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Volume inequalities for L_0-Minkowski combination of convex bodies

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Abstract

Recently Böröczky, Lutwak, Yang and Zhang proved the L_0-Brunn-Minkowski inequality for two origin-symmetric convex bodies in the plane. This paper extends their results to m ($m \geq 2$) origin-symmetric convex bodies in the plane. Moreover, relying on the recent results of Schuster and Weberndorfer, volume inequalities for L_0-Minkowski combination of origin-symmetric convex bodies in $\mathbb{R}^n$ and its dual form are established in this paper.

MSC: 52A20; 52A40

Keywords: L_0-Minkowski combination; L_0-Brunn-Minkowski inequality; cone-volume probability measure; normalized L_0-mixed volume; Wulff shape

1 Introduction

The setting for this article is an Euclidean space $\mathbb{R}^n$, $n \geq 2$. A convex body is a compact convex subset of $\mathbb{R}^n$ with a non-empty interior. For a compact convex set $K \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, the support function $h_K : \mathbb{R}^n \to \mathbb{R}$ is defined by $h_K(x) = \max\{x \cdot y : y \in K\}$, where $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$. The polar body of a convex body K is given by $K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}$. The Minkowski addition of two convex bodies K and L is defined as $K + L = \{x + y : x \in K, y \in L\}$, and the scalar multiplication λK of K, where $\lambda \geq 0$, is defined as $\lambda K = \{\lambda x : x \in K\}$.

In the early 1960s, Firey [1] extended the Minkowski combination of convex bodies to L_p-Minkowski combination for each $p \geq 1$. Furthermore, he established the L_p-Brunn-Minkowski inequality which states the following: If K_i ($i = 1, 2, \ldots, m$) are convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors, and $\lambda_i \in [0, 1]$ satisfying $\sum_{i=1}^m \lambda_i = 1$, then the volumes of the bodies K_i and their L_p-Minkowski combination $\lambda_1 \cdot K_1 +_p \cdots +_p \lambda_m \cdot K_m$ are related by

$$V(\lambda_1 \cdot K_1 +_p \cdots +_p \lambda_m \cdot K_m) \geq V(K_1)^{\lambda_1} \cdots V(K_m)^{\lambda_m}, \tag{1.1}$$

with equality if and only if K_i are equal.

Recently, Böröczky *et al.* [2] defined the L_0-Minkowski combination of convex bodies and proved the L_0-Brunn-Minkowski inequality, which is stronger than (1.1), for two origin-symmetric convex bodies in the plane.

Theorem 1.1 [2] *If K and L are origin-symmetric convex bodies in the plane, then for all real $\lambda \in [0,1]$,*

$$V\big((1-\lambda)\cdot K +_0 \lambda \cdot L\big) \geq V(K)^{1-\lambda} V(L)^{\lambda}.$$

When $\lambda \in (0,1)$, equality in the inequality holds if and only if K and L are dilates or K and L are parallelograms with parallel sides.

Our first main result of this paper is to extend Theorem 1.1 to m ($m \geq 2$) origin-symmetric convex bodies in the plane.

Definition If K_i ($i = 1,2,\dots,m$) are convex bodies that contain the origin in their interiors, then for real $\lambda_i \geq 0$ (not all zero), the L_0-Minkowski combination $\lambda_1 \cdot K_1 +_0 \cdots +_0 \lambda_m \cdot K_m$ of K_i is defined by

$$\lambda_1 \cdot K_1 +_0 \cdots +_0 \lambda_m \cdot K_m = \bigcap_{u \in S^{n-1}} \left\{ x \in \mathbb{R}^n : x \cdot u \leq \prod_{i=1}^{m} h_{K_i}(u)^{\lambda_i} \right\}. \tag{1.2}$$

Theorem 1.2 *If K_i ($i = 1,2,\dots,m$) are origin-symmetric convex bodies in the plane, then for all real $\lambda_i \in [0,1]$ satisfying $\sum_{i=1}^m \lambda_i = 1$, we have*

$$V(\lambda_1 \cdot K_1 +_0 \cdots +_0 \lambda_m \cdot K_m) \geq V(K_1)^{\lambda_1} \cdots V(K_m)^{\lambda_m}. \tag{1.3}$$

However, the L_0-Brunn-Minkowski inequality in $\mathbb{R}^n$ is still an open problem, even for origin-symmetric convex bodies.

Now, with our second main result we focus on the volume estimate for L_0-Minkowski combination of origin-symmetric convex bodies in $\mathbb{R}^n$.

In [3], Schuster and Weberndorfer established two powerful volume inequalities of the Wulff shape $W_{\nu,f}$ determined by an f-centered isotropic measure ν (see Section 2 for details). Using their results, we establish the following two inequalities.

Theorem 1.3 *If K_i ($i = 1,2,\dots,m$) are origin-symmetric convex bodies in $\mathbb{R}^n$, then for all real $\lambda_i \in [0,1]$ satisfying $\sum_{i=1}^m \lambda_i = 1$, we have*

$$V(\lambda_1 \cdot K_1 +_0 \cdots +_0 \lambda_m \cdot K_m) \leq \frac{n^{n/2}(n+1)^{(n+1)/2}}{n!} \prod_{i=1}^{m} \Big(\sup_{u \in S^{n-1}} h_{K_i}(u) \Big)^{n\lambda_i}.$$

Theorem 1.4 *If K_i ($i = 1,2,\dots,m$) are origin-symmetric convex bodies in $\mathbb{R}^n$, then for all real $\lambda_i \in [0,1]$ satisfying $\sum_{i=1}^m \lambda_i = 1$, we have*

$$V\big((\lambda_1 \cdot K_1 +_0 \cdots +_0 \lambda_m \cdot K_m)^*\big) \geq \frac{(n+1)^{(n+1)/2}}{n! n^{n/2}} \prod_{i=1}^{m} \inf_{u \in S^{n-1}} \big(h_{K_i}(u)^{-n\lambda_i} \big).$$

Furthermore, inequalities of mixed volume and normalized L_0-mixed volume (given in this paper) for L_0-Minkowski combination of not necessarily origin-symmetric convex bodies in $\mathbb{R}^n$ are established in the following theorems.

Theorem 1.5 *If K_i ($i = 1, 2, \ldots, m$) and L_j ($j = 2, \ldots, n$) are convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors, then for all real $\lambda_i \in [0,1]$ satisfying $\sum_{i=1}^m \lambda_i = 1$, we have*

$$V(\lambda_1 \cdot K_1 +_0 \cdots +_0 \lambda_m \cdot K_m, L_2, \ldots, L_n) \leq \prod_{i=1}^m V(K_i, L_2, \ldots, L_n)^{\lambda_i}.$$

If K and L are convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors, then for $p \neq 0$, the L_p-mixed volume $V_p(K, L)$ can be defined as

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L^p h_K^{1-p} \, dS_K = \int_{S^{n-1}} \left(\frac{h_L}{h_K} \right)^p dV_K,$$

where S_K and V_K are the surface area measure and the cone-volume measure of K, respectively (see Section 2 for the definitions).

The normalized L_p-mixed volume $\bar{V}_p(K, L)$ is defined by

$$\bar{V}_p(K, L) = \left(\frac{V_p(K, L)}{V(K)} \right)^{1/p} = \left(\int_{S^{n-1}} \left(\frac{h_L}{h_K} \right)^p d\bar{V}_K \right)^{1/p},$$

where $\bar{V}_K$ is the cone-volume probability measure of K (also see Section 2 for the definition).

Note that when p converges to zero, the normalized L_0-mixed volume $\bar{V}_0(K, L)$ can naturally be given as

$$\bar{V}_0(K, L) = \exp\left(\int_{S^{n-1}} \log \frac{h_L}{h_K} \, d\bar{V}_K \right). \tag{1.4}$$

Theorem 1.6 *Suppose that K, L and Q are convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors, then for real $\lambda \in [0,1]$, we have*

$$\bar{V}_0\big(Q, (1-\lambda) \cdot K +_0 \lambda \cdot L\big) \leq \bar{V}_0(Q, K)^{1-\lambda} \bar{V}_0(Q, L)^{\lambda}.$$

Combining the famous variant (proved in [4]) of Aleksandrov's lemma and the representation of (1.4), we obtain a limit form of $\bar{V}_0(K, L)$ in the following theorem.

Theorem 1.7 *Suppose that K and L are convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors, then we have*

$$nV(K) \log \bar{V}_0(K, L) = \lim_{\lambda \to 0} \frac{V((1-\lambda) \cdot K +_0 \lambda \cdot L) - V(K)}{\lambda}.$$

The paper is organized as follows. In Section 2 some of the basic notations and preliminaries are provided. Section 3 contains the proofs of the main theorems. Some properties of normalized L_0-mixed volume and Wulff shape are discussed in Section 4.

2 Notations and preliminaries

Good general references for the theory of convex bodies are provided by the books [5–9] and the articles [2, 4, 7, 10–25].

The group of nonsingular linear transformations is denoted by $GL(n)$; its members are, in particular, bijections of $\mathbb{R}^n$ onto itself. The group of special linear transformations of $\mathbb{R}^n$ is denoted by $SL(n)$. These are the members of $GL(n)$ whose determinant is one.

For $\phi \in GL(n)$, let ϕ^t, ϕ^{-1} and ϕ^{-t} denote the transpose, inverse and inverse of the transpose of ϕ, respectively.

For $x \in \mathbb{R}^n$, then

$$h_K(\lambda x) = \lambda h_K(x), \quad \text{for } \lambda \geq 0,$$

and

$$h_{\phi K}(x) = h_K\left(\phi^t x\right), \quad \text{for } \phi \in GL(n).$$

Recall that for a Borel set $\omega \subseteq S^{n-1}$, the surface area measure of a convex body K in $\mathbb{R}^n$ $S_K(\omega)$ is the $(n-1)$-dimensional Hausdorff measure of the set of all boundary points of K at which there exists a normal vector of K belonging to ω.

Let K be a convex body in $\mathbb{R}^n$ that contains the origin in its interior. The cone-volume measure V_K of K is a Borel measure on the unit sphere S^{n-1} defined by

$$dV_K = \frac{1}{n} h_K \, dS_K. \tag{2.1}$$

Obviously,

$$V(K) = \frac{1}{n} \int_{S^{n-1}} h_K(u) \, dS_K(u).$$

The cone-volume probability measure $\bar{V}_K$ of K is defined by

$$\bar{V}_K = \frac{1}{V(K)} V_K. \tag{2.2}$$

Let S^{n-1} and B denote the unit sphere centered at the origin and the unit ball in $\mathbb{R}^n$, respectively. The n-dimensional volume κ_n of B and the $(n-1)$-dimensional volume ω_n of S^{n-1} are

$$\kappa_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})},$$

and

$$\omega_n = n\kappa_n.$$

If K_i $(i = 1, 2, \dots, n)$ are convex bodies in $\mathbb{R}^n$, the mixed volume $V(K_1, \dots, K_n)$ is given by (see [5], [8, Theorem 5.1.6] or [26, Section 29])

$$V(K_1, \dots, K_n) = \frac{1}{n} \int_{S^{n-1}} h_{K_1}(u) \, dS(K_2, \dots, K_n, u), \tag{2.3}$$

where $S(K_2, \dots, K_n, \cdot)$ is the mixed area measure of K_i $(i = 2, \dots, n)$.

If K is a convex body in $\mathbb{R}^n$, the quermassintegrals $W_i(K)$ of K are defined for $0 \le i \le n$ by

$$W_i(K) = V(K, n-i; B, i),$$

where the notation $V(K, n-i; B, i)$ signifies that K appears $(n-i)$ times and B appears i times.

The mean width $W(K)$ of a convex body K in $\mathbb{R}^n$ is defined by

$$W(K) = \frac{2}{\omega_n} \int_{S^{n-1}} h_K(u)\, d\mathcal{H}^{n-1}(u),$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure.

It can be shown that

$$\frac{2}{\kappa_n} W_{n-1}(K) = W(K). \tag{2.4}$$

Throughout, all Borel measures are understood to be non-negative and finite. We write $\operatorname{supp} \nu$ for the support of a measure ν.

Suppose that ν is a Borel measure on S^{n-1} and f is a positive continuous function on S^{n-1}. The Wulff shape $W_{\nu,f}$ determined by ν and f is defined by

$$W_{\nu,f} = \{x \in \mathbb{R}^n : x \cdot u \le f(u) \text{ for all } u \in \operatorname{supp} \nu\}. \tag{2.5}$$

Obviously,

$$h_{W_{\nu,f}}(u) \le f(u) \quad \text{for all } u \in \operatorname{supp} \nu. \tag{2.6}$$

Let f be a positive continuous function on S^{n-1}. A Borel measure ν on S^{n-1} is called f-centered if

$$\int_{S^{n-1}} f(u) u\, d\nu(u) = o.$$

The measure ν is called isotropic if

$$\int_{S^{n-1}} u \otimes u\, d\nu(u) = I_n,$$

where $u \otimes u$ is the orthogonal projection onto the line spanned by u and I_n denotes the identity map on $\mathbb{R}^n$. Thus, ν is isotropic if

$$\int_{S^{n-1}} |v \cdot u|^2\, d\nu(u) = 1 \quad \text{for all } v \in S^{n-1}.$$

The displacement of $W_{\nu,f}$ is defined by

$$\operatorname{disp} W_{\nu,f} = \operatorname{cd} W_{\nu,f} \cdot \int_{S^{n-1}} \frac{u}{f(u)}\, d\nu(u),$$

where $\operatorname{cd} W_{\nu,f}$ denotes the centroid of $W_{\nu,f}$.

3 Proof of main results

The following lemma will be used in the proof of Theorem 1.2.

Lemma 3.1 *If K_i ($i = 1, 2, \dots, m$) are convex bodies that contain the origin in their interiors, then for real $\lambda_i \in (0,1)$ and $\sum_{i=1}^m \lambda_i = 1$, we have*

$$\begin{aligned}
&\lambda_1 \cdot K_1 +_0 \cdots +_0 \lambda_m \cdot K_m \\
&\quad\supseteq (1-\lambda_m) \cdot \left(\frac{\lambda_1}{1-\lambda_m} \cdot K_1 +_0 \cdots +_0 \frac{\lambda_{m-1}}{1-\lambda_m} \cdot K_{m-1}\right) +_0 \lambda_m \cdot K_m.
\end{aligned}$$

Proof Since K_i ($i = 1, 2, \dots, m$) contain the origin in their interiors, thus it is easy to see that the L_0-Minkowski combination

$$\frac{\lambda_1}{1-\lambda_m} \cdot K_1 +_0 \cdots +_0 \frac{\lambda_{m-1}}{1-\lambda_m} \cdot K_{m-1} = \bigcap_{u \in S^{n-1}} \left\{ x \in \mathbb{R}^n : x \cdot u \le \prod_{i=1}^{m-1} h_{K_i}(u)^{\frac{\lambda_i}{1-\lambda_m}} \right\}, \tag{3.1}$$

also contains the origin in its interior.

Let $x \in (1-\lambda_m) \cdot (\frac{\lambda_1}{1-\lambda_m} \cdot K_1 +_0 \cdots +_0 \frac{\lambda_{m-1}}{1-\lambda_m} \cdot K_{m-1}) +_0 \lambda_m \cdot K_m$, then combining (2.6) and (3.1) we have

$$\begin{aligned}
x \cdot u &\le h_{\frac{\lambda_1}{1-\lambda_m} \cdot K_1 +_0 \cdots +_0 \frac{\lambda_{m-1}}{1-\lambda_m} \cdot K_{m-1}}(u)^{1-\lambda_m} h_{K_m}(u)^{\lambda_m} \\
&\le \left(\prod_{i=1}^{m-1} h_{K_i}(u)^{\frac{\lambda_i}{1-\lambda_m}} \right)^{1-\lambda_m} h_{K_m}(u)^{\lambda_m} \\
&= \prod_{i=1}^{m} h_{K_i}(u)^{\lambda_i}
\end{aligned}$$

for all $u \in S^{n-1}$.

Hence, by (1.2), we have $x \in \lambda_1 \cdot K_1 +_0 \cdots +_0 \lambda_m \cdot K_m$, which yields the lemma directly. □

Proof of Theorem 1.2 We will prove Theorem 1.2 by induction on m.

Obviously, it is true by Theorem 1.1 when $m = 2$.

Suppose that the result holds on $(m-1)$. Thus, for real $\lambda_i \in (0,1)$ satisfying $\sum_{i=1}^m \lambda_i = 1$, we have

$$V\left(\frac{\lambda_1}{1-\lambda_m} \cdot K_1 +_0 \cdots +_0 \frac{\lambda_{m-1}}{1-\lambda_m} \cdot K_{m-1}\right) \ge V(K_1)^{\frac{\lambda_1}{1-\lambda_m}} \cdots V(K_{m-1})^{\frac{\lambda_{m-1}}{1-\lambda_m}}.$$

We now consider the situation on m. In fact, since K_i ($i = 1, 2, \dots, m$) are origin-symmetric convex bodies in the plane, then by (3.1) we have $\frac{\lambda_1}{1-\lambda_m} \cdot K_1 +_0 \cdots +_0 \frac{\lambda_{m-1}}{1-\lambda_m} \cdot K_{m-1}$ is also an origin-symmetric convex body in the plane.

Then by Lemma 3.1, Theorem 1.1 and the induction hypothesis, we have

$$\begin{aligned}
&V(\lambda_1 \cdot K_1 +_0 \cdots +_0 \lambda_m \cdot K_m) \\
&\quad\ge V\left((1-\lambda_m) \cdot \left(\frac{\lambda_1}{1-\lambda_m} \cdot K_1 +_0 \cdots +_0 \frac{\lambda_{m-1}}{1-\lambda_m} \cdot K_{m-1}\right) +_0 \lambda_m \cdot K_m\right)
\end{aligned}$$

$$\begin{aligned}
&\geq V\left(\frac{\lambda_1}{1-\lambda_m}\cdot K_1 +_0 \cdots +_0 \frac{\lambda_{m-1}}{1-\lambda_m}\cdot K_{m-1}\right)^{1-\lambda_m} V(K_m)^{\lambda_m} \\
&\geq \left(V(K_1)^{\frac{\lambda_1}{1-\lambda_m}} \cdots V(K_{m-1})^{\frac{\lambda_{m-1}}{1-\lambda_m}}\right)^{1-\lambda_m} V(K_m)^{\lambda_m} \\
&= V(K_1)^{\lambda_1} \cdots V(K_{m-1})^{\lambda_{m-1}} V(K_m)^{\lambda_m}.
\end{aligned}$$

Note that Theorem 1.2 also holds when $\lambda_m = 0$ and $\lambda_m = 1$, respectively.
Thus Theorem 1.2 holds for all $\lambda_i \in [0,1]$ satisfying $\sum_{i=1}^m \lambda_i = 1$. □

In [3], Schuster and Weberndorfer established a sharp bound for the volume of the Wulff shape $W_{\nu,f}$ determined by an f-centered isotropic measure ν as follows.

Lemma 3.2 [3] *Suppose that f is a positive continuous function on S^{n-1} and that ν is an isotropic f-centered measure. If* disp $W_{\nu,f} = 0$, *then*

$$V(W_{\nu,f}) \leq \frac{(n+1)^{(n+1)/2}}{n!}\|f\|^n_{L^2(\nu)},$$

with equality if and only if conv supp ν *is a regular simplex inscribed in S^{n-1} and f is constant on* supp ν.

Proof of Theorem 1.3 Let $d\nu = \frac{1}{\kappa_n}\,du$ and $f(u) = \prod_{i=1}^m h_{K_i}(u)^{\lambda_i}$ for all $u \in S^{n-1}$ in Lemma 3.2. Obviously, f is a positive continuous function on S^{n-1} and ν is isotropic. By (2.5) we have

$$W_{\nu,f} = \left\{x \in \mathbb{R}^n : x\cdot u \leq \prod_{i=1}^m h_{K_i}(u)^{\lambda_i} \text{ for all } u \in \operatorname{supp}\nu\right\}.$$

Combining it with (1.2), we get

$$W_{\nu,f} = \lambda_1\cdot K_1 +_0 \cdots +_0 \lambda_m\cdot K_m. \tag{3.2}$$

Furthermore, since K_i ($i = 1,2,\dots,m$) are origin-symmetric convex bodies, then we have

$$\frac{1}{\kappa_n}\int_{S^{n-1}} h_{K_1}(u)^{\lambda_1}\cdots h_{K_m}(u)^{\lambda_m} u\,du = o,$$

and

$$\frac{1}{\kappa_n}\int_{S^{n-1}} \frac{u}{h_{K_1}(u)^{\lambda_1}\cdots h_{K_m}(u)^{\lambda_m}}\,du = o.$$

Thus ν is f-centered, and

$$\operatorname{disp} W_{\nu,f} = \operatorname{cd} W_{\nu,f}\cdot\frac{1}{\kappa_n}\int_{S^{n-1}} \frac{u}{h_{K_1}(u)^{\lambda_1}\cdots h_{K_m}(u)^{\lambda_m}}\,du = 0.$$

Combining (3.2) and Lemma 3.2, we have

$$V(\lambda_1\cdot K_1 +_0 \cdots +_0 \lambda_m\cdot K_m) \leq \frac{(n+1)^{(n+1)/2}}{n!}\|f\|^n_{L^2(\nu)}. \tag{3.3}$$

Since $\lambda_i \in [0,1]$ satisfying $\sum_{i=1}^m \lambda_i = 1$, then from Hölder's inequality (see [27]) we have

$$\begin{aligned}\|f\|^2_{L^2(\nu)} &= \frac{1}{\kappa_n}\int_{S^{n-1}} f(u)^2\,du\\ &= \frac{1}{\kappa_n}\int_{S^{n-1}} \prod_{i=1}^m h_{K_i}(u)^{2\lambda_i}\,du\\ &\le n\prod_{i=1}^m\Bigl(\sup_{u\in S^{n-1}} h_{K_i}(u)\Bigr)^{2\lambda_i}. \end{aligned}\tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$V(\lambda_1\cdot K_1 +_0 \cdots +_0 \lambda_m\cdot K_m) \le \frac{n^{n/2}(n+1)^{(n+1)/2}}{n!}\prod_{i=1}^m\Bigl(\sup_{u\in S^{n-1}} h_{K_i}(u)\Bigr)^{n\lambda_i}. \qquad \square$$

A natural dual to Lemma 3.2 is also given in [3], which provided a sharp lower bound for the volume of the polar of the Wulff shape $W_{\nu,f}$.

Lemma 3.3 [3] *Suppose that f is a positive continuous function on S^{n-1} and that ν is an isotropic f-centered measure. Then*

$$V\bigl(W^*_{\nu,f}\bigr) \ge \frac{(n+1)^{(n+1)/2}}{n!}\|f\|^{-n}_{L^2(\nu)},$$

with equality if and only if conv supp ν *is a regular simplex inscribed in S^{n-1} and f is constant on* supp ν.

Proof of Theorem 1.4 Let $d\nu = \frac{1}{\kappa_n}\,du$ and $f(u) = \prod_{i=1}^m h_{K_i}(u)^{\lambda_i}$ for all $u \in S^{n-1}$ in Lemma 3.3. Similarly, from the proof of Theorem 1.3, we know that f is a positive continuous function on S^{n-1} and ν is an isotropic f-centered measure.

Thus, combining (3.2) and Lemma 3.3, we have

$$V\bigl((\lambda_1\cdot K_1 +_0 \cdots +_0 \lambda_m\cdot K_m)^*\bigr) \ge \frac{(n+1)^{(n+1)/2}}{n!}\|f\|^{-n}_{L^2(\nu)}. \tag{3.5}$$

Now, combining (3.4) and (3.5), we obtain

$$\begin{aligned}V\bigl((\lambda_1\cdot K_1 +_0 \cdots +_0 \lambda_m\cdot K_m)^*\bigr) &\ge \frac{(n+1)^{(n+1)/2}}{n!n^{n/2}}\prod_{i=1}^m\Bigl(\sup_{u\in S^{n-1}} h_{K_i}(u)\Bigr)^{-n\lambda_i}\\ &= \frac{(n+1)^{(n+1)/2}}{n!n^{n/2}}\prod_{i=1}^m \inf_{u\in S^{n-1}}\bigl(h_{K_i}(u)^{-n\lambda_i}\bigr). \qquad \square\end{aligned}$$

Proof of Theorem 1.5 Since $\lambda_i \in [0,1]$ satisfying $\sum_{i=1}^m \lambda_i = 1$, then by (2.3), (2.6), Hölder's inequality and again (2.3), we have

$$\begin{aligned}&V(\lambda_1\cdot K_1 +_0 \cdots +_0 \lambda_m\cdot K_m, L_2,\ldots,L_n)\\ &\quad= \frac{1}{n}\int_{S^{n-1}} h_{\lambda_1\cdot K_1+_0\cdots+_0\lambda_m\cdot K_m}(u)\,dS(L_2,\ldots,L_n,u)\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{n}\int_{S^{n-1}} h_{K_1}(u)^{\lambda_1}\cdots h_{K_m}(u)^{\lambda_m}\, dS(L_2,\ldots,L_n,u)\\
&\leq \prod_{i=1}^{m}\left(\frac{1}{n}\int_{S^{n-1}} h_{K_i}(u)^{\lambda_i\cdot\frac{1}{\lambda_i}}\, dS(L_2,\ldots,L_n,u)\right)^{\lambda_i}\\
&= \prod_{i=1}^{m} V(K_i,L_2,\ldots,L_n)^{\lambda_i}.
\end{aligned}$$

□

Letting $L_2 = \cdots = L_n = \lambda_1 \cdot K_1 +_0 \cdots +_0 \lambda_m \cdot K_m$ in Theorem 1.5 gives the following inequality of mixed volumes.

Corollary 3.4 *If K_i $(i = 1,2,\ldots,m)$ are convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors, then for all real $\lambda_i \in [0,1]$ satisfying $\sum_{i=1}^{m}\lambda_i = 1$, we have*

$$V(\lambda_1 \cdot K_1 +_0 \cdots +_0 \lambda_m \cdot K_m) \leq \prod_{i_1=1}^{m}\cdots\prod_{i_n=1}^{m} V(K_{i_1},\ldots,K_{i_n})^{\lambda_{i_1}\cdots\lambda_{i_n}}.$$

Letting $L_2 = \cdots = L_n = B$ in Theorem 1.5 gives the following inequality of quermassintegrals.

Corollary 3.5 *If K_i $(i = 1,2,\ldots,m)$ are convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors, then for all real $\lambda_i \in [0,1]$ satisfying $\sum_{i=1}^{m}\lambda_i = 1$, we have*

$$W_{n-1}(\lambda_1 \cdot K_1 +_0 \cdots +_0 \lambda_m \cdot K_m) \leq \prod_{i=1}^{m} W_{n-1}(K_i)^{\lambda_i}.$$

In view of (2.4), we also obtain the following inequality of mean widths.

Corollary 3.6 *If K_i $(i = 1,2,\ldots,m)$ are convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors, then for all real $\lambda_i \in [0,1]$ satisfying $\sum_{i=1}^{m}\lambda_i = 1$, we have*

$$W(\lambda_1 \cdot K_1 +_0 \cdots +_0 \lambda_m \cdot K_m) \leq \prod_{i=1}^{m} W(K_i)^{\lambda_i}.$$

Proof of Theorem 1.6 Since K, L and Q are convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors, then from (1.4), (1.2), (2.6) and again (1.4), we have

$$\begin{aligned}
&\bar{V}_0\big(Q,(1-\lambda)\cdot K +_0 \lambda\cdot L\big)\\
&\quad= \exp\left(\int_{S^{n-1}} \log\frac{h_{(1-\lambda)\cdot K+_0\lambda\cdot L}}{h_Q}\, d\bar{V}_Q\right)\\
&\quad\leq \exp\left(\int_{S^{n-1}} \log\frac{h_K^{1-\lambda}h_L^{\lambda}}{h_Q}\, d\bar{V}_Q\right)\\
&\quad= \exp\left[\int_{S^{n-1}} \log\left(\frac{h_K}{h_Q}\right)^{1-\lambda} d\bar{V}_Q + \int_{S^{n-1}} \log\left(\frac{h_L}{h_Q}\right)^{\lambda} d\bar{V}_Q\right]\\
&\quad= \exp\left[\int_{S^{n-1}} \log\left(\frac{h_K}{h_Q}\right)^{1-\lambda} d\bar{V}_Q\right]\exp\left[\int_{S^{n-1}} \log\left(\frac{h_L}{h_Q}\right)^{\lambda} d\bar{V}_Q\right]
\end{aligned}$$

$$= \left[\exp\left(\int_{S^{n-1}} \log \frac{h_K}{h_Q}\, d\bar{V}_Q\right)\right]^{1-\lambda} \left[\exp\left(\int_{S^{n-1}} \log \frac{h_L}{h_Q}\, d\bar{V}_Q\right)\right]^{\lambda}$$
$$= \bar{V}_0(Q,K)^{1-\lambda}\,\bar{V}_0(Q,L)^{\lambda}. \qquad \square$$

The following variant of Aleksandrov's lemma (see [28, p.103]) will be needed in proving Theorem 1.7.

Lemma 3.7 [4] *Suppose that* $q_\lambda(u) = q(\lambda,u) : I \times S^{n-1} \to (0,\infty)$ *is a continuous function, where* $I \subset \mathbb{R}$ *is an open interval. Suppose also that the convergence in*

$$\frac{\partial q(\lambda,u)}{\partial \lambda} = \lim_{\gamma\to 0} \frac{q(\lambda+\gamma,u) - q(\lambda,u)}{\gamma}$$

is uniform on S^{n-1}. *If* $\{Q_\lambda\}_{\lambda\in I}$ *is the family of Wulff shapes associated with* q_λ, *i.e., for fixed* $\lambda \in I$,

$$Q_\lambda = \bigcap_{u\in S^{n-1}} \{x \in \mathbb{R}^n : x\cdot u \le q(\lambda,u)\},$$

then

$$\frac{dV(Q_\lambda)}{d\lambda} = \int_{S^{n-1}} \frac{\partial q(\lambda,u)}{\partial\lambda}\, dS_{Q_\lambda}(u).$$

Proof of Theorem 1.7 Since K and L are convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors, let $q(\lambda,u) = h_K(u)^{1-\lambda} h_L(u)^{\lambda}$ in Lemma 3.7, then the convergence in

$$\begin{aligned}\frac{\partial(h_K(u)^{1-\lambda}h_L(u)^{\lambda})}{\partial\lambda} &= \lim_{\gamma\to 0} \frac{h_K(u)^{1-(\lambda+\gamma)}h_L(u)^{(\lambda+\gamma)} - h_K(u)^{1-\lambda}h_L(u)^{\lambda}}{\gamma} \\ &= h_K(u)\left(\frac{h_L(u)}{h_K(u)}\right)^{\lambda} \log\frac{h_L(u)}{h_K(u)}\end{aligned}$$

is uniform on S^{n-1}, and

$$Q_\lambda = \bigcap_{u\in S^{n-1}} \{x \in \mathbb{R}^n : x\cdot u \le h_K(u)^{1-\lambda}h_L(u)^{\lambda}\}. \tag{3.6}$$

Observe that $q(0,u)$ is the support function of K, hence $Q_0 = K$.

On the one hand, from Lemma 3.7, (2.1), (2.2) and (1.4) we have

$$\begin{aligned}\left.\frac{dV(Q_\lambda)}{d\lambda}\right|_{\lambda=0} &= \int_{S^{n-1}} \left.\frac{\partial(h_K(u)^{1-\lambda}h_L(u)^{\lambda})}{\partial\lambda}\right|_{\lambda=0} dS_{Q_0}(u) \\ &= \int_{S^{n-1}} h_K(u)\log\frac{h_L(u)}{h_K(u)}\, dS_K(u) \\ &= nV(K)\int_{S^{n-1}} \log\frac{h_L(u)}{h_K(u)}\, d\bar{V}_K(u) \\ &= nV(K)\log\bar{V}_0(K,L). \end{aligned}\tag{3.7}$$

On the other hand, by (3.6) and (1.2), we have

$$Q_\lambda = (1-\lambda)\cdot K +_0 \lambda \cdot L.$$

Therefore

$$\left.\frac{dV(Q_\lambda)}{d\lambda}\right|_{\lambda=0} = \lim_{\lambda\to 0}\frac{V((1-\lambda)\cdot K +_0 \lambda\cdot L) - V(K)}{\lambda}. \tag{3.8}$$

Hence, combining (3.7) and (3.8), we obtain

$$nV(K)\log \bar{V}_0(K,L) = \lim_{\lambda\to 0}\frac{V((1-\lambda)\cdot K +_0 \lambda\cdot L) - V(K)}{\lambda}. \qquad \square$$

4 Other results and comments

Firstly, we prove that L_0-Minkowski combination and normalized L_0-mixed volume are invariant under simultaneous unimodular centro-affine transformations.

Proposition 4.1 *Suppose that K and L are convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors, and $\lambda \in [0,1]$. If $\phi \in SL(n)$, then*

$$\phi\big((1-\lambda)\cdot K +_0 \lambda\cdot L\big) = (1-\lambda)\cdot \phi K +_0 \lambda\cdot \phi L.$$

Proof For $x \in \mathbb{R}^n$ and $u \in S^{n-1}$, let $y = \phi x$ and $\phi^{-t}u = |\phi^{-t}u|v$, then $y \in \mathbb{R}^n$ and $v \in S^{n-1}$. Thus we have

$$\begin{aligned}
&\phi\big((1-\lambda)\cdot K +_0 \lambda\cdot L\big)\\
&\quad = \phi\Bigg(\bigcap_{u\in S^{n-1}}\big\{x\in\mathbb{R}^n : x\cdot u \le h_K(u)^{1-\lambda}h_L(u)^{\lambda}\big\}\Bigg)\\
&\quad = \phi\Bigg(\bigcap_{u\in S^{n-1}}\big\{\phi^{-1}y\in\mathbb{R}^n : \phi^{-1}y\cdot u \le h_K(u)^{1-\lambda}h_L(u)^{\lambda}\big\}\Bigg)\\
&\quad = \bigcap_{u\in S^{n-1}}\big\{y\in\mathbb{R}^n : y\cdot \phi^{-t}u \le h_K(u)^{1-\lambda}h_L(u)^{\lambda}\big\}\\
&\quad = \bigcap_{v\in S^{n-1}}\big\{y\in\mathbb{R}^n : y\cdot v \le h_K\big(\phi^t v\big)^{1-\lambda}h_L\big(\phi^t v\big)^{\lambda}\big\}\\
&\quad = \bigcap_{v\in S^{n-1}}\big\{y\in\mathbb{R}^n : y\cdot v \le h_{\phi K}(v)^{1-\lambda}h_{\phi L}(v)^{\lambda}\big\}\\
&\quad = (1-\lambda)\cdot \phi K +_0 \lambda\cdot \phi L. \qquad \square
\end{aligned}$$

Proposition 4.2 *Suppose that K and L are convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors, then for $\phi \in SL(n)$ we have*

$$\bar{V}_0(\phi K,\phi L) = \bar{V}_0(K,L).$$

Proof From Theorem 1.7, we have

$$\bar{V}_0(K,L) = \exp\left\{\frac{1}{nV(K)}\lim_{\lambda\to 0}\frac{V((1-\lambda)\cdot K +_0 \lambda\cdot L) - V(K)}{\lambda}\right\}.$$

Then, by Proposition 4.1, we obtain

$$\begin{aligned}\bar{V}_0(\phi K,\phi L) &= \exp\left\{\frac{1}{nV(\phi K)}\lim_{\lambda\to 0}\frac{V((1-\lambda)\cdot\phi K +_0 \lambda\cdot\phi L) - V(\phi K)}{\lambda}\right\}\\ &= \exp\left\{\frac{1}{nV(\phi K)}\lim_{\lambda\to 0}\frac{V(\phi((1-\lambda)\cdot K +_0 \lambda\cdot L)) - V(\phi K)}{\lambda}\right\}\\ &= \exp\left\{\frac{1}{nV(K)}\lim_{\lambda\to 0}\frac{V((1-\lambda)\cdot K +_0 \lambda\cdot L) - V(K)}{\lambda}\right\}\\ &= \bar{V}_0(K,L).\end{aligned}$$

□

The following proposition shows the property of weak convergence of the cone-volume probability measure.

Proposition 4.3 *If K_i is a sequence of convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors, and $\lim_{i\to\infty} K_i = K_0$, where K_0 is a convex body that also contains the origin in its interior, then $\lim_{i\to\infty} \bar{V}_{K_i} = \bar{V}_{K_0}$ weakly.*

Proof Suppose that $f \in C(S^{n-1})$. Since $K_i \to K_0$, by definition, $h_{K_i} \to h_{K_0}$ uniformly on S^{n-1}. Since the continuous function h_{K_0} is positive, the h_{K_i} are uniformly bounded away from zero, and thus

$$fh_{K_i} \to fh_{K_0} \quad \text{uniformly on } S^{n-1}.$$

But $K_i \to K_0$ also implies (see [8]) that

$$S(K_i,\cdot) \to S(K_0,\cdot) \quad \text{weakly on } S^{n-1}.$$

By the continuity of the volume, that is, if $K_i \to K_0$ then $V(K_i) \to V(K_0)$, we have

$$\frac{1}{nV(K_i)}\int_{S^{n-1}} f(u)h_{K_i}(u)\,dS(K_i,u) \to \frac{1}{nV(K_0)}\int_{S^{n-1}} f(u)h_{K_0}(u)\,dS(K_0,u)$$

or, equivalently,

$$\int_{S^{n-1}} f(u)\,d\bar{V}_{K_i}(u) \to \int_{S^{n-1}} f(u)\,d\bar{V}_{K_0}(u).$$

□

The continuity of the normalized L_0-mixed volume is contained in the following proposition.

Proposition 4.4 *Suppose that K_i and L_i are two sequences of convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors, and $\lim_{i\to\infty} K_i = K$, $\lim_{i\to\infty} L_i = L$, where K and L are convex bodies that also contain the origin in its interior, then $\lim_{i\to\infty} \bar{V}_0(K_i,L_i) = \bar{V}_0(K,L)$.*

Proof Since $K_i \to K$ and $L_i \to L$, by definition, $h_{K_i} \to h_K$ and $h_{L_i} \to h_L$ uniformly on S^{n-1}. Since the continuous functions h_K and h_L are positive, the h_{K_i} and h_{L_i} are uniformly bounded away from zero. It follows that $\frac{h_{L_i}}{h_{K_i}} \to \frac{h_L}{h_K}$ uniformly on S^{n-1}, and thus that

$$\log\frac{h_{L_i}}{h_{K_i}} \to \log\frac{h_L}{h_K} \quad \text{uniformly on } S^{n-1}.$$

By Proposition 4.3, $K_i \to K$ implies that

$$\bar{V}_{K_i} \to \bar{V}_K \quad \text{weakly on } S^{n-1}.$$

Hence

$$\exp\left(\int_{S^{n-1}} \log \frac{h_{L_i}}{h_{K_i}}\, d\bar{V}_{K_i}\right) \to \exp\left(\int_{S^{n-1}} \log \frac{h_L}{h_K}\, d\bar{V}_K\right). \qquad \square$$

Next, we show some properties of the Wulff shape in the following propositions.

Proposition 4.5 [8] *Suppose that ν is a Borel measure on S^{n-1} and that f_i, f are positive continuous functions on S^{n-1}. If $\lim_{i\to\infty} f_i(u) = f(u)$ uniformly on $u \in \operatorname{supp}\nu$, then $\lim_{i\to\infty} W_{\nu,f_i} = W_{\nu,f}$.*

Proposition 4.6 *Suppose that ν is a Borel measure on S^{n-1} and that f, g are positive continuous functions on S^{n-1}, then $W_{\nu,\min\{f,g\}} = W_{\nu,f} \cap W_{\nu,g}$.*

Proof Assume that $x \in W_{\nu,\min\{f,g\}}$, then

$$x \cdot u \le \min\{f(u), g(u)\} \quad \text{for all } u \in \operatorname{supp}\nu.$$

Thus

$$x \cdot u \le f(u) \quad \text{and} \quad x \cdot u \le g(u) \quad \text{for all } u \in \operatorname{supp}\nu.$$

Hence, by (2.5), we get

$$x \in W_{\nu,f} \quad \text{and} \quad x \in W_{\nu,g}.$$

Then $x \in W_{\nu,f} \cap W_{\nu,g}$. Therefore, $W_{\nu,\min\{f,g\}} \subseteq W_{\nu,f} \cap W_{\nu,g}$.

Conversely, assume that $x \in W_{\nu,f} \cap W_{\nu,g}$, then

$$x \cdot u \le f(u) \quad \text{and} \quad x \cdot u \le g(u) \quad \text{for all } u \in \operatorname{supp}\nu.$$

Fix $u_0 \in \operatorname{supp}\nu$, then $x \cdot u_0 \le \min\{f(u_0), g(u_0)\}$. By the arbitrariness of u_0, we have $x \cdot u \le \min\{f(u), g(u)\}$ for all $u \in \operatorname{supp}\nu$. Hence $x \in W_{\nu,\min\{f,g\}}$. Therefore, $W_{\nu,f} \cap W_{\nu,g} \subseteq W_{\nu,\min\{f,g\}}$. $\square$

Proposition 4.7 *Suppose that ν is a Borel measure on S^{n-1} and that f, g are positive continuous functions on S^{n-1}, then $W_{\nu,\max\{f,g\}} \supseteq W_{\nu,f} \cup W_{\nu,g}$.*

Proof Assume that $x \in W_{\nu,f} \cup W_{\nu,g}$. Fix $u_0 \in \operatorname{supp}\nu$, then $x \cdot u_0 \le f(u_0)$ or $x \cdot u_0 \le g(u_0)$, thus $x \cdot u_0 \le \max\{f(u_0), g(u_0)\}$. By the arbitrariness of u_0, we have $x \cdot u \le \max\{f(u), g(u)\}$ for all $u \in \operatorname{supp}\nu$. Hence $x \in W_{\nu,\max\{f,g\}}$. $\square$

Proposition 4.8 *Suppose that ν is a Borel measure on S^{n-1} and that f, g are positive continuous functions on S^{n-1}, then for real number $\lambda \in [0,1]$ we have $W_{\nu,f^{1-\lambda}\cdot g^{\lambda}} \supseteq (1-\lambda)\cdot W_{\nu,f} +_0 \lambda \cdot W_{\nu,g}$.*

Proof Let $x \in (1-\lambda)\cdot W_{\nu,f} +_0 \lambda \cdot W_{\nu,g}$. By (1.2), we have $x \cdot u \leq h_{W_{\nu,f}}(u)^{1-\lambda} h_{W_{\nu,g}}(u)^{\lambda}$ for all $u \in \operatorname{supp}\nu$. From (2.6), we know that $h_{W_{\nu,f}}(u) \leq f(u)$ and $h_{W_{\nu,g}}(u) \leq g(u)$ for all $u \in \operatorname{supp}\nu$. Since f and g are positive continuous functions on S^{n-1}, then

$$h_{W_{\nu,f}}(u)^{1-\lambda} h_{W_{\nu,g}}(u)^{\lambda} \leq f(u)^{1-\lambda} g(u)^{\lambda}.$$

Hence $x \cdot u \leq f(u)^{1-\lambda} g(u)^{\lambda}$ for all $u \in \operatorname{supp}\nu$. Therefore, by (2.5), we have $x \in W_{\nu,f^{1-\lambda}\cdot g^{\lambda}}$. □

Proposition 4.9 *Suppose that* ν *is a Borel measure on* S^{n-1} *and that* f, g *are positive continuous functions on* S^{n-1}, *then* $W_{\nu,f+g} \supseteq W_{\nu,f} + W_{\nu,g}$.

Proof Let $x \in W_{\nu,f} + W_{\nu,g}$. Assume $x = x_1 + x_2$, where $x_1 \in W_{\nu,f}$ and $x_2 \in W_{\nu,g}$, then by (2.5) we have $x_1 \cdot u \leq f(u)$ and $x_2 \cdot u \leq g(u)$ for all $u \in \operatorname{supp}\nu$. Thus $x_1 \cdot u + x_2 \cdot u \leq f(u) + g(u)$ for all $u \in \operatorname{supp}\nu$. That is, $x \cdot u \leq f(u) + g(u)$ for all $u \in \operatorname{supp}\nu$. Therefore, $x \in W_{\nu,f+g}$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
QL and GL jointly contributed to the main results Theorem 1.2, Theorem 1.3, Theorem 1.4, Theorem 1.5, Theorem 1.6, Theorem 1.7. QL drafted the manuscript and made the text file. Both authors read and approved the final manuscript.

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