Wu and Chen *Journal of Inequalities and Applications* 2013, **2013**:129
http://www.journalofinequalitiesandapplications.com/content/2013/1/129

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

An improved result in almost sure central limit theorem for self-normalized products of partial sums

Qunying Wu[1*] and Pingyan Chen[2]

*Correspondence:
wqy666@glut.edu.cn
[1]College of Science, Guilin University of Technology, Guilin, 541004, P.R. China
Full list of author information is available at the end of the article

Abstract

Let $X, X_1, X_2, \dots$ be a sequence of independent and identically distributed random variables in the domain of attraction of the normal law. A universal result in an almost sure limit theorem for the self-normalized products of partial sums is established.

MSC: 60F15

Keywords: domain of attraction of the normal law; self-normalized partial sums; almost sure central limit theorem

1 Introduction

Let $\{X, X_n\}_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed (i.i.d.) positive random variables with a non-degenerate distribution function and $\mathbb{E}X = \mu > 0$. For each $n \ge 1$, the symbol S_n/V_n denotes self-normalized partial sums, where $S_n = \sum_{i=1}^n X_i$, $V_n^2 = \sum_{i=1}^n (X_i - \mu)^2$. We say that the random variable X belongs to the domain of attraction of the normal law if there exist constants $a_n > 0$, $b_n \in \mathbb{R}$ such that

$$\frac{S_n - b_n}{a_n} \xrightarrow{d} \mathcal{N}. \tag{1}$$

Here and in the sequel, $\mathcal{N}$ is a standard normal random variable, and $\xrightarrow{d}$ denotes the convergence in distribution. We say that $\{X_n\}_{n\in\mathbb{N}}$ satisfies the central limit theorem (CLT).

It is known that (1) holds if and only if

$$\lim_{x\to\infty} \frac{x^2\mathbb{P}(|X| > x)}{\mathbb{E}X^2 I(|X| \le x)} = 0. \tag{2}$$

In contrast to the well-known classical central limit theorem, Gine *et al.* [7] obtained the following self-normalized version of the central limit theorem: $(S_n - \mathbb{E}S_n)/V_n \xrightarrow{d} \mathcal{N}$ as $n \to \infty$ if and only if (2) holds.

The limit theorem of products $\Pi_{j=1}^n S_j$ was initiated by Arnold and Villaseñor [1]. Their result was generalized by Wu [20], Ye and Wu [22], and Rempala and Wesolowski [16] who proved that if $\{X_n; n \ge 1\}$ is a sequence of i.i.d. positive and finite second moment random variables with $\mathbb{E}X_1 = \mu$, $\operatorname{Var} X_1 = \sigma^2 > 0$ and the coefficient of variation $\gamma = \sigma/\mu$, then

$$\left(\frac{\prod_{i=1}^n S_i}{n!\mu^n}\right)^{1/(\gamma\sqrt{n})} \xrightarrow{d} \mathrm{e}^{\sqrt{2}\mathcal{N}} \quad \text{as } n \to \infty. \tag{3}$$

Recently Pang *et al.* [14] obtained the following self-normalized products of sums for i.i.d. sequences: Let $\{X, X_n\}_{n\in\mathbb{N}}$ be a sequence of i.i.d. positive random variables with $\mathbb{E}X = \mu > 0$, and assume that X is in the domain of attraction of the normal law. Then

$$\left(\frac{\prod_{i=1}^{n} S_i}{n!\mu^n}\right)^{\mu/V_n} \xrightarrow{d} \mathrm{e}^{\sqrt{2}\mathcal{N}} \quad \text{as } n \to \infty. \tag{4}$$

Brosamler [4] and Schatte [17] obtained the following almost sure central limit theorem (ASCLT): Let $\{X_n\}_{n\in\mathbb{N}}$ be i.i.d. random variables with mean 0, variance $\sigma^2 > 0$, and partial sums S_n. Then

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\left\{\frac{S_k}{\sigma\sqrt{k}} < x\right\} = \Phi(x) \quad \text{a.s. for all } x \in \mathbb{R}, \tag{5}$$

with $d_k = 1/k$ and $D_n = \sum_{k=1}^{n} d_k$; here and in the sequel, I denotes an indicator function, and $\Phi(x)$ is the standard normal distribution function. Some ASCLT results for partial sums were obtained by Lacey and Philipp [12], Ibragimov and Lifshits [11], Miao [13], Berkes and Csáki [2], Hörmann [9], Wu [18, 19]. Gonchigdanzan and Rempala [8] gave ASCLT for products of partial sums. Huang and Pang [10], Wu [21], and Zhang and Yang [23] obtained ASCLT results for self-normalized version.

Under mild moment conditions, ASCLT follows from the ordinary CLT, but in general, the validity of ASCLT is a delicate question of a totally different character as CLT. The difference between CLT and ASCLT lies in the weight in ASCLT.

The terminology of summation procedures (see, *e.g.*, Chandrasekharan and Minakshisundaram [5], p.35) shows that the larger the weight sequence $\{d_k; k \geq 1\}$ in (5) is, the stronger the relation becomes. By this argument, one should also expect to get stronger results if we use larger weights. It would be of considerable interest to determine the optimal weights.

On the other hand, by Theorem 1 of Schatte [17], (5) fails for weight $d_k = 1$. The optimal weight sequence remains unknown.

The purpose of this paper is to study and establish the ASCLT for self-normalized products of partial sums of random variables in the domain of attraction of the normal law. We show that the ASCLT holds under a fairly general growth condition on $d_k = k^{-1}\exp((\ln k)^\alpha)$, $0 \leq \alpha < 1/2$.

In the following, we assume that $\{X, X_n\}_{n\in\mathbb{N}}$ is a sequence of i.i.d. positive random variables in the domain of attraction of the normal law with $\mathbb{E}X = \mu > 0$. Let $b_{k,n} = \sum_{j=k}^{n} 1/j$, $S_k = \sum_{i=1}^{k} X_i$, $V_k^2 = \sum_{i=1}^{k}(X_i - \mu)^2$, $S_{k,k} = \sum_{i=1}^{k} b_{i,k}(X_i - \mu)$ for $1 \leq k \leq n$. $a_n \sim b_n$ denotes $\lim_{n\to\infty} a_n/b_n = 1$. The symbol c stands for a generic positive constant which may differ from one place to another.

Our theorem is formulated in a general setting.

Theorem 1.1 *Let $\{X, X_n\}_{n\in\mathbb{N}}$ be a sequence of i.i.d. positive random variables in the domain of attraction of the normal law with mean $\mu > 0$. Suppose $0 \leq \alpha < 1/2$ and set*

$$d_k = \frac{\exp(\ln^\alpha k)}{k}, \qquad D_n = \sum_{k=1}^{n} d_k. \tag{6}$$

Then

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k I\left(\left(\frac{\prod_{i=1}^{k} S_i}{k!\mu^k}\right)^{\mu/V_k}\le x\right) = F(x) \quad a.s.\ x\in\mathbb{R}. \tag{7}$$

Here and in the sequel, F is the distribution function of the random variable $\mathrm{e}^{\sqrt{2}\mathcal{N}}$.

By the terminology of summation procedures, we have the following corollary.

Corollary 1.2 *Theorem* 1.1 *remains valid if we replace the weight sequence* $\{d_k\}_{k\in\mathbb{N}}$ *by* $\{d_k^*\}_{k\in\mathbb{N}}$ *such that* $0\le d_k^*\le d_k$, $\sum_{k=1}^{\infty} d_k^* = \infty$.

Remark 1.3 Our results give substantial improvements for weight sequence in Theorem 1.1 obtained by Zhang and Yang [23].

Remark 1.4 If X is in the domain of attraction of the normal law, then $\mathbb{E}|X|^p < \infty$ for $0 < p < 2$. On the contrary, if $\mathbb{E}X^2 < \infty$, then X is in the domain of attraction of the normal law. Therefore, the class of random variables in Theorem 1.1 is of very broad range.

Remark 1.5 Essentially, the problem whether Theorem 1.1 holds for $1/2 \le \alpha < 1$ remains open.

2 Proofs

Furthermore, the following three lemmas will be useful in the proof, and the first is due to Csörgo *et al.* [6].

Lemma 2.1 *Let X be a random variable with $\mathbb{E}X = \mu$, and denote $l(x) = \mathbb{E}(X-\mu)^2 I\{|X-\mu|\le x\}$. The following statements are equivalent.*

(i) *X is in the domain of attraction of the normal law.*
(ii) $x^2\mathbb{P}(|X-\mu| > x) = o(l(x))$.
(iii) $x\mathbb{E}(|X-\mu| I(|X-\mu| > x)) = o(l(x))$.
(iv) $\mathbb{E}(|X-\mu|^\alpha I(|X-\mu|\le x)) = o(x^{\alpha-2} l(x))$ *for* $\alpha > 2$.
(v) *$l(x)$ is a slowly varying function at ∞.*

Lemma 2.2 *Let $\{\xi,\xi_n\}_{n\in\mathbb{N}}$ be a sequence of uniformly bounded random variables. If there exist constants $c > 0$ and $\delta > 0$ such that*

$$|\mathbb{E}\xi_k\xi_j| \le c\left(\frac{k}{j}\right)^{\delta} \quad \text{for } 1\le k < j, \tag{8}$$

then

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k\xi_k = 0 \quad a.s., \tag{9}$$

where d_k and D_n are defined by (6).

Proof Since

$$\begin{aligned}
\mathbb{E}\left(\sum_{k=1}^{n} d_k\xi_k\right)^2 &\le \sum_{k=1}^{n} d_k^2\mathbb{E}\xi_k^2 + 2\sum_{1\le k<j\le n} d_k d_j|\mathbb{E}\xi_k\xi_j| \\
&= \sum_{k=1}^{n} d_k^2\mathbb{E}\xi_k^2 + 2\sum_{1\le k<j\le n; j/k\ge \ln^{2/\delta} D_n} d_k d_j|\mathbb{E}\xi_k\xi_j| \\
&\quad + 2\sum_{1\le k<j\le n; j/k<\ln^{2/\delta} D_n} d_k d_j|\mathbb{E}\xi_k\xi_j| \\
&:= T_{n1} + 2(T_{n2} + T_{n3}).
\end{aligned} \tag{10}$$

By the assumption of Lemma 2.2, there exists a constant $c > 0$ such that $|\xi_k| \le c$ for any k. Noting that $\exp(\ln^\alpha x) = \exp(\int_1^x \frac{\alpha(\ln u)^{\alpha-1}}{u}\,\mathrm{d}u)$, we have that $\exp(\ln^\alpha x), \alpha < 1$, is a slowly varying function at infinity. Hence,

$$T_{n1} \le c\sum_{k=1}^{n} \frac{\exp(2\ln^\alpha k)}{k^2} \le c\sum_{k=1}^{\infty} \frac{\exp(2\ln^\alpha k)}{k^2} < \infty.$$

By (8),

$$\begin{aligned}
T_{n2} &\le c\sum_{1\le k<j\le n; j/k\ge \ln^{2/\delta} D_n} d_k d_j\left(\frac{k}{j}\right)^\delta \\
&\le c\sum_{1\le k<j\le n; j/k\ge \ln^{2/\delta} D_n} \frac{d_k d_j}{\ln^2 D_n} \le \frac{cD_n^2}{\ln^2 D_n}.
\end{aligned} \tag{11}$$

On the other hand, if $\alpha = 0$, we have $d_k = \mathrm{e}/k$, $D_n \sim \mathrm{e}\ln n$, and hence, for sufficiently large n,

$$T_{n3} \le c\sum_{k=1}^{n} \frac{1}{k}\sum_{j=k}^{k\ln^{2/\delta} D_n} \frac{1}{j} \le cD_n \ln\ln D_n \le \frac{D_n^2}{\ln^2 D_n}. \tag{12}$$

If $0 < \alpha < 1/2$, then by $y^{-\alpha} \to 0$, $y \to \infty$, for arbitrary small $\varepsilon > 0$, there exists n_0 such that for $y \ge \ln n_0$, $(1-\alpha)y^{-\alpha}/\alpha < \varepsilon$. Therefore

$$\begin{aligned}
1 &\le \frac{\int_0^{\ln n}(\exp(y^\alpha) + \frac{1-\alpha}{\alpha}y^{-\alpha}\exp(y^\alpha))\,\mathrm{d}y}{\int_0^{\ln n}\exp(y^\alpha)\,\mathrm{d}y} \\
&\le \frac{\int_0^{\ln n_0}\exp(y^\alpha)(1 + \frac{1-\alpha}{\alpha}y^{-\alpha})\,\mathrm{d}y + (1+\varepsilon)\int_{\ln n_0}^{\ln n}\exp(y^\alpha)\,\mathrm{d}y}{\int_0^{\ln n}\exp(y^\alpha)\,\mathrm{d}y} \to 1+\varepsilon.
\end{aligned}$$

This implies

$$\int_0^{\ln n}\exp(y^\alpha)\,\mathrm{d}y \sim \int_0^{\ln n}\left(\exp(y^\alpha) + \frac{1-\alpha}{\alpha}y^{-\alpha}\exp(y^\alpha)\right)\mathrm{d}y$$

from the arbitrariness of ε.

Hence,

$$\begin{aligned}
D_n &\sim \int_1^n \frac{\exp(\ln^\alpha x)}{x}\,\mathrm{d}x = \int_0^{\ln n} \exp(y^\alpha)\,\mathrm{d}y \\
&\sim \int_0^{\ln n} \left(\exp(y^\alpha) + \frac{1-\alpha}{\alpha} y^{-\alpha} \exp(y^\alpha)\right)\mathrm{d}y \\
&= \int_0^{\ln n} \frac{1}{\alpha}\left(y^{1-\alpha}\exp(y^\alpha)\right)'\,\mathrm{d}y \\
&= \frac{1}{\alpha}\ln^{1-\alpha} n \exp(\ln^\alpha n), \quad n \to \infty.
\end{aligned} \tag{13}$$

This implies

$$\ln D_n \sim \ln^\alpha n, \qquad \exp(\ln^\alpha n) \sim \frac{\alpha D_n}{(\ln D_n)^{\frac{1-\alpha}{\alpha}}}, \qquad \ln\ln D_n \sim \alpha \ln\ln n.$$

Thus combining $|\xi_k| \le c$ for any k,

$$\begin{aligned}
T_{n3} &\le c\sum_{k=1}^{n} \sum_{1\le k<j\le n; j/k<(\ln D_n)^{2/\delta}} d_k d_j \\
&\le c\sum_{k=1}^{n} d_k \sum_{k<j\le k(\ln D_n)^{2/\delta}} \exp(\ln^\alpha n)\frac{1}{j} \\
&\le c\exp(\ln^\alpha n)\ln\ln D_n \sum_{k=1}^{n} d_k \\
&\le c\frac{D_n^2 \ln\ln D_n}{(\ln D_n)^{(1-\alpha)/\alpha}}.
\end{aligned}$$

Since $\alpha < 1/2$ implies $(1-2\alpha)/(2\alpha) > 0$ and $\varepsilon_1 := 1/(2\alpha) - 1 > 0$, thus for sufficiently large n, we get

$$T_{n3} \le c\frac{D_n^2}{(\ln D_n)^{1/(2\alpha)}}\frac{\ln\ln D_n}{(\ln D_n)^{(1-2\alpha)/(2\alpha)}} \le \frac{D_n^2}{(\ln D_n)^{1/(2\alpha)}} = \frac{D_n^2}{(\ln D_n)^{1+\varepsilon_1}}. \tag{14}$$

Let $T_n := \frac{1}{D_n}\sum_{k=1}^n d_k \xi_k$, $\varepsilon_2 := \min(1, \varepsilon_1)$. Combining (10)-(12) and (14), for sufficiently large n, we get

$$\mathbb{E}T_n^2 \ll \frac{c}{(\ln D_n)^{1+\varepsilon_2}}.$$

By (13), we have $D_{n+1} \sim D_n$. Let $0 < \eta < \frac{\varepsilon_2}{1+\varepsilon_2}$, $n_k = \inf\{n; D_n \ge \exp(k^{1-\eta})\}$, then $D_{n_k} \ge \exp(k^{1-\eta})$, $D_{n_k-1} < \exp(k^{1-\eta})$. Therefore

$$1 \le \frac{D_{n_k}}{\exp(k^{1-\eta})} \sim \frac{D_{n_k-1}}{\exp(k^{1-\eta})} < 1,$$

that is,

$$D_{n_k} \sim \exp(k^{1-\eta}).$$

Since $(1-\eta)(1+\varepsilon_2) > 1$ from the definition of η, thus for any $\varepsilon > 0$, we have

$$\sum_{k=1}^{\infty} P\big(|T_{n_k}| > \varepsilon\big) \le c \sum_{k=1}^{\infty} ET_{n_k}^2 \le c \sum_{k=1}^{\infty} \frac{1}{k^{(1-\eta)(1+\varepsilon_2)}} < \infty.$$

By the Borel-Cantelli lemma,

$$T_{n_k} \to 0 \quad \text{a.s.}$$

Now, for $n_k < n \le n_{k+1}$, by $|\xi_k| \le c$ for any k,

$$|T_n| \le |T_{n_k}| + \frac{c}{D_{n_k}} \sum_{i=n_k+1}^{n_{k+1}} d_i \le |T_{n_k}| + c\left(\frac{D_{n_{k+1}}}{D_{n_k}} - 1\right) \to 0 \quad \text{a.s.}$$

from $\frac{D_{n_{k+1}}}{D_{n_k}} \sim \frac{\exp((k+1)^{1-\eta})}{\exp(k^{1-\eta})} = \exp(k^{1-\eta}((1+1/k)^{1-\eta} - 1)) \sim \exp((1-\eta)k^{-\eta}) \to 1$, *i.e.*, (9) holds. This completes the proof of Lemma 2.2. □

Let $l(x) = \mathbb{E}(X-\mu)^2 I\{|X-\mu| \le x\}$, $b = \inf\{x \ge 1; l(x) > 0\}$ and

$$\eta_j = \inf\left\{s; s \ge b+1, \frac{l(s)}{s^2} \le \frac{1}{j}\right\} \quad \text{for } j \ge 1.$$

By the definition of η_j, we have $jl(\eta_j) \le \eta_j^2$ and $jl(\eta_j - \varepsilon) > (\eta_j - \varepsilon)^2$ for any $\varepsilon > 0$. It implies that

$$nl(\eta_n) \sim \eta_n^2 \quad \text{as } n \to \infty. \tag{15}$$

For every $1 \le i \le k \le n$, let

$$\bar{X}_{ki} = (X_i - \mu) I\big(|X_i - \mu| \le \eta_k\big), \qquad \bar{V}_k^2 = \sum_{i=1}^{k} \bar{X}_{ki}^2, \qquad \bar{S}_{k,k} = \sum_{i=1}^{k} b_{i,k} \bar{X}_{ki}.$$

Lemma 2.3 *Suppose that the assumptions of Theorem* 1.1 *hold. Then*

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\left\{\frac{\bar{S}_{k,k} - \mathbb{E}\bar{S}_{k,k}}{\sqrt{2kl(\eta_k)}} \le x\right\} = \Phi(x) \quad \textit{a.s. for any } x \in \mathbb{R}, \tag{16}$$

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \left(I\left(\bigcup_{i=1}^{k}(|X_i - \mu| > \eta_k)\right) - \mathbb{E}I\left(\bigcup_{i=1}^{k}(|X_i - \mu| > \eta_k)\right)\right) = 0 \quad \textit{a.s.}, \tag{17}$$

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \left(f\left(\frac{\bar{V}_k^2}{kl(\eta_k)}\right) - \mathbb{E}f\left(\frac{\bar{V}_k^2}{kl(\eta_k)}\right)\right) = 0 \quad \textit{a.s.}, \tag{18}$$

where d_k and D_n are defined by (6) *and f is a non-negative, bounded Lipschitz function.*

Proof By the central limit theorem for i.i.d. random variables and $\operatorname{Var}\bar{S}_{n,n} \sim 2nl(\eta_n)$ as $n \to \infty$ from $\sum_{k=1}^{n} b_{k,n}^2 \sim 2n$, it follows that

$$\frac{\bar{S}_{n,n} - \mathbb{E}\bar{S}_{n,n}}{\sqrt{2nl(\eta_n)}} \xrightarrow{d} \mathcal{N} \quad \text{as } n \to \infty,$$

where $\mathcal{N}$ denotes the standard normal random variable. This implies that for any $g(x)$ which is a non-negative, bounded Lipschitz function,

$$\mathbb{E}g\left(\frac{\bar{S}_{n,n} - \mathbb{E}\bar{S}_{n,n}}{\sqrt{2nl(\eta_n)}}\right) \longrightarrow \mathbb{E}g(\mathcal{N}) \quad \text{as } n \to \infty.$$

Hence, we obtain

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \mathbb{E}g\left(\frac{\bar{S}_{k,k} - \mathbb{E}\bar{S}_{k,k}}{\sqrt{2kl(\eta_k)}}\right) = \mathbb{E}g(\mathcal{N})$$

from the Toeplitz lemma.

On the other hand, note that (16) is equivalent to

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k g\left(\frac{\bar{S}_{k,k} - \mathbb{E}\bar{S}_{k,k}}{\sqrt{2kl(\eta_k)}}\right) = \mathbb{E}g(\mathcal{N}) \quad \text{a.s.}$$

from Theorem 7.1 of Billingsley [3] and Section 2 of Peligrad and Shao [15]. Hence, to prove (16), it suffices to prove

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \left(g\left(\frac{\bar{S}_{k,k} - \mathbb{E}\bar{S}_{k,k}}{\sqrt{2kl(\eta_k)}}\right) - \mathbb{E}g\left(\frac{\bar{S}_{k,k} - \mathbb{E}\bar{S}_{k,k}}{\sqrt{2kl(\eta_k)}}\right)\right) = 0 \quad \text{a.s.} \tag{19}$$

for any $g(x)$ which is a non-negative, bounded Lipschitz function.

For any $k \geq 1$, let

$$\xi_k = g\left(\frac{\bar{S}_{k,k} - \mathbb{E}\bar{S}_{k,k}}{\sqrt{2kl(\eta_k)}}\right) - \mathbb{E}g\left(\frac{\bar{S}_{k,k} - \mathbb{E}\bar{S}_{k,k}}{\sqrt{2kl(\eta_k)}}\right).$$

For any $1 \leq k < j$, note that $g(\frac{\bar{S}_{k,k} - \mathbb{E}\bar{S}_{k,k}}{\sqrt{2kl(\eta_k)}})$ and $g(\frac{\bar{S}_{j,j} - \mathbb{E}\bar{S}_{j,j} - \sum_{i=1}^{k} b_{i,j}(\bar{X}_{ji} - \mathbb{E}\bar{X}_{ji})}{\sqrt{2jl(\eta_j)}})$ are independent and $g(x)$ is a non-negative, bounded Lipschitz function. By the definition of η_j, we get

$$\begin{aligned}
|\mathbb{E}\xi_k\xi_j| &= \left|\operatorname{Cov}\left(g\left(\frac{\bar{S}_{k,k} - \mathbb{E}\bar{S}_{k,k}}{\sqrt{2kl(\eta_k)}}\right), g\left(\frac{\bar{S}_{j,j} - \mathbb{E}\bar{S}_{j,j}}{\sqrt{2jl(\eta_j)}}\right)\right)\right| \\
&= \left|\operatorname{Cov}\left(g\left(\frac{\bar{S}_{k,k} - \mathbb{E}\bar{S}_{k,k}}{\sqrt{2kl(\eta_k)}}\right), g\left(\frac{\bar{S}_{j,j} - \mathbb{E}\bar{S}_{j,j}}{\sqrt{2jl(\eta_j)}}\right)\right.\right. \\
&\quad \left.\left. - g\left(\frac{\bar{S}_{j,j} - \mathbb{E}\bar{S}_{j,j} - \sum_{i=1}^{k} b_{i,j}(\bar{X}_{ji} - \mathbb{E}\bar{X}_{ji})}{\sqrt{2jl(\eta_j)}}\right)\right)\right| \\
&\leq c\frac{\mathbb{E}|\sum_{i=1}^{k} b_{i,j}(\bar{X}_{ji} - \mathbb{E}\bar{X}_{ji})|}{\sqrt{jl(\eta_j)}} \leq c\frac{\sqrt{\mathbb{E}(\sum_{i=1}^{k} b_{i,j}(\bar{X}_{ji} - \mathbb{E}\bar{X}_{ji}))^2}}{\sqrt{jl(\eta_j)}} \leq c\frac{\sqrt{\sum_{i=1}^{k} b_{i,j}^2 \mathbb{E}\bar{X}_{ji}^2}}{\sqrt{jl(\eta_j)}} \\
&\leq c\frac{\sqrt{\sum_{i=1}^{k} (b_{i,k} + b_{k+1,j})^2 l(\eta_j)}}{\sqrt{jl(\eta_j)}} \leq c\frac{\sqrt{\sum_{i=1}^{k} b_{i,k}^2 + \sum_{i=1}^{k} b_{k+1,j}^2}}{\sqrt{j}} \\
&\leq c\frac{\sqrt{k + k\ln^2(j/k)}}{\sqrt{j}} \leq c\left(\frac{k}{j}\right)^{1/4}.
\end{aligned}$$

By Lemma 2.2, (19) holds.

Now we prove (17). Let

$$Z_k = I\left(\bigcup_{i=1}^{k}\bigl(|X_i - \mu| > \eta_k\bigr)\right) - \mathbb{E}I\left(\bigcup_{i=1}^{k}\bigl(|X_i - \mu| > \eta_k\bigr)\right) \quad \text{for any } k \geq 1.$$

It is known that $I(A \cup B) - I(B) \leq I(A)$ for any sets A and B. Then for $1 \leq k < j$, by Lemma 2.1(ii) and (15), we get

$$\mathbb{P}\bigl(|X - \mu| > \eta_j\bigr) = o(1)\frac{l(\eta_j)}{\eta_j^2} = \frac{o(1)}{j}. \tag{20}$$

Hence

$$\begin{aligned}
|\mathbb{E}Z_kZ_j| &= \left|\operatorname{Cov}\left(I\left(\bigcup_{i=1}^{k}\bigl(|X_i - \mu| > \eta_k\bigr)\right), I\left(\bigcup_{i=1}^{j}\bigl(|X_i - \mu| > \eta_j\bigr)\right)\right)\right| \\
&= \left|\operatorname{Cov}\left(I\left(\bigcup_{i=1}^{k}\bigl(|X_i - \mu| > \eta_k\bigr)\right), I\left(\bigcup_{i=1}^{j}\bigl(|X_i - \mu| > \eta_j\bigr)\right)\right.\right. \\
&\quad \left.\left. - I\left(\bigcup_{i=k+1}^{j}\bigl(|X_i - \mu| > \eta_j\bigr)\right)\right)\right| \\
&\leq \mathbb{E}\left|I\left(\bigcup_{i=1}^{j}\bigl(|X_i - \mu| > \eta_j\bigr)\right) - I\left(\bigcup_{i=k+1}^{j}\bigl(|X_i - \mu| > \eta_j\bigr)\right)\right| \\
&\leq \mathbb{E}I\left(\bigcup_{i=1}^{k}\bigl(|X_i - \mu| > \eta_j\bigr)\right) \leq k\mathbb{P}\bigl(|X - \mu| > \eta_j\bigr) \\
&\leq \frac{k}{j}.
\end{aligned}$$

By Lemma 2.2, (17) holds.

Finally, we prove (18). Let

$$\zeta_k = f\left(\frac{\bar{V}_k^2}{kl(\eta_k)}\right) - \mathbb{E}f\left(\frac{\bar{V}_k^2}{kl(\eta_k)}\right) \quad \text{for any } k \geq 1.$$

For $1 \leq k < j$,

$$\begin{aligned}
|\mathbb{E}\zeta_k\zeta_j| &= \left|\operatorname{Cov}\left(f\left(\frac{\bar{V}_k^2}{kl(\eta_k)}\right), f\left(\frac{\bar{V}_j^2}{jl(\eta_j)}\right)\right)\right| \\
&= \left|\operatorname{Cov}\left(f\left(\frac{\bar{V}_k^2}{kl(\eta_k)}\right), f\left(\frac{\bar{V}_j^2}{jl(\eta_j)}\right) - f\left(\frac{\bar{V}_j^2 - \sum_{i=1}^{k}(X_i - \mu)^2 I(|X_i - \mu| \leq \eta_j)}{jl(\eta_j)}\right)\right)\right| \\
&\leq c\frac{\mathbb{E}(\sum_{i=1}^{k}(X_i - \mu)^2 I(|X_i - \mu| \leq \eta_j))}{jl(\eta_j)} = c\frac{k\mathbb{E}(X - \mu)^2 I(|X - \mu| \leq \eta_j)}{jl(\eta_j)} = c\frac{kl(\eta_j)}{jl(\eta_j)} \\
&= c\frac{k}{j}.
\end{aligned}$$

By Lemma 2.2, (18) holds. This completes the proof of Lemma 2.3. □

Proof of Theorem 1.1 Let $U_i = S_i/(i\mu)$; then (7) is equivalent to

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k I\left(\frac{\mu}{\sqrt{2}V_k}\sum_{i=1}^{k}\ln U_i \le x\right) = \Phi(x) \quad \text{a.s. } x\in\mathbb{R}. \tag{21}$$

Let $q \in (4/3, 2)$, then $\mathbb{E}|X| < \infty$ and $\mathbb{E}|X|^q < \infty$ from Remark 1.4. Using the Marcinkiewicz-Zygmund strong large number law, we have

$$U_k - 1 = \frac{S_k - \mu k}{k\mu} \to 0 \quad \text{a.s.}$$

$$S_k - \mu k = o\big(k^{1/q}\big) \quad \text{a.s.}$$

Hence let $a_k = \sqrt{2(1\pm\varepsilon)kl(\eta_k)}$ for any given $0 < \varepsilon < 1$, by $|\ln(1+x) - x| = O(x^2)$ for $|x| < 1/2$,

$$\begin{aligned}\left|\frac{\mu}{a_k}\sum_{i=1}^{k}\ln U_i - \frac{\mu}{a_k}\sum_{i=1}^{k}(U_i - 1)\right| &\le c\frac{1}{\sqrt{kl(\eta_k)}}\sum_{i=1}^{k}(U_i-1)^2 \\ &\le \frac{c}{\sqrt{kl(\eta_k)}}\sum_{i=1}^{k} i^{2(1/q-1)} \\ &\le c\frac{1}{k^{3/2-2/q}\sqrt{l(\eta_k)}} \to 0 \quad \text{a.s. } k\to\infty,\end{aligned}$$

from $3/2 - 2/q > 0$, $l(x)$ is a slowly varying function at ∞, and $\eta_k \le k + 1$.

Therefore, for any $\delta > 0$ and almost every event ω, there exists $k_0 = k_0(\omega, \delta, x)$ such that for $k > k_0$,

$$\begin{aligned} I\left(\frac{\mu}{a_k}\sum_{i=1}^{k}(U_i-1) \le x - \delta\right) &\le I\left(\frac{\mu}{a_k}\sum_{i=1}^{k}\ln U_i \le x\right) \\ &\le I\left(\frac{\mu}{a_k}\sum_{i=1}^{k}(U_i-1) \le x + \delta\right).\end{aligned} \tag{22}$$

Note that under the condition $|X_j - \mu| \le \eta_k$, $1 \le j \le k$,

$$\begin{aligned}\mu\sum_{i=1}^{k}(U_i - 1) &= \sum_{i=1}^{k}\frac{S_i - i\mu}{i} = \sum_{i=1}^{k}\frac{1}{i}\sum_{j=1}^{i}(X_j - \mu) \\ &= \sum_{j=1}^{k}\sum_{i=j}^{k}\frac{1}{i}\bar{X}_{kj} = \sum_{j=1}^{k} b_{j,k}\bar{X}_{kj} = \bar{S}_{k,k}.\end{aligned} \tag{23}$$

Thus, by (22) and (23), for any given $0 < \varepsilon < 1$, $\delta > 0$, we have for $k > k_0$,

$$\begin{aligned} I\left(\frac{\mu}{\sqrt{2}V_k}\sum_{i=1}^{k}\ln U_i \le x\right) &\le I\left(\frac{\bar{S}_{k,k}}{\sqrt{2(1+\varepsilon)kl(\eta_k)}} \le x + \delta\right) + I\big(\bar{V}_k^2 > (1+\varepsilon)kl(\eta_k)\big) \\ &\quad + I\left(\bigcup_{i=1}^{k}\big(|X_i - \mu| > \eta_k\big)\right) \quad \text{for } x \ge 0,\end{aligned}$$

$$
\begin{aligned}
I\left(\frac{\mu}{\sqrt{2}V_k}\sum_{i=1}^k \ln U_i \le x\right) &\le I\left(\frac{\bar{S}_{k,k}}{\sqrt{2(1-\varepsilon)kl(\eta_k)}} \le x+\delta\right) + I\big(\bar{V}_k^2 < (1-\varepsilon)kl(\eta_k)\big) \\
&\quad + I\left(\bigcup_{i=1}^k \big(|X_i-\mu| > \eta_k\big)\right) \quad \text{for } x < 0,
\end{aligned}
$$

and

$$
\begin{aligned}
I\left(\frac{\mu}{\sqrt{2}V_k}\sum_{i=1}^k \ln U_i \le x\right) &\ge I\left(\frac{\bar{S}_{k,k}}{\sqrt{2(1-\varepsilon)kl(\eta_k)}} \le x-\delta\right) - I\big(\bar{V}_k^2 < (1-\varepsilon)kl(\eta_k)\big) \\
&\quad - I\left(\bigcup_{i=1}^k \big(|X_i-\mu| > \eta_k\big)\right) \quad \text{for } x \ge 0, \\
I\left(\frac{\mu}{\sqrt{2}V_k}\sum_{i=1}^k \ln U_i \le x\right) &\ge I\left(\frac{\bar{S}_{k,k}}{\sqrt{2(1+\varepsilon)kl(\eta_k)}} \le x-\delta\right) - I\big(\bar{V}_k^2 > (1+\varepsilon)kl(\eta_k)\big) \\
&\quad - I\left(\bigcup_{i=1}^k \big(|X_i-\mu| > \eta_k\big)\right) \quad \text{for } x < 0.
\end{aligned}
$$

Hence, to prove (21), it suffices to prove

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^n d_k I\left(\frac{\bar{S}_{k,k}}{\sqrt{2kl(\eta_k)}} \le \sqrt{1\pm\varepsilon}x \pm \delta_1\right) = \Phi(\sqrt{1\pm\varepsilon}x\pm\delta_1) \quad \text{a.s.,} \tag{24}$$

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^n d_k I\left(\bigcup_{i=1}^k \big(|X_i-\mu| > \eta_k\big)\right) = 0 \quad \text{a.s.,} \tag{25}$$

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^n d_k I\big(\bar{V}_k^2 > (1+\varepsilon)kl(\eta_k)\big) = 0 \quad \text{a.s.,} \tag{26}$$

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^n d_k I\big(\bar{V}_k^2 < (1-\varepsilon)kl(\eta_k)\big) = 0 \quad \text{a.s.} \tag{27}$$

for any $0 < \varepsilon < 1$ and $\delta_1 > 0$.

Firstly, we prove (24). Let $0 < \beta < 1/2$, and let $h(\cdot)$ be a real function such that for any given $x \in \mathbb{R}$,

$$I(y \le \sqrt{1\pm\varepsilon}x \pm \delta_1 - \beta) \le h(y) \le I(y \le \sqrt{1\pm\varepsilon}x \pm \delta_1 + \beta). \tag{28}$$

By $\mathbb{E}(X_i-\mu) = 0$, Lemma 2.1(iii) and (15), we have

$$
\begin{aligned}
|\mathbb{E}\bar{S}_{k,k}| &= \left|\mathbb{E}\sum_{i=1}^k b_{i,k}(X_i-\mu)I\big(|X_i-\mu| \le \eta_k\big)\right| \le \sum_{i=1}^k b_{i,k}\mathbb{E}|X_i-\mu|I\big(|X_i-\mu| > \eta_k\big) \\
&= \sum_{i=1}^k\sum_{j=i}^k \frac{1}{j}\mathbb{E}|X-\mu|I\big(|X-\mu| > \eta_k\big) = \sum_{j=1}^k\sum_{i=1}^j \frac{1}{j}\frac{o(l(\eta_k))}{\eta_k} \\
&= o\big(\sqrt{kl(\eta_k)}\big).
\end{aligned}
$$

This, combining with (16), (28) and the arbitrariness of β in (28), (24), holds.

By (17), (20) and the Toeplitz lemma,

$$\begin{aligned}0 &\le \frac{1}{D_n}\sum_{k=1}^{n} d_k I\left(\bigcup_{i=1}^{k}\bigl(|X_i-\mu|>\eta_k\bigr)\right) \sim \frac{1}{D_n}\sum_{k=1}^{n} d_k \mathbb{E} I\left(\bigcup_{i=1}^{k}\bigl(|X_i-\mu|>\eta_k\bigr)\right)\\ &\le \frac{1}{D_n}\sum_{k=1}^{n} d_k k\mathbb{P}\bigl(|X-\mu|>\eta_k\bigr)\to 0 \quad \text{a.s.}\end{aligned}$$

Hence, (25) holds.

Now we prove (26). For any $\lambda>0$, let f be a non-negative, bounded Lipschitz function such that

$$I(x>1+\lambda)\le f(x)\le I(x>1+\lambda/2).$$

From $\mathbb{E}\bar{V}_k^2 = kl(\eta_k)$, $\bar{X}_{ni}$ is i.i.d., Lemma 2.1(iv), and (15),

$$\begin{aligned}\mathbb{P}\left(\bar{V}_k^2 > \left(1+\frac{\lambda}{2}\right)kl(\eta_k)\right) &= \mathbb{P}\bigl(\bar{V}_k^2-\mathbb{E}\bar{V}_k^2>\lambda kl(\eta_k)/2\bigr)\\ &\le c\frac{\mathbb{E}(\bar{V}_k^2-\mathbb{E}\bar{V}_k^2)^2}{k^2l^2(\eta_k)} \le c\frac{\mathbb{E}(X-\mu)^4I(|X-\mu|\le\eta_k)}{kl^2(\eta_k)}\\ &= \frac{o(1)\eta_k^2}{kl(\eta_k)} = o(1)\to 0.\end{aligned}$$

Therefore, from (18) and the Toeplitz lemma,

$$\begin{aligned}0 \ &\le\ \frac{1}{D_n}\sum_{k=1}^{n} d_k I\bigl(\bar{V}_k^2>(1+\lambda)kl(\eta_k)\bigr) \le \frac{1}{D_n}\sum_{k=1}^{n} d_k f\left(\frac{\bar{V}_k^2}{kl(\eta_k)}\right)\\ &\sim\ \frac{1}{D_n}\sum_{k=1}^{n} d_k \mathbb{E} f\left(\frac{\bar{V}_k^2}{kl(\eta_k)}\right) \le \frac{1}{D_n}\sum_{k=1}^{n} d_k \mathbb{E} I\bigl(\bar{V}_k^2>(1+\lambda/2)kl(\eta_k)\bigr)\\ &=\ \frac{1}{D_n}\sum_{k=1}^{n} d_k \mathbb{P}\bigl(\bar{V}_k^2>(1+\lambda/2)kl(\eta_k)\bigr)\\ &\to 0 \quad \text{a.s.}\end{aligned}$$

Hence, (26) holds. By similar methods used to prove (26), we can prove (27). This completes the proof of Theorem 1.1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
QW conceived of the study and drafted, complete the manuscript. PC participated in the discussion of the manuscript. QW and PC read and approved the final manuscript.

Author details
[1]College of Science, Guilin University of Technology, Guilin, 541004, P.R. China. [2]Department of Mathematics, Ji'nan University, Guangzhou, 510630, P.R. China.

Authors' information
Qunying Wu, Professor, Doctor, working in the field of probability and statistics.

Acknowledgements
The authors are very grateful to the referees and the editors for their valuable comments and some helpful suggestions that improved the clarity and readability of the paper. Supported by the National Natural Science Foundation of China (11061012), and the support Program of the Guangxi China Science Foundation (2012GXNSFAA053010).

Received: 16 November 2012 Accepted: 5 March 2013 Published: 27 March 2013

References
1. Arnold, BC, Villaseñor, JA: The asymptotic distribution of sums of records. Extremes **1**(3), 351-363 (1998)
2. Berkes, I, Csáki, E: A universal result in almost sure central limit theory. Stoch. Process. Appl. **94**, 105-134 (2001)
3. Billingsley, P: Convergence of Probability Measures. Wiley, New York (1968)
4. Brosamler, GA: An almost everywhere central limit theorem. Math. Proc. Camb. Philos. Soc. **104**, 561-574 (1988)
5. Chandrasekharan, K, Minakshisundaram, S: Typical Means. Oxford University Press, Oxford (1952)
6. Csörgo, M, Szyszkowicz, B, Wang, QY: Donsker's theorem for self-normalized partial sums processes. Ann. Probab. **31**(3), 1228-1240 (2003)
7. Gine, E, Götze, F, Mason, DM: When is the Student *t*-statistic asymptotically standard normal? Ann. Probab. **25**, 1514-1531 (1997)
8. Gonchigdanzan, K, Rempala, G: A note on the almost sure central limit theorem for the product of partial sums. Appl. Math. Lett. **19**, 191-196 (2006)
9. Hörmann, S: Critical behavior in almost sure central limit theory. J. Theor. Probab. **20**, 613-636 (2007)
10. Huang, SH, Pang, TX: An almost sure central limit theorem for self-normalized partial sums. Comput. Math. Appl. **60**, 2639-2644 (2010)
11. Ibragimov, IA, Lifshits, M: On the convergence of generalized moments in almost sure central limit theorem. Stat. Probab. Lett. **40**, 343-351 (1998)
12. Lacey, MT, Philipp, W: A note on the almost sure central limit theorem. Stat. Probab. Lett. **9**, 201-205 (1990)
13. Miao, Y: Central limit theorem and almost sure central limit theorem for the product of some partial sums. Proc. Indian Acad. Sci. Math. Sci. **118**(2), 289-294 (2008)
14. Pang, TX, Lin, ZY, Hwang, KS: Asymptotics for self-normalized random products of sums of i.i.d. random variables. J. Math. Anal. Appl. **334**, 1246-1259 (2007)
15. Peligrad, M, Shao, QM: A note on the almost sure central limit theorem for weakly dependent random variables. Stat. Probab. Lett. **22**, 131-136 (1995)
16. Rempala, G, Wesolowski, J: Asymptotics for products of sums and *U*-statistics. Electron. Commun. Probab. **7**, 47-54 (2002)
17. Schatte, P: On strong versions of the central limit theorem. Math. Nachr. **137**, 249-256 (1988)
18. Wu, QY: Almost sure limit theorems for stable distribution. Stat. Probab. Lett. **81**(6), 662-672 (2011)
19. Wu, QY: An almost sure central limit theorem for the weight function sequences of NA random variables. Proc. Indian Acad. Sci. Math. Sci. **121**(3), 369-377 (2011)
20. Wu, QY: Almost sure central limit theory for products of sums of partial sums. Appl. Math. J. Chin. Univ. Ser. B **27**(2), 169-180 (2012)
21. Wu, QY: A note on the almost sure limit theorem for self-normalized partial sums of random variables in the domain of attraction of the normal law. J. Inequal. Appl. **2012**, 17 (2012). 10.1186/1029-242X-2012-17
22. Ye, DX, Wu, QY: Almost sure central limit theorem for product of partial sums of strongly mixing random variables. J. Inequal. Appl. **2011**, Article ID 576301 (2011)
23. Zhang, Y, Yang, XY: An almost sure central limit theorem for self-normalized products of sums of i.i.d. random variables. J. Math. Anal. Appl. **376**, 29-41 (2011)

doi:10.1186/1029-242X-2013-129
Cite this article as: Wu and Chen: **An improved result in almost sure central limit theorem for self-normalized products of partial sums.** *Journal of Inequalities and Applications* 2013 **2013**:129.