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# $p$ -subordination chains and $p$ -valence criteria

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## Abstract

The main object of this investigation is to give some sufficient conditions for analytic functions, by the method of  $p$ -subordination chains, to be the  $p$ th power of a univalent function in the open unit disk  $\mathcal{U}$ . Also, the significant relationships and relevance to other results are also given. A number of known univalent conditions would follow upon specializing the parameters involved in our main results.

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**Keywords:**  $p$ -valent function;  $p$ -subordination chain;  $p$ -valence criterion

## 1 Introduction

Denote by  $\mathcal{U}_r = \{z \in \mathbb{C} : |z| < r\}$  ( $0 < r \leq 1$ ) the disk of radius  $r$ , and let  $\mathcal{U} = \mathcal{U}_1$ . Let  $\mathcal{A}$  denote the class of analytic functions in the open unit disk  $\mathcal{U}$  which satisfy the usual normalization condition  $f(0) = f'(0) - 1 = 0$ . Traditionally, the subclass of  $\mathcal{A}$  consisting of univalent functions is denoted by  $\mathcal{S}$ . Let  $\mathcal{P}$  denote the class of functions  $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ ,  $z \in \mathcal{U}$ , that satisfy the condition  $\Re p(z) > 0$ . Also, let  $\mathcal{A}_p$  denote the class of analytic functions in the open unit disk  $\mathcal{U}$  which satisfy the normalizations  $f^{(k)}(0) = 0$  for  $k = 1, 2, \dots, p - 1$  ( $p \in \mathbb{N} = \{1, 2, \dots\}$ ) and  $f^{(p)}(0) \neq 0$ , and let  $\mathcal{A}_p^*$  be the subclass of  $\mathcal{A}_p$  consisting of functions of the form  $f(z) = z^p + \sum_{n=1+p}^{\infty} a_n z^n$  in  $\mathcal{U}$ . These classes have been one of the most important subjects of research in geometric function theory for a long time (see [1]). For analytic functions  $f(z)$  and  $g(z)$  in  $\mathcal{U}$ ,  $f$  is said to be subordinate to  $g$ , denoted by  $f(z) \prec g(z)$ , if there exists an analytic function  $w$  satisfying  $w(0) = 0$ ,  $|w(z)| < 1$ , such that  $f(z) = g(w(z))$  ( $z \in \mathcal{U}$ ). In particular, if the function  $g$  is univalent in  $\mathcal{U}$ , the above subordination is equivalent to  $f(0) = g(0)$  and  $f(\mathcal{U}) \subset g(\mathcal{U})$ .

## 2 $p$ -normalized subordination chain and related theorem

Before proving our main theorem, we need a brief summary of the method of  $p$ -subordination chains.

**Definition 2.1** (see Hallenbeck and Livingston [2]) Let  $\mathcal{L}(z, t)$  be a function defined on  $\mathcal{U} \times \mathcal{I}$ , where  $\mathcal{I} := [0, \infty)$ .  $\mathcal{L}(z, t)$  is called a  $p$ -subordination chain if  $\mathcal{L}(z, t)$  satisfies the following conditions:

1.  $\mathcal{L}(z, t)$  is analytic in  $\mathcal{U}$  for all  $t \in \mathcal{I}$ ,
2.  $\mathcal{L}^{(k)}(0, t) = 0$ ,  $k = 1, 2, \dots, p - 1$ , and  $\mathcal{L}^{(p)}(0, t) \neq 0$ ,
3.  $\mathcal{L}(z, t) \prec \mathcal{L}(z, s)$  for all  $0 \leq t \leq s < \infty$ ,  $z \in \mathcal{U}$ .

$p$ -subordination chain is said to be normalized if  $\mathcal{L}(0, t) = 0$  and  $\mathcal{L}^{(p)}(0, t) = p!e^{pt}$  for all  $t \in \mathcal{I}$ .

In order to prove our main results, we need the following lemma due to Hallenbeck and Livingston [2].

**Lemma 2.1** *Let  $\mathcal{L}(z, t) = a_p(t)z^p + a_{p+1}(t)z^{p+1} + \dots$ ,  $a_p(t) \neq 0$ , be analytic in  $\mathcal{U}_r$  for all  $t \in \mathcal{I}$ . Suppose that*

- (i)  $\mathcal{L}(z, t)$  is a locally absolutely continuous function in the interval  $\mathcal{I}$  and locally uniform with respect to  $\mathcal{U}_r$ .
- (ii)  $a_p(t)$  is a complex-valued continuous function on  $\mathcal{I}$  such that  $a_p(t) \neq 0$ ,  $|a_p(t)| \rightarrow \infty$  for  $t \rightarrow \infty$  and

$$\left\{ \frac{\mathcal{L}(z, t)}{a_p(t)} \right\}_{t \in \mathcal{I}}$$

forms a normal family of functions in  $\mathcal{U}_r$ .

- (iii) There exists an analytic function  $h : \mathcal{U} \times \mathcal{I} \rightarrow \mathbb{C}$  satisfying  $\Re h(z, t) > 0$  for all  $z \in \mathcal{U}$ ,  $t \in \mathcal{I}$  and

$$p \frac{\partial \mathcal{L}(z, t)}{\partial t} = z \frac{\partial \mathcal{L}(z, t)}{\partial z} h(z, t) \quad (z \in \mathcal{U}_r, t \in \mathcal{I}). \tag{2.1}$$

Then, for each  $t \in \mathcal{I}$ , the function  $\mathcal{L}(z, t)$  is the  $p$ th power of a univalent function in  $\mathcal{U}$ .

Pommerenke’s theory of subordination chains [3, 4] corresponds to  $p = 1$ .

The univalence of complex functions is an important property, but, unfortunately, it is difficult and in many cases impossible to show directly that a certain complex function, especially a function belonging to the class  $\mathcal{A}$ , is univalent. Pommerenke [3, 4] and Becker [5] have used the idea of normalized 1-subordination chains, or briefly subordination chains, to obtain sufficient conditions for univalence of the functions belonging to the class  $\mathcal{A}$ . There are three very important criteria for univalence of the function  $f \in \mathcal{A}$ . Two of them are the well-known criteria of Becker [5] and Ahlfors [6] which were obtained by a clever use of the theory of subordination chains and the generalized Loewner differential equation. The other, Nehari’s univalence criterion (see [7]), was obtained without using the subordination chains for the analytic functions. Then Epstein [8] generalized this criterion by using the hyperbolic geometry, and his proof was quite different from the subordination chains method. By using the subordination chains methods, Pommerenke [9] gave a simplified proof of a univalence criterion obtained earlier by Epstein [8]. But in some cases, these criteria may not be sufficient for learning the univalence of the function  $f \in \mathcal{A}$ . For example, although the function  $f(z) = z - \frac{1}{2}z^2$  is univalent, this function is satisfied neither by Becker and Ahlfors nor by Nehari criteria. This situation is deficiency for these criteria. For this reason, we need to find new criteria or generalize the current criteria. During the time many mathematicians have studied on this problem and have obtained some results (see [10–19] and [20]).

On the other hand, Hallenbeck and Livingston [2] defined  $p$ -subordination chains and gave Lemma 2.1 for the functions  $f \in \mathcal{A}_p^*$ . In the same paper, they obtained some results for  $f \in \mathcal{A}_p^*$  to be the  $p$ th power of a univalent function in  $\mathcal{U}$ . Their criteria are a  $p$ -valence version of Becker and Ahlfors’s criteria. Recently Deniz *et al.* [10] submitted a paper which

includes sufficient conditions for a integral operator to be the  $p$ th power of a univalent function in  $\mathcal{U}$ .

In the present paper, we obtain sufficient conditions for the functions  $f$  belonging to the class  $\mathcal{A}_p^*$  in terms of the Schwarz derivative defined by

$$S_f(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

to be the  $p$ th power of a univalent function by using  $p$ -subordination chains. Our main result is a  $p$ -valence version of Nehari [7] and Epstein's [8] criteria.

### 3 $p$ -valence criteria

Making use of Lemma 2.1, we can prove now our main result related to the Schwarz derivative.

**Theorem 3.1** *Let  $f, g \in \mathcal{A}_p^*$ . If*

$$\left| (1 - |z|^{2p}) \left[ 1 - p + \frac{zg''(z)}{g'(z)} \right] + \frac{(1 - |z|^{2p})^2}{2p} \frac{z^2}{|z|^{2p}} [S_f(z) - S_g(z)] \right| \leq p \tag{3.1}$$

for all  $z \in \mathcal{U}$ , then  $f$  is the  $p$ th power of a univalent function in  $\mathcal{U}$ .

*Proof* Consider the functions defined by

$$v(z) = \sqrt{\frac{g'(z)}{f'(z)}} = 1 + v_2z + \dots, \tag{3.2}$$

where we choose the branch of the power  $(\cdot)^{1/2}$ , which for  $z = 0$  has value 1, and

$$u(z) = f(z)v(z) = z^p + u_2z^{p+1} + \dots. \tag{3.3}$$

The functions  $u$  and  $v$  are analytic in  $\mathcal{U}$  since  $f$  and  $g$  analytic.

For all  $t \in \mathcal{I}$  and  $z \in \mathcal{U}_{r_1}$  ( $0 < r_1 \leq 1$ ), the function  $\mathcal{L} : \mathcal{U}_{r_1} \times \mathcal{I} \rightarrow \mathbb{C}$  defined formally by

$$\begin{aligned} \mathcal{L}(z, t) &= \frac{u(e^{-t}z) + \frac{(e^{2pt}-1)}{p} e^{-t}zu'(e^{-t}z)}{v(e^{-t}z) + \frac{(e^{2pt}-1)}{p} e^{-t}zv'(e^{-t}z)} \\ &= e^{pt}z^p + \Phi(e^{-mt}, z^{p+1}), \quad m = 1, 2, \dots \end{aligned} \tag{3.4}$$

is analytic in  $\mathcal{U}_{r_1}$  since  $\Phi(e^{-mt}, z^{p+1})$  is an analytic function in  $\mathcal{U}$  for each fixed  $t \in \mathcal{I}$  and  $m = 1, 2, \dots$ . From (3.4) we have  $a_p(t) = e^{pt}$  and

$$\lim_{t \rightarrow \infty} |a_p(t)| = \lim_{t \rightarrow \infty} e^{pt} = \infty.$$

After simple calculation, we obtain, for each  $z \in \mathcal{U}_{r_1}$ ,

$$\lim_{t \rightarrow \infty} \frac{\mathcal{L}(z, t)}{e^{pt}} = \lim_{t \rightarrow \infty} \{z^p + \Phi(e^{-(m+p)t}, z^{p+1})\} = z^p.$$

The limit function  $k(z) = z^p$  belongs to the family  $\{\mathcal{L}(z, t)/e^{pt} : t \in \mathcal{I}\}$ ; then there exists a number  $r_0$  ( $0 < r_0 < r_1$ ) such that in every closed disk  $\mathcal{U}_{r_0}$ , there exists a constant  $K_0 > 0$  such that

$$\left| \frac{\mathcal{L}(z, t)}{e^{pt}} \right| < K_0 \quad (z \in \mathcal{U}_{r_0}, t \in \mathcal{I})$$

uniformly in this disk, provided that  $t$  is sufficiently large. Thus, by Montel's theorem,  $\{\mathcal{L}(z, t)/e^{pt}\}$  forms a normal family in each disk  $\mathcal{U}_{r_0}$ .

Since the function  $\Phi(e^{-mt}, z^{p+1})$  is analytic in  $\mathcal{U}$ , for  $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ , the function  $\Phi^{(k)}(e^{-mt}, z^{p+1})$  is continuous on the compact set, so  $\Phi^{(k)}(e^{-mt}, z^{p+1})$ ,  $k \in \mathbb{N}_0$ , is a bounded function. Thus, for all fixed  $T > 0$ , we can write  $e^t < e^T$ , and we obtain that for all fixed numbers  $t \in [0, T] \subset \mathcal{I}$ , there exists a constant  $K_1 > 0$  such that

$$\left| \frac{\partial \mathcal{L}(z, t)}{\partial t} \right| < K_1, \quad \forall z \in \mathcal{U}_{r_0}, t \in [0, T].$$

Therefore, the function  $\mathcal{L}(z, t)$  is locally absolutely continuous in  $\mathcal{I}$ ; locally uniform with respect to  $\mathcal{U}_{r_0}$ .

After simple calculations, from (3.4) we obtain

$$\begin{aligned} \frac{\partial \mathcal{L}(z, t)}{\partial z} &= e^{-t} \left\{ \left( 1 + \frac{e^{2pt} - 1}{p} \right) (u'v - v'u) \right. \\ &\quad \left. + \frac{e^{2pt} - 1}{p} e^{-t} z (u''v - v''u) + \left( \frac{e^{2pt} - 1}{p} \right)^2 e^{-2t} z^2 (u''v' - v''u') \right\} \\ &\quad / \left[ v(e^{-t}z) + \frac{e^{2pt} - 1}{p} e^{-t} z v'(e^{-t}z) \right]^2, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \frac{\partial \mathcal{L}(z, t)}{\partial t} &= e^{-t} z \left\{ \left( -1 - \frac{e^{2pt} - 1}{p} + 2e^{2pt} \right) (u'v - v'u) \right. \\ &\quad \left. - \frac{e^{2pt} - 1}{p} e^{-t} z (u''v - v''u) - \left( \frac{e^{2pt} - 1}{p} \right)^2 e^{-2t} z^2 (u''v' - v''u') \right\} \\ &\quad / \left[ v(e^{-t}z) + \frac{e^{2pt} - 1}{p} e^{-t} z v'(e^{-t}z) \right]^2, \end{aligned} \tag{3.6}$$

where

$$u'v - v'u = g', \tag{3.7}$$

$$u''v - v''u = g'', \tag{3.8}$$

$$u''v' - v''u' = \frac{g'}{2} (S_f - S_g), \tag{3.9}$$

and  $u, v, u', v', u'', v''$  are calculated at  $e^{-t}z$ .

Consider the function  $h : \mathcal{U}_r \times \mathcal{I} \rightarrow \mathbb{C}$  for  $0 < r < r_0$  and  $t \in \mathcal{I}$  defined by

$$h(z, t) = p \frac{\partial \mathcal{L}(z, t)}{\partial t} / z \frac{\partial \mathcal{L}(z, t)}{\partial z}.$$

From (3.5) to (3.9), we can easily see that the function  $h(z, t)$  is analytic in  $\mathcal{U}_r$ ,  $0 < r < r_0$ . If the function

$$w(z, t) = \frac{h(z, t) - 1}{h(z, t) + 1} = \frac{p \frac{\partial \mathcal{L}(z, t)}{\partial t} - z \frac{\partial \mathcal{L}(z, t)}{\partial z}}{p \frac{\partial \mathcal{L}(z, t)}{\partial t} + z \frac{\partial \mathcal{L}(z, t)}{\partial z}} \quad (z \in \mathcal{U}_r, t \in \mathcal{I}) \tag{3.10}$$

is analytic in  $\mathcal{U}$  and  $|w(z, t)| < 1$  for all  $z \in \mathcal{U}$  and  $t \in \mathcal{I}$ , then  $h(z, t)$  has an analytic extension with a positive real part ( $\Re h(z, t) > 0$ ) in  $\mathcal{U}$  for all  $t \in \mathcal{I}$ .

From equality (3.10) we have

$$w(z, t) = \frac{(p + 1)\Psi(z, t) - 1}{(p - 1)\Psi(z, t) - 1}, \tag{3.11}$$

where

$$\begin{aligned} \Psi(z, t) = & \frac{1 + (p - 1)e^{-2pt}}{2p^2} + \frac{(1 - e^{-2pt})e^{-t}z}{2p^2} \left( \frac{u''v - v''u}{u'v - v'u} \right) \\ & + \frac{(1 - e^{-2pt})^2 e^{2pt} (e^{-t}z)^2}{2p^3} \left( \frac{u''v' - v''u'}{u'v - v'u} \right) \end{aligned} \tag{3.12}$$

for  $z \in \mathcal{U}$  and  $t \in \mathcal{I}$ .

The inequality  $|w(z, t)| < 1$  for all  $z \in \mathcal{U}$  and  $t \in \mathcal{I}$ , where  $w(z, t)$  is defined by (3.11), is equivalent to

$$\left| \Psi(z, t) - \frac{1}{2p} \right| < \frac{1}{2p}, \quad \forall z \in \mathcal{U}, t \in \mathcal{I}. \tag{3.13}$$

From the hypothesis of theorem, (3.12) and (3.13), we have

$$\left| \Psi(z, 0) - \frac{1}{2p} \right| < p \quad \text{for all } z \in \mathcal{U} \tag{3.14}$$

and

$$\left| \Psi(0, t) - \frac{1}{2p} \right| < p \quad \text{for all } t \in \mathcal{I}. \tag{3.15}$$

Since  $|e^{-t}z| \leq e^{-t} < 1$  for all  $z \in \overline{\mathcal{U}}$  and  $t > 0$ , we find that  $w(z, t)$  is an analytic function in  $\overline{\mathcal{U}}$ . By the maximum modulus principle, it follows that for all  $z \in \mathcal{U} - \{0\}$  and each  $t > 0$  arbitrarily fixed, there exists  $\theta = \theta(t) \in \mathbb{R}$  such that

$$\left| \Psi(z, t) - \frac{1}{2p} \right| < \lim_{|z|=1} \left| \Psi(z, t) - \frac{1}{2p} \right| = \left| \Psi(e^{i\theta}, t) - \frac{1}{2p} \right|. \tag{3.16}$$

Denote  $\zeta = e^{-t}e^{i\theta}$ . Then  $|\zeta| = e^{-t}$ , and from (3.7)-(3.9) and (3.12), we have

$$\begin{aligned} \left| \Psi(e^{i\theta}, t) - \frac{1}{2p} \right| = & \frac{1}{2p^2} \left| (1 - |\zeta|^{2p}) \left( 1 - p + \frac{\zeta g''(\zeta)}{g'(\zeta)} \right) \right. \\ & \left. + \frac{(1 - |\zeta|^{2p})^2}{2p} \frac{\zeta^2}{|\zeta|^{2p}} (S_f(\zeta) - S_g(\zeta)) \right|. \end{aligned} \tag{3.17}$$

Because  $\zeta \in \mathcal{U}$ , the inequality (3.1) implies that  $|\Psi(e^{i\theta}, t) - \frac{1}{2p}| \leq \frac{1}{2p}$ , and from (3.14), (3.15) and (3.16), we conclude that  $|\Psi(e^{i\theta}, t) - \frac{1}{2p}| < \frac{1}{2p}$  for all  $z \in \mathcal{U}$  and  $t \in \mathcal{I}$ . Therefore  $|w(z, t)| < 1$  for all  $z \in \mathcal{U}$  and  $t \in \mathcal{I}$ . Since all the conditions of Lemma 2.1 are satisfied, we obtain that the function  $\mathcal{L}(z, t)$  is the  $p$ th power of a univalent function in the whole unit disk  $\mathcal{U}$  for all  $t \in \mathcal{I}$ .  $\square$

Theorem 3.1 is a  $p$ -valence version of the univalence criterion in the unit disk obtained earlier by Epstein [8].

If we take  $g(z) = z^p$  in Theorem 3.1, we obtain a  $p$ -valence version of Nehari's [7] univalence criterion.

**Corollary 3.2** *Let  $f \in \mathcal{A}_p^*$ . If*

$$(1 - |z|^{2p})^2 \left| z^2 S_f(z) + \frac{p^2 - 1}{2} \right| \leq 2p^2 |z|^{2p}$$

*for all  $z \in \mathcal{U}$ , then  $f$  is the  $p$ th power of a univalent function in  $\mathcal{U}$ .*

If we take  $g(z) = f(z)$  in Theorem 3.1, we obtain a  $p$ -valence version of Beckers's [5] univalence criterion which was proved in [2].

**Corollary 3.3** *Let  $f \in \mathcal{A}_p^*$ . If*

$$\left| (1 - |z|^{2p}) \left[ 1 - p + \frac{zf''(z)}{f'(z)} \right] \right| \leq p$$

*for all  $z \in \mathcal{U}$ , then  $f$  is the  $p$ th power of a univalent function in  $\mathcal{U}$ .*

The following theorem contains another sufficient condition for analytic functions to be univalent in the open unit disk  $\mathcal{U}$ .

**Theorem 3.4** *Let  $F, G \in \mathcal{A}$ . If*

$$\begin{aligned} & \left| (1 - |z|^{2p}) \left[ \frac{zG''(z)}{G'(z)} + (p-1) \left( 1 - \frac{zG'(z)}{G(z)} \right) \right] \right. \\ & \quad \left. + \frac{(1 - |z|^{2p})^2}{2p} \frac{z^2}{|z|^{2p}} \left[ S_F(z) - S_G(z) - \frac{p^2 - 1}{2} \left( \left( \frac{F'(z)}{F(z)} \right)^2 - \left( \frac{G'(z)}{G(z)} \right)^2 \right) \right] \right| \\ & \leq p \end{aligned}$$

*for all  $z \in \mathcal{U}$ , then the function  $F$  is a univalent function in  $\mathcal{U}$ .*

*Proof* Let  $f(z) = [F(z)]^p$  and  $g(z) = [G(z)]^p$ . Thus we obtain

$$S_f(z) = S_F(z) - \left( \frac{F'(z)}{F(z)} \right)^2 \left( \frac{p^2 - 1}{2} \right).$$

It is easy to see that  $f$  and  $g$  satisfy the assumption of Theorem 3.1 if they satisfy the assumption of this theorem. Thus  $F$  is a univalent function in  $\mathcal{U}$  because  $f$  in view of Theorem 3.1 is the  $p$ th power of a univalent function.  $\square$

#### Competing interests

The author declares that they have no competing interests.

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#### References

1. Srivastava, HM, Owa, S (eds.): *Current Topics in Analytic Function Theory*. World Scientific, Singapore (1992)
2. Hallenbeck, DJ, Livingston, AE: Subordination chains and  $p$ -valent functions. Preprint (1975)
3. Pommerenke, C: Über die Subordination analytischer Funktionen. *J. Reine Angew. Math.* **218**, 159-173 (1965)
4. Pommerenke, C: *Univalent Functions*. Vandenhoeck Ruprecht, Göttingen (1975)
5. Becker, J: Löwnersche differentialgleichung und quasikonform fortsetzbare schlichte functionen. *J. Reine Angew. Math.* **255**, 23-43 (1972) (in German)
6. Ahlfors, LV: Sufficient conditions for quasiconformal extension. *Ann. Math. Stud.* **79**, 23-29 (1974)
7. Nehari, Z: The Schwarzian derivative and schlicht functions. *Bull. Am. Math. Soc.* **55**, 545-551 (1949)
8. Epstein, CL: Univalence criteria and surfaces in hyperbolic space. *J. Reine Angew. Math.* **380**, 196-214 (1987)
9. Pommerenke, C: On the Epstein univalence criterion. *Results Math.* **10**, 143-146 (1986)
10. Deniz, E, Orhan, H, Çağlar, M: Sufficient conditions for  $p$ -valence of an integral operator (submitted)
11. Deniz, E, Orhan, H: Some notes on extensions of basic univalence criteria. *J. Korean Math. Soc.* **48**(1), 179-189 (2011)
12. Deniz, E, Orhan, H: Univalence criterion for meromorphic functions and Loewner chains. *Appl. Math. Comput.* **218**(6), 751-755 (2011)
13. Deniz, E: Sufficient conditions for univalence and quasiconformal extensions of meromorphic functions. *Georgian Math. J.* **19**(4), 639-653 (2012)
14. Kanas, S, Lecko, A: Univalence criteria connected with arithmetic and geometric means, II. *Folia Sci. Univ. Tech. Resov.* **20**, 49-59 (1996)
15. Kanas, S, Lecko, A: Univalence criteria connected with arithmetic and geometric means, II. In: *Proceedings of the Second Int. Workshop of Transform Methods and Special Functions, Varna '96*, pp. 201-209. *Bulgar. Acad. Sci., Sofia* (1996)
16. Kanas, S, Srivastava, HM: Some criteria for univalence related to Ruscheweyh and Salagean derivatives. *Complex Var. Elliptic Equ.* **38**, 263-275 (1997)
17. Lewandowski, Z: On a univalence criterion. *Bull. Acad. Pol. Sci., Sér. Sci. Math.* **29**, 123-126 (1981)
18. Ovesea, H: A generalization of Ruscheweyh's univalence criterion. *J. Math. Anal. Appl.* **258**, 102-109 (2001)
19. Raducanu, D, Orhan, H, Deniz, E: On some sufficient conditions for univalence. *An. Univ. 'Ovidius' Constanța, Ser. Mat.* **18**(2), 217-222 (2010)
20. Ruscheweyh, S: An extension of Becker's univalence condition. *Math. Ann.* **220**, 285-290 (1976)

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