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Convergence theorems for total asymptotically nonexpansive non-self mappings in CAT(0) spaces

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Abstract
In this paper, we introduce the concept of total asymptotically nonexpansive nonself mappings and prove the demiclosed principle for this kind of mappings in CAT(0) spaces. As a consequence, we obtain a Δ-convergence theorem of total asymptotically nonexpansive nonself mappings in CAT(0) spaces. Our results extend and improve the corresponding recent results announced by many authors.
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1 Introduction

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. That is to say, let (X,d) be a metric space, and let $x,y \in X$ with $d(x,y) = l$. A *geodesic path* from x to y is an isometry $c : [0,l] \to X$ such that $c(0) = x$ and $c(l) = y$. The image of a geodesic path is called a *geodesic segment*. A metric space X is a (uniquely) *geodesic space* if every two points of X are joined by only one geodesic segment. A *geodesic triangle* $\Delta(x_1,x_2,x_3)$ in a geodesic space X consists of three points x_1, x_2, x_3 of X and three geodesic segments joining each pair of vertices. A *comparison triangle* of the geodesic triangle $\Delta(x_1,x_2,x_3)$ is the triangle $\bar{\Delta}(x_1,x_2,x_3) := \Delta(\bar{x_1},\bar{x_2},\bar{x_3})$ in the Euclidean space $\mathscr{R}^2$ such that

$$d(x_i,x_j) = d_{\mathscr{R}^2}(\bar{x}_i,\bar{x}_j), \quad \forall i,j = 1,2,3.$$

A geodesic space X is a CAT(0) *space* if for each geodesic triangle $\Delta(x_1,x_2,x_3)$ in X and its comparison triangle $\bar{\Delta} := \Delta(\bar{x_1},\bar{x_2},\bar{x_3})$ in $\mathscr{R}^2$, the CAT(0) *inequality*

$$d(x,y) \le d_{\mathscr{R}^2}(\bar{x},\bar{y}) \tag{1.1}$$

is satisfied for all $x,y \in \Delta$ and $\bar{x},\bar{y} \in \bar{\Delta}$.

A thorough discussion on these spaces and their important role in various branches of mathematics is given in [1–5].

In 1976, Lim [6] introduced the concept of Δ-convergence in a general metric space. Fixed point theory in a CAT(0) space was first studied by Kirk [7, 8]. He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a

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complete CAT(0) space always has a fixed point. In 2008, Kirk and Panyanak [9] specialized Lim's concept to CAT(0) spaces and proved that it is very similar to the weak convergence in the Banach space setting. So, the fixed point and Δ-convergence theorems for single-valued and multivalued mappings in CAT(0) spaces have been rapidly developed and many papers have appeared [10–25].

Let (X,d) be a metric space. Recall that a mapping $T: X \to X$ is said to be nonexpansive if

$$d(Tx,Ty) \le d(x,y), \quad \forall x,y \in X. \tag{1.2}$$

T is said to be asymptotically nonexpansive, if there is a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ such that

$$d\big(T^n x, T^n y\big) \le k_n d(x,y), \quad \forall n \ge 1, x,y \in X. \tag{1.3}$$

T is said to be $(\{\mu_n\},\{\nu_n\},\zeta)$-total asymptotically nonexpansive, if there exist nonnegative sequences $\{\mu_n\}$, $\{\nu_n\}$ with $\mu_n \to 0$, $\nu_n \to 0$ and a strictly increasing continuous function $\zeta : [0,\infty) \to [0,\infty)$ with $\zeta(0) = 0$ such that

$$d\big(T^n x, T^n y\big) \le d(x,y) + \nu_n \zeta\big(d(x,y)\big) + \mu_n, \quad \forall n \ge 1, x,y \in X. \tag{1.4}$$

Let (X,d) be a metric space, and let C be a nonempty and closed subset of X. Recall that C is said to be a *retract of* X if there exists a continuous map $P: X \to C$ such that $Px = x$, $\forall x \in C$. A map $P: X \to C$ is said to be a *retraction* if $P^2 = P$. If P is a retraction, then $Py = y$ for all y in the range of P.

Definition 1.1 Let X and C be the same as above. A mapping $T: C \to X$ is said to be $(\{\mu_n\},\{\nu_n\},\zeta)$-total asymptotically nonexpansive nonself mapping if there exist nonnegative sequences $\{\mu_n\}$, $\{\nu_n\}$ with $\mu_n \to 0$, $\nu_n \to 0$ and a strictly increasing continuous function $\zeta : [0,\infty) \to [0,\infty)$ with $\zeta(0) = 0$ such that

$$d\big(T(PT)^{n-1}x, T(PT)^{n-1}y\big) \le d(x,y) + \nu_n \zeta\big(d(x,y)\big) + \mu_n, \quad \forall n \ge 1, x,y \in C, \tag{1.5}$$

where P is a nonexpansive retraction of X onto C.

Remark 1.2 From the definitions, it is to know that each nonexpansive nonself mapping is an asymptotically nonexpansive nonself mapping with a sequence $\{k_n = 1\}$, and each asymptotically nonexpansive mapping is a $(\{\mu_n\},\{\nu_n\},\zeta)$-total asymptotically nonexpansive mapping with $\mu_n = 0$, $\nu_n = k_n - 1$, $\forall n \ge 1$ and $\zeta(t) = t$, $t \ge 0$.

Definition 1.3 A nonself mapping $T: C \to X$ is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$d\big(T(PT)^{n-1}x, T(PT)^{n-1}y\big) \le L d(x,y), \quad \forall n \ge 1, x,y \in C. \tag{1.6}$$

Recently, Chang *et al.* [26] introduced the following Krasnoselskii-Mann type iteration for finding a fixed point of a total asymptotically nonexpansive mappings in CAT(0) spaces.

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T^n x_n, \quad n \ge 1. \tag{1.7}$$

Under some limit conditions, they proved that the sequence $\{x_n\}$ Δ-converges to a fixed point of T.

Inspired and motivated by the recent work of Chang *et al.* [26], Tang *et al.* [27], Laowang *et al.* [19] and so on, the purpose of this paper is to introduce the concept of total asymptotically nonexpansive nonself mappings and prove the demiclosed principle for this kind of mappings in CAT(0) spaces. As a consequence, we obtain a Δ-convergence theorem of total asymptotically nonexpansive nonself mappings in CAT(0) spaces. The results presented in this paper improve and extend the corresponding recent results in [19, 26, 27].

2 Preliminaries

The following lemma plays an important role in our paper.

In this paper, we write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that

$$d(z,x) = td(x,y), \qquad d(z,y) = (1-t)d(x,y). \tag{2.1}$$

We also denote by $[x,y]$ the geodesic segment joining from x to y, that is, $[x,y] = \{(1-t)x \oplus ty : t \in [0,1]\}$.

A subset C of a CAT(0) space is convex if $[x,y] \subset C$ for all $x, y \in C$.

Lemma 2.1 [18] *A geodesic space X is a* CAT(0) *space if and only if the following inequality holds*:

$$d^2\bigl((1-t)x \oplus ty, z\bigr) \le (1-t)d^2(x,z) + td^2(y,z) - t(1-t)d^2(x,y) \tag{2.2}$$

for all $x, y, z \in X$ and all $t \in [0,1]$. In particular, if x, y, z are points in a CAT(0) *space and $t \in [0,1]$, then*

$$d\bigl((1-t)x \oplus ty, z\bigr) \le (1-t)d(x,z) + td(y,z). \tag{2.3}$$

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r\bigl(x, \{x_n\}\bigr) = \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r\bigl(\{x_n\}\bigr) = \inf\bigl\{r\bigl(x, \{x_n\}\bigr) : x \in X\bigr\}. \tag{2.4}$$

The *asymptotic radius* $r_C(\{x_n\})$ *of* $\{x_n\}$ *with respect to* $C \subset X$ is given by

$$r_C\bigl(\{x_n\}\bigr) = \inf\bigl\{r\bigl(x, \{x_n\}\bigr) : x \in C\bigr\}. \tag{2.5}$$

The *asymptotic center* $A(\{x_n\})$ *of* $\{x_n\}$ is the set

$$A\bigl(\{x_n\}\bigr) = \bigl\{x \in X : r\bigl(x, \{x_n\}\bigr) = r\bigl(\{x_n\}\bigr)\bigr\}. \tag{2.6}$$

And the *asymptotic center* $A_C(\{x_n\})$ *of* $\{x_n\}$ *with respect to* $C \subset X$ is the set

$$A_C(\{x_n\}) = \{x \in C : r(x, \{x_n\}) = r_C(\{x_n\})\}. \tag{2.7}$$

Recall that a bounded sequence $\{x_n\}$ in X is said to be regular if $r(\{x_n\}) = r(\{u_n\})$ for every subsequence $\{u_n\}$ of $\{x_n\}$.

Proposition 2.2 [14] *Let* X *be a complete* CAT(0) *space, let* $\{x_n\}$ *be a bounded sequence in* X, *and let* C *be a closed convex subset of* X. *Then*

(1) *there exists a unique point* $u \in C$ *such that*

$$r(u, \{x_n\}) = \inf_{x \in C} r(x, \{x_n\});$$

(2) $A(\{x_n\})$ *and* $A_C(\{x_n\})$ *both are singleton.*

Definition 2.3 [6, 9] Let X be a CAT(0) space. A sequence $\{x_n\}$ in X is said to Δ*-converge to* $p \in X$ if p is the unique asymptotic center of $\{u_n\}$ for each subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta\text{-}\lim_{n\to\infty} x_n = p$ and call p *the* Δ*-limit of* $\{x_n\}$.

Lemma 2.4 [9] *Every bounded sequence in a complete* CAT(0) *space always has a* Δ*-convergent subsequence.*

Lemma 2.5 [11] *Let* X *be a complete* CAT(0) *space, and let* C *be a closed convex subset of* X. *If* $\{x_n\}$ *is a bounded sequence in* C, *then the asymptotic center of* $\{x_n\}$ *is in* C.

Remark 2.6 Let X be a CAT(0) space, and let C be a closed convex subset of X. Let $\{x_n\}$ be a bounded sequence in C. In what follows, we denote

$$\{x_n\} \rightharpoonup w \quad \Leftrightarrow \quad \Phi(w) = \inf_{x \in C} \Phi(x),$$

where $\Phi(x) := \limsup_{n\to\infty} d(x_n, x)$.

Now we give a connection between the '$\rightharpoonup$' convergence and Δ-convergence.

Proposition 2.7 [26] *Let* X *be a* CAT(0) *space, let* C *be a closed convex subset of* X, *and let* $\{x_n\}$ *be a bounded sequence in* C. *Then* $\Delta\text{-}\lim_{n\to\infty} x_n = p$ *implies that* $\{x_n\} \rightharpoonup p$.

Lemma 2.8 *Let* C *be a closed and convex subset of a complete* CAT(0) *space* X, *and let* $T : C \to X$ *be a uniformly* L*-Lipschitzian and* $(\{\mu_n\}, \{\nu_n\}, \zeta)$*-total asymptotically nonexpansive nonself mapping. Let* $\{x_n\}$ *be a bounded sequence in* C *such that* $\{x_n\} \rightharpoonup p$ *and* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. *Then* $Tp = p$.

Proof By the definition, $\{x_n\} \rightharpoonup p$ if and only if $A_C(\{x_n\}) = \{p\}$. By Lemma 2.5, we have $A(\{x_n\}) = \{p\}$.

Since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, by induction we can prove that

$$\lim_{n\to\infty} d(x_n, T(PT)^m x_n) = 0 \quad \text{for each } m \geq 0. \tag{2.8}$$

In fact, it is obvious that the conclusion is true for $m = 0$. Suppose that the conclusion holds for $m \geq 1$, now we prove that the conclusion is also true for $m + 1$.

Indeed, since T is uniformly L-Lipschitzian, we have

$$\begin{aligned} d\bigl(x_n, T(PT)^m x_n\bigr) &\leq d\bigl(x_n, T(PT)^{m-1} x_n\bigr) + d\bigl(T(PT)^{m-1} x_n, T(PT)^m x_n\bigr) \\ &\leq d\bigl(x_n, T(PT)^{m-1} x_n\bigr) + Ld(x_n, PTx_n) \\ &= d\bigl(x_n, T(PT)^{m-1} x_n\bigr) + Ld(Px_n, PTx_n) \\ &\leq d\bigl(x_n, T(PT)^{m-1} x_n\bigr) + Ld(x_n, Tx_n) \to 0 \quad (\text{as } n \to \infty). \end{aligned}$$

(2.8) is proved. Hence for each $x \in C$ and $m \geq 1$, from (2.8), we have

$$\Phi(x) := \limsup_{n\to\infty} d(x_n, x) = \limsup_{n\to\infty} d\bigl(T(PT)^{m-1}(x_n), x\bigr). \tag{2.9}$$

In (2.9) taking $x = T(PT)^{m-1}p$, $m \geq 1$, we have

$$\begin{aligned} \Phi\bigl(T(PT)^{m-1}p\bigr) &= \limsup_{n\to\infty} d\bigl(T(PT)^{m-1}(x_n), T(PT)^{m-1}p\bigr) \\ &\leq \limsup_{n\to\infty} \bigl\{d(x_n, p) + \nu_m \zeta\bigl(d(x_n, p)\bigr) + \mu_m\bigr\}. \end{aligned}$$

Letting $m \to \infty$ and taking the superior limit on the both sides, we get that

$$\limsup_{m\to\infty} \Phi\bigl(T(PT)^{m-1}p\bigr) \leq \Phi(p). \tag{2.10}$$

Furthermore, for any $n, m \geq 1$, it follows from inequality (2.2) with $t = \frac{1}{2}$ that

$$\begin{aligned} &d^2\left(x_n, \frac{p \oplus T(PT)^{m-1}(p)}{2}\right) \\ &\quad\leq \frac{1}{2} d^2(x_n, p) + \frac{1}{2} d^2\bigl(x_n, T(PT)^{m-1}(p)\bigr) - \frac{1}{4} d^2\bigl(p, T(PT)^{m-1}(p)\bigr). \end{aligned} \tag{2.11}$$

Letting $n \to \infty$ and taking the superior limit on the both sides of the above inequality, for any $m \geq 1$, we get

$$\begin{aligned} &\Phi\left(\frac{p \oplus T(PT)^{m-1}(p)}{2}\right)^2 \\ &\quad\leq \frac{1}{2}\Phi(p)^2 + \frac{1}{2}\Phi\bigl(T(PT)^{m-1}(p)\bigr)^2 - \frac{1}{4} d^2\bigl(p, T(PT)^{m-1}(p)\bigr). \end{aligned} \tag{2.12}$$

Since $A(\{x_n\}) = \{p\}$, for any $m \geq 1$, we have

$$\begin{aligned} \Phi(p)^2 &\leq \Phi\left(\frac{p \oplus T(PT)^{m-1}(p)}{2}\right)^2 \\ &\leq \frac{1}{2}\Phi(p)^2 + \frac{1}{2}\Phi\bigl(T(PT)^{m-1}(p)\bigr)^2 - \frac{1}{4} d^2\bigl(p, T(PT)^{m-1}(p)\bigr), \end{aligned} \tag{2.13}$$

which implies that

$$d^2\bigl(p, T(PT)^{m-1}(p)\bigr) \leq 2\Phi\bigl(T(PT)^{m-1}(p)\bigr)^2 - 2\Phi(p)^2. \tag{2.14}$$

By (2.10) and (2.14), we have $\lim_{m\to\infty} d(p, T(PT)^{m-1}p) = 0$. Hence we have

$$\begin{aligned} d(Tp,p) &\le d\bigl(Tp, T(PT)^m p\bigr) + d\bigl(T(PT)^m p, p\bigr) \\ &\le Ld\bigl(p, T(PT)^{m-1}p\bigr) + d\bigl(T(PT)^m p, p\bigr) \to 0 \quad (\text{as } m \to \infty), \end{aligned}$$

i.e., $p = Tp$, as desired. □

The following result can be obtained from Lemma 2.8 immediately.

Lemma 2.9 *Let C be a closed and convex subset of a complete* CAT(0) *space X, and let $T : C \to X$ be an asymptotically nonexpansive nonself mapping with a sequence $\{k_n\} \subset [1,\infty)$, $k_n \to 1$. Let $\{x_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\Delta\text{-}\lim_{n\to\infty} x_n = p$. Then $Tp = p$.*

Lemma 2.10 [26] *Let X be a* CAT(0) *space, let $x \in X$ be a given point, and let $\{t_n\}$ be a sequence in $[b,c]$ with $b, c \in (0,1)$ and $0 < b(1-c) \le \frac{1}{2}$. Let $\{x_n\}$ and $\{y_n\}$ be any sequences in X such that*

$$\limsup_{n\to\infty} d(x_n,x) \le r, \qquad \limsup_{n\to\infty} d(y_n,x) \le r \quad \textit{and}$$
$$\lim_{n\to\infty} d\bigl((1-t_n)x_n \oplus t_n y_n, x\bigr) = r,$$

for some $r \ge 0$. Then

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{2.15}$$

Lemma 2.11 [26] *Let $\{a_n\}$, $\{\lambda_n\}$ and $\{c_n\}$ be the sequences of nonnegative numbers such that*

$$a_{n+1} \le (1+\lambda_n)a_n + c_n, \quad \forall n \ge 1.$$

If $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If there exists a subsequence of $\{a_n\}$ which converges to zero, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.12 [18] *Let X be a complete* CAT(0) *space, and let $\{x_n\}$ be a bounded sequence in X with $A(\{x_n\}) = \{p\}$; $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$, and the sequence $\{d(x_n, u)\}$ converges, then $p = u$.*

3 Main results

Theorem 3.1 *Let C be a nonempty, closed and convex subset of a complete* CAT(0) *space E. Let $T_i : C \to E$ be a uniformly L-Lipschitzian and total asymptotically nonexpansive nonself mapping with sequences $\{\mu_n^{(i)}\}$ and $\{\upsilon_n^{(i)}\}$ satisfying $\lim_{n\to\infty} \mu_n^{(i)} = 0$ and $\lim_{n\to\infty} \upsilon_n^{(i)} = 0$, and strictly increasing function $\xi^{(i)} : [0,\infty) \to [0,\infty)$ with $\xi^{(i)}(0) = 0$, $i = 1, 2$. For arbitrarily chosen $x_1 \in K$, $\{x_n\}$ is defined as follows*:

$$\begin{cases} y_n = P((1-\beta_n)x_n \oplus \beta_n T_2(PT_2)^{n-1}x_n), \\ x_{n+1} = P((1-\alpha_n)x_n \oplus \alpha_n T_1(PT_1)^{n-1}y_n), \end{cases} \tag{3.1}$$

where $\{\mu_n^{(1)}\}$, $\{\mu_n^{(2)}\}$, $\{\upsilon_n^{(1)}\}$, $\{\upsilon_n^{(2)}\}$, $\xi^{(1)}$, $\xi^{(2)}$, $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy the following conditions*:

(1) $\sum_{n=1}^{\infty}\mu_n^{(i)} < \infty$, $\sum_{n=1}^{\infty}\upsilon_n^{(i)} < \infty$, $i = 1,2$;

(2) *there exist constants* $a, b \in (0,1)$ *with* $0 < b(1-a) \le \frac{1}{2}$ *such that* $\{\alpha_n\} \subset [a,b]$ *and* $\{\beta_n\} \subset [a,b]$;

(3) *there exists a constant* $M^* > 0$ *such that* $\xi^{(i)}(r) \le M^* r$, $r \ge 0$, $i = 1,2$.

Then the sequence $\{x_n\}$ *defined in* (3.1) Δ*-converges to a common fixed point of* T_1 *and* T_2.

Proof We divide the proof into three steps.

Step 1. We first show that $\lim_{n\to\infty} d(x_n, q)$ exists for each $q \in F(T_1) \cap F(T_2)$.

Set $\mu_n = \max\{\mu_n^{(1)}, \mu_n^{(2)}\}$ and $\upsilon_n = \{\upsilon_n^{(1)}, \upsilon_n^{(2)}\}$, $n = 1, 2, \ldots, \infty$. Since $\sum_{n=1}^{\infty}\mu_n^{(i)} < \infty$, $\sum_{n=1}^{\infty}\upsilon_n^{(i)} < \infty$, $i = 1,2$, we know that $\sum_{n=1}^{\infty}\mu_n < \infty$ and $\sum_{n=1}^{\infty}\upsilon_n < \infty$. For any $q \in F(T_1) \cap F(T_2)$, we have

$$\begin{aligned} d(x_{n+1}, q) &= d\bigl(P\bigl((1-\alpha_n)x_n \oplus \alpha_n T_1(PT_1)^{n-1}y_n\bigr), q\bigr) \\ &\le d\bigl((1-\alpha_n)x_n \oplus \alpha_n T_1(PT_1)^{n-1}y_n, q\bigr) \\ &\le (1-\alpha_n)d(x_n,q) + \alpha_n d\bigl(T_1(PT_1)^{n-1}y_n, q\bigr) \\ &\le (1-\alpha_n)d(x_n,q) + \alpha_n\bigl[d(y_n,q) + \upsilon_n\xi^{(1)}\bigl(d(y_n,q)\bigr) + \mu_n\bigr] \\ &\le (1-\alpha_n)d(x_n,q) + \alpha_n\bigl[\bigl(1+\upsilon_n M^*\bigr)d(y_n,q) + \mu_n\bigr], \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} d(y_n, q) &= d\bigl(P\bigl((1-\beta_n)x_n \oplus \beta_n T_2(PT_2)^{n-1}x_n\bigr), q\bigr) \\ &\le d\bigl((1-\beta_n)x_n \oplus \beta_n T_2(PT_2)^{n-1}x_n, q\bigr) \\ &\le (1-\beta_n)d(x_n,q) + \beta_n d\bigl(T_2(PT_2)^{n-1}x_n, q\bigr) \\ &\le (1-\beta_n)d(x_n,q) + \beta_n\bigl[d(x_n,q) + \upsilon_n\xi^{(2)}\bigl(d(x_n,q)\bigr) + \mu_n\bigr] \\ &\le \bigl(1+\beta_n\upsilon_n M^*\bigr)d(x_n,q) + \beta_n\mu_n. \end{aligned} \tag{3.3}$$

Substituting (3.3) into (3.2), we have

$$\begin{aligned} d(x_{n+1}, q) &\le (1-\alpha_n)d(x_n,q) + \alpha_n\bigl[\bigl(1+\upsilon_n M^*\bigr)\bigl(\bigl(1+\beta_n\upsilon_n M^*\bigr)d(x_n,q) + \beta_n\mu_n\bigr) + \mu_n\bigr] \\ &\le \bigl[1 + \bigl(1+\beta_n+\beta_n\upsilon_n M^*\bigr)\alpha_n M^*\upsilon_n\bigr]d(x_n,q) + (1+\beta_n)\bigl(1+\upsilon_n M^*\bigr)\alpha_n\mu_n. \end{aligned} \tag{3.4}$$

Since $\sum_{n=1}^{\infty}\mu_n < \infty$ and $\sum_{n=1}^{\infty}\upsilon_n < \infty$, it follows from Lemma 2.11 that $\lim_{n\to\infty} d(x_n,q)$ exists for each $q \in F(T_1) \cap F(T_2)$.

Step 2. We show that $\lim_{n\to\infty} d(x_n, T_1x_n) = \lim_{n\to\infty} d(x_n, T_2x_n) = 0$.

For each $q \in F(T_1) \cap F(T_2)$, from the proof of Step 1, we know that $\lim_{n\to\infty} d(x_n,q)$ exists. We may assume that $\lim_{n\to\infty} d(x_n,q) = k$. From (3.3), we have

$$d(y_n, q) \le \bigl(1+\beta_n\upsilon_n M^*\bigr)d(x_n,q) + \beta_n\mu_n. \tag{3.5}$$

Taking lim sup on both sides in (3.5), we have

$$\limsup_{n\to\infty} d(y_n, q) \le k. \tag{3.6}$$

In addition, since

$$\begin{aligned}d\big(T_1(PT_1)^{n-1}y_n,q\big) &\le d(y_n,q)+\upsilon_n\xi^{(2)}\big(d(y_n,q)\big)+\mu_n\\ &\le \big(1+\upsilon_n M^*\big)d(y_n,q)+\mu_n,\end{aligned}$$

we have

$$\limsup_{n\to\infty} d\big(T_1(PT_1)^{(n-1)}y_n,q\big)\le k. \tag{3.7}$$

Since $\lim_{n\to\infty} d(x_{n+1},q)=k$, it is easy to prove that

$$\lim_{n\to\infty} d\big((1-\alpha_n)x_n\oplus\alpha_n T_1(PT_1)^{(n-1)}y_n,q\big)=k. \tag{3.8}$$

It follows from Lemma 2.10 that

$$\lim_{n\to\infty} d\big(x_n,T_1(PT_1)^{n-1}y_n\big)=0. \tag{3.9}$$

On the other hand, since

$$\begin{aligned}d(x_n,q) &\le d\big(x_n,T_1(PT_1)^{n-1}y_n\big)+d\big(T_1(PT_1)^{n-1}y_n,q\big)\\ &\le d\big(x_n,T_1(PT_1)^{n-1}y_n\big)+d(y_n,q)+\upsilon_n M^* d(y_n,q)+\mu_n\\ &= d\big(x_n,T_1(PT_1)^{n-1}y_n\big)+\big(1+\upsilon_n M^*\big)d(y_n,q)+\mu_n,\end{aligned}$$

we have $\liminf_{n\to\infty} d(y_n,q)\ge k$. Combined with (3.6), it yields that

$$\lim_{n\to\infty} d(y_n,q)=k. \tag{3.10}$$

This implies that

$$\lim_{n\to\infty} d\big((1-\beta_n)x_n\oplus\beta_n T_2(PT_2)^{n-1}x_n,q\big)=k. \tag{3.11}$$

It is easy to show that

$$\limsup_{n\to\infty} d\big(T_2(PT_2)^{n-1}x_n,q\big)\le k. \tag{3.12}$$

So, it follows from (3.12) and Lemma 2.10 that

$$\lim_{n\to\infty} d\big(x_n,T_2(PT_2)^{n-1}x_n\big)=0. \tag{3.13}$$

Observe that

$$\begin{aligned}d\big(x_n,T_1(PT_1)^{n-1}x_n\big) &\le d\big(x_n,T_1(PT_1)^{n-1}y_n\big)+d\big(T_1(PT_1)^{n-1}y_n,T_1(PT_1)^{n-1}x_n\big)\\ &\le d\big(x_n,T_1(PT_1)^{n-1}y_n\big)+d(x_n,y_n)+\upsilon_n\xi^{(1)}\big(d(x_n,y_n)\big)+\mu_n\\ &= d\big(x_n,T_1(PT_1)^{n-1}y_n\big)+\big(1+\upsilon_n M^*\big)d(x_n,y_n)+\mu_n,\end{aligned} \tag{3.14}$$

where

$$\begin{aligned} d(x_n, y_n) &= d\bigl(P\bigl((1-\beta_n)x_n \oplus \beta_n T_2(PT_2)^{n-1}x_n\bigr), x_n\bigr) \\ &\le \beta_n d\bigl(x_n, T_2(PT_2)^{n-1}x_n\bigr). \end{aligned} \tag{3.15}$$

It follows from (3.13) that

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{3.16}$$

Thus, from (3.9), (3.14) and (3.16), we have

$$\lim_{n\to\infty} d\bigl(x_n, T_1(PT_1)^{n-1}x_n\bigr) = 0. \tag{3.17}$$

In addition, since

$$\begin{aligned} d(x_{n+1}, x_n) &= d\bigl(P\bigl((1-\alpha_n)x_n \oplus \alpha_n T_1(PT_1)^{n-1}y_n\bigr), x_n\bigr) \\ &\le d\bigl((1-\alpha_n)x_n \oplus \alpha_n T_1(PT_1)^{n-1}y_n, x_n\bigr) \\ &\le \alpha_n d\bigl(T_1(PT_1)^{n-1}y_n, x_n\bigr), \end{aligned}$$

from (3.9), we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.18}$$

Finally, since

$$\begin{aligned} d(x_n, T_1x_n) &\le d(x_n, x_{n+1}) + d\bigl(x_{n+1}, T_1(PT_1)^n x_{n+1}\bigr) \\ &\quad + d\bigl(T_1(PT_1)^n x_{n+1}, T_1(PT_1)^n x_n\bigr) + d\bigl(T_1(PT_1)^n x_n, T_1x_n\bigr) \\ &\le (1+L)d(x_n, x_{n+1}) + d\bigl(x_{n+1}, T_1(PT_1)^n x_{n+1}\bigr) + Ld\bigl(T_1(PT_1)^{n-1}x_n, x_n\bigr), \end{aligned}$$

it follows from (3.17) and (3.18) that $\lim_{n\to\infty} d(x_n, T_1x_n) = 0$. Similarly, we also can show that $\lim_{n\to\infty} d(x_n, T_2x_n) = 0$.

Step 3. We show that $\{x_n\}$ Δ-converges to a common fixed point of T_1 and T_2.

Let $W_\omega(x_n) = \bigcup_{\{u_n\}\subset\{x_n\}} A(\{u_n\})$. Firstly, we show that $W_\omega \subset F(T_1) \cap F(T_2)$. Let $u \in W_\omega$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.4 and Lemma 2.5, there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_{i\to\infty} u_{n_i} = p \in K$. Since $\lim_{n\to\infty} d(x_n, T_1x_n) = \lim_{n\to\infty} d(x_n, T_2x_n) = 0$, it follows from Lemma 2.8 that $p \in F(T_1) \cap F(T_2)$. So, $\lim_{n\to\infty} d(x_n, p)$ exists. By Lemma 2.12, we know that $p = u \in F(T_1) \cap F(T_2)$. This implies that $W_\omega(x_n) \subset F(T_1) \cap F(T_2)$. Next, let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and $A\{x_n\} = \{v\}$. Since $u \in W_\omega(x_n) \subset F(T_1) \cap F(T_2)$, $\lim_{n\to\infty} d(x_n, u)$ exists. By Lemma 2.12, we know that $v = u$. This implies that $W_\omega(x_n)$ contains only one point. Thus, since $W_\omega(x_n) \subset F(T_1) \cap F(T_2)$, $W_\omega(x_n)$ contains only one point and $\lim_{n\to\infty} d(x_n, q)$ exists for each $q \in F(T_1) \cap F(T_2)$, we know that $\{x_n\}$ Δ-converges to a common fixed point of T_1 and T_2. The proof is completed. $\square$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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