m^l(\alpha_1)f(z)} \quad (z \in \mathbb{U}) \end{aligned} \tag{3.15}$$

for $g(z) \in \mathcal{R}(\frac{1}{2}(1-\frac{\mu}{\cos\alpha}))$.

In view of (3.14) and (3.15), an application of the lemma leads to

$$\frac{z(H_m^l(\alpha_1)\phi(z))'}{H_m^l(\alpha_1)\phi(z)} \prec e^{-i\alpha}\bigl(h(z)\cos\alpha + i\sin\alpha\bigr).$$

Consequently, by applying Theorem 1, $\phi(z) \in \mathcal{S}_m^l(\alpha_1, \alpha, \mu)$ and the proof of (3.13) is completed. □

Note that $\mathcal{R}(\frac{1}{2}) = \mathcal{S}^*(\frac{1}{2})$. Since $\mathcal{R}(\beta_1) \subset \mathcal{R}(\beta_2)$ for $\beta_1 \le \beta_2 \le 1$ (see [15, p.49], we have

$$\mathcal{K} = \mathcal{R}(0) \subset \mathcal{R}\left(\frac{1}{2}\left(1 - \frac{\mu}{\cos\alpha}\right)\right) \quad (-\cos\alpha < \mu \le \cos\alpha).$$

Thus Theorem 5 yields the following.

Corollary 4

(i) *If* $f(z) \in \mathcal{S}_m^l(\alpha_1, \alpha, 0)$ *and* $g(z) \in \mathcal{S}^*(\frac{1}{2})$, *then*

$$(f * g)(z) \in \mathcal{S}_m^l(\alpha_1, \alpha, 0).$$

(ii) *If* $f(z) \in \mathcal{S}_m^l(\alpha_1, \alpha, \mu)$ *with* $-\cos\alpha < \mu \le \cos\alpha$ *and* $g(z) \in \mathcal{K}$, *then*

$$(f * g)(z) \in \mathcal{S}_m^l(\alpha_1, \alpha, \mu).$$

Theorem 6 *The function* $f(z) \in \mathcal{A}$ *defined by*

$$H_m^l(\alpha_1)f(z) = \frac{z}{(1 - bz)^{2\cos\alpha e^{-i\alpha}}} \quad (z \in \mathbb{U}) \tag{3.16}$$

belongs to the class $\mathcal{S}_m^l(\alpha_1, \alpha, \mu)$, *where*

$$\alpha_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\} \quad (j = 1, 2, \ldots, l),$$

b is complex and

$$|b| \le \begin{cases} \frac{\cos\alpha + \mu}{3\cos\alpha - \mu} & (-\cos\alpha < \mu < \frac{\cos\alpha}{3}), \\ \sqrt{\frac{\mu}{\cos\alpha + \mu}} & (\mu \ge \frac{\cos\alpha}{3}). \end{cases} \tag{3.17}$$

The result is sharp, that is, $|b|$ *cannot be increased.*

Proof For $f(z) \in \mathcal{A}$ defined by (3.16) and

$$\alpha_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\} \quad (j = 1, 2, \ldots, l),$$

we easily have

$$e^{i\alpha} \frac{z(H_m^l(\alpha_1)f(z))'}{H_m^l(\alpha_1)f(z)} = \frac{1 + bz}{1 - bz} \cos\alpha + i\sin\alpha \quad (z \in \mathbb{U}). \tag{3.18}$$

Hence, by Theorem 1, $f(z) \in \mathcal{S}_m^l(\alpha_1, \alpha, \mu)$ if and only if

$$\frac{1 + bz}{1 - bz} \prec h(z), \tag{3.19}$$

where $h(z)$ is given by (2.2). Clearly, (3.19) is equivalent to

$$\left\{ w : \left| w - \frac{1 + |b|^2}{1 - |b|^2} \right| < \frac{2|b|}{1 - |b|^2} \right\} \subset h(\mathbb{U}) \tag{3.20}$$

for $0 < |b| < 1$. Let

$$\delta = \min\left\{\left|w - \frac{1+|b|^2}{1-|b|^2}\right| : w \in \partial h(\mathbb{U})\right\}, \tag{3.21}$$

where $\partial h(\mathbb{U})$ is given by (3.2). Then we have

$$\begin{cases} \delta = \min\{\sqrt{g(u)} : u \ge \frac{1}{2}(1 - \frac{\mu}{\cos\alpha})\}, \\ g(u) = (u - \frac{1+|b|^2}{1-|b|^2})^2 + 2(1 + \frac{\mu}{\cos\alpha})(u - \frac{\cos\alpha-\mu}{2\cos\alpha}) \quad (u \ge \frac{\cos\alpha-\mu}{2\cos\alpha}). \end{cases} \tag{3.22}$$

Note that

$$\frac{1}{2}\left(1 - \frac{\mu}{\cos\alpha}\right) < \frac{1+|b|^2}{1-|b|^2}, \qquad g'(u) = 2\left(u - \left(\frac{2|b|^2}{1-|b|^2} - \frac{\mu}{\cos\alpha}\right)\right). \tag{3.23}$$

(i) If

$$-\cos\alpha < \mu < \frac{\cos\alpha}{3} \quad \text{and} \quad |b| = \frac{\cos\alpha + \mu}{3\cos\alpha - \mu}, \tag{3.24}$$

then

$$\frac{1-|b|}{1+|b|} = \frac{1}{2}\left(1 - \frac{\mu}{\cos\alpha}\right), \qquad |b|^2 = \left(\frac{\cos\alpha + \mu}{3\cos\alpha - \mu}\right)^2 < \frac{\cos\alpha + \mu}{5\cos\alpha + \mu},$$

and so

$$\frac{2|b|^2}{1-|b|^2} - \frac{\mu}{\cos\alpha} < \frac{1}{2}\left(1 - \frac{\mu}{\cos\alpha}\right). \tag{3.25}$$

From (3.22), (3.23) and (3.25), we have $g'(u) > 0 (u \ge \frac{1}{2}(1 - \frac{\mu}{\cos\alpha}))$, and hence

$$\delta = \sqrt{g\left(\frac{1}{2}\left(1 - \frac{\mu}{\cos\alpha}\right)\right)} = \frac{1+|b|^2}{1-|b|^2} - \frac{1}{2}\left(1 - \frac{\mu}{\cos\alpha}\right) = \frac{2|b|}{1-|b|^2}. \tag{3.26}$$

(ii) If

$$-\cos\alpha < \mu < \frac{\cos\alpha}{3} \quad \text{and} \quad \frac{\cos\alpha + \mu}{3\cos\alpha - \mu} < |b| < \sqrt{\frac{\cos\alpha + \mu}{5\cos\alpha + \mu}}, \tag{3.27}$$

then

$$\frac{1-|b|}{1+|b|} < \frac{1}{2}\left(1 - \frac{\mu}{\cos\alpha}\right) \quad \text{and} \quad g'(u) > 0\left(u \ge \frac{1}{2}\left(1 - \frac{\mu}{\cos\alpha}\right)\right).$$

Hence

$$\delta = \sqrt{g\left(\frac{1}{2}\left(1 - \frac{\mu}{\cos\alpha}\right)\right)} < \frac{2|b|}{1-|b|^2}. \tag{3.28}$$

(iii) If

$$\mu \geq \frac{\cos\alpha}{3} \quad \text{and} \quad |b| = \sqrt{\frac{\mu}{\cos\alpha + \mu}}, \tag{3.29}$$

then

$$|b|^2 = \frac{\mu}{\cos\alpha + \mu} \geq \frac{\cos\alpha + \mu}{5\cos\alpha + \mu},$$

and so

$$\frac{2|b|^2}{1-|b|^2} - \frac{\mu}{\cos\alpha} \geq \frac{1}{2}\left(1 - \frac{\mu}{\cos\alpha}\right).$$

Thus $g(u)$ attains its minimum value at

$$u_0 = \frac{2|b|^2}{1-|b|^2} - \frac{\mu}{\cos\alpha}$$

and

$$\delta = \sqrt{g(u_0)} = 2|b|\sqrt{\frac{\cos\alpha + \mu}{\cos\alpha(1-|b|^2)}} = \frac{2|b|}{1-|b|^2}. \tag{3.30}$$

(iv) If

$$\mu \geq \frac{\cos\alpha}{3} \quad \text{and} \quad \sqrt{\frac{\mu}{\cos\alpha + \mu}} < |b| < 1, \tag{3.31}$$

then from (iii) we easily have

$$\delta = \sqrt{g(u_0)} < \frac{2|b|}{1-|b|^2}. \tag{3.32}$$

Now, by virtue of (3.19), (3.20), (3.21), and (i)-(iv), we have proved the theorem. □

Theorem 7 *Let*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_m^l(\alpha_1, \alpha, \mu),$$

where

$$\alpha_j \in \mathbb{C} \setminus \{0, -1, -2, -3, \ldots\} \quad (j = 1, 2, \ldots, l).$$

Then

$$|a_2| \leq \frac{8(\cos\alpha + \mu)}{\pi^2}\left|\frac{\beta_1 \cdots \beta_m}{\alpha_1 \cdots \alpha_l}\right|. \tag{3.33}$$

The result is sharp.

Proof It can be easily verified that, for $z \in \mathbb{U}$,

$$\frac{z(H_m^l(\alpha_1)f(z))'}{H_m^l(\alpha_1)f(z)} = 1 + \frac{\alpha_1 \cdots \alpha_l}{\beta_1 \cdots \beta_m} a_2 z + \cdots \tag{3.34}$$

and

$$\begin{aligned} h(z) &= 1 + \frac{8z}{\pi^2}\left(1 + \frac{\mu}{\cos\alpha}\right)\left(\sum_{n=1}^{\infty} \frac{z^{n-1}}{2n-1}\right)^2 \\ &= 1 + \frac{8}{\pi^2}\left(1 + \frac{\mu}{\cos\alpha}\right)\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{\nu=0}^{n-1} \frac{1}{2\nu+1}\right) z^n \\ &= 1 + \frac{8}{\pi^2}\left(1 + \frac{\mu}{\cos\alpha}\right) z + \cdots, \end{aligned} \tag{3.35}$$

where

$$f(z) = z + a_2 z^2 + \cdots \in \mathcal{S}_m^l(\alpha_1, \alpha, \mu)$$

and $h(z)$ is given by (2.2). From (3.34), (3.35) and Theorem 1, we obtain

$$\begin{aligned} \frac{\pi^2 e^{i\alpha}}{8(\cos\alpha + \mu)}\left(\frac{z(H_m^l(\alpha_1)f(z))'}{H_m^l(\alpha_1)f(z)} - 1\right) &= \frac{\pi^2 e^{i\alpha}\alpha_1 \cdots \alpha_l}{8(\cos\alpha + \mu)\beta_1 \cdots \beta_m} a_2 z + \cdots \\ &\prec \frac{\pi^2 \cos\alpha}{8(\cos\alpha + \mu)}\big(h(z) - 1\big) \in \mathcal{K}. \end{aligned} \tag{3.36}$$

It is the well-known Rogosinski result (*cf.* [22, p.195]) that if

$$g(z) = \sum_{n=1}^{\infty} b_n z^n$$

is analytic in $\mathbb{U}$, $g(z) \prec \phi(z)$ and $\phi(z) \in \mathcal{K}$, then $|b_n| \leq 1$ ($n \in \mathbb{N}$). Hence (3.33) follows from (3.36) at once. □

The estimate (3.33) is sharp since equality is attained for the function $f_0(z)$ defined by (2.11).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors did not provide this information

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