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Strong convergence theorems for modifying Halpern iterations for a totally quasi-ϕ-asymptotically nonexpansive multi-valued mapping in reflexive Banach spaces

Hong Bo Liu* and Yi Li

*Correspondence:
liuhongbo@swust.edu.cn
School of Science, Southwest University of Science and Technology, Mianyang, Sichuan 621010, P.R. China

Abstract
In this paper, we discuss an iterative sequence for a totally quasi-ϕ-asymptotically nonexpansive multi-valued mapping for modifying Halpern's iterations and establish some strong convergence theorems under certain conditions. We utilize the theorems to study a modified Halpern iterative algorithm for a system of equilibrium problems. The results improve and extend the corresponding results of Chang *et al.* (Appl. Math. Comput. 218:6489-6497, 2012).

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1 Introduction

Throughout this paper, we denote by N and R the sets of positive integers and real numbers, respectively. Let D be a nonempty closed subset of a real Banach space X. A mapping $T : D \to D$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in D$. Let $N(D)$ and $\mathrm{CB}(D)$ denote the family of nonempty subsets and nonempty bounded closed subsets of D, respectively. The Hausdorff metric on $\mathrm{CB}(D)$ is defined by

$$H(A_1, A_2) = \max\Big\{\sup_{x\in A_1} d(x, A_2), \sup_{y\in A_2} d(y, A_1)\Big\}$$

for $A_1, A_2 \in \mathrm{CB}(D)$, where $d(x, A_2) = \inf\{\|x - y\|, y \in A_2\}$. The multi-valued mapping $T : D \to \mathrm{CB}(D)$ is called nonexpansive if $H(Tx, Ty) \leq \|x - y\|$ for all $x, y \in D$. An element $p \in D$ is called a fixed point of $T : D \to \mathrm{CB}(D)$ if $p \in T(p)$. The set of fixed points of T is represented by $F(T)$.

In the sequel, denote $S(X) = \{x \in X : \|x\| = 1\}$. A Banach space X is said to be strictly convex if $\|\frac{x+y}{2}\| \leq 1$ for all $x, y \in S(X)$ and $x \neq y$. A Banach space is said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\} \subset S(X)$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 0$. The norm of the Banach space X is said to be Gâteaux differentiable if for each $x, y \in S(X)$, the limit

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \tag{1.1}$$

exists. In this case, X is said to be smooth. The norm of the Banach space X is said to be Fréchet differentiable if for each $x\in S(X)$, the limit (1.1) is attained uniformly for $y\in S(x)$, and the norm is uniformly Fréchet differentiable if the limit (1.1) is attained uniformly for $x,y\in S(X)$. In this case, X is said to be uniformly smooth.

Let X be a real Banach space with dual X^*. We denote by J the normalized duality mapping from X to 2^{X^*} which is defined by

$$J(x)=\left\{x^*\in X^*:\langle x,x^*\rangle=\|x\|^2=\left\|x^*\right\|^2\right\},\quad x\in X,$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing.

Remark 1.1 The following basic properties for the Banach space X and for the normalized duality mapping J can be found in Cioranescu [1].

(1) X (X^*, resp.) is uniformly convex if and only if X^* (X, resp.) is uniformly smooth.
(2) If X is smooth, then J is single-valued and norm-to-weak* continuous.
(3) If X is reflexive, then J is onto.
(4) If X is strictly convex, then $Jx\cap Jy\neq\Phi$ for all $x,y\in X$.
(5) If X has a Fréchet differentiable norm, then J is norm-to-norm continuous.
(6) If X is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of X.
(7) Each uniformly convex Banach space X has the Kadec-Klee property, *i.e.*, for any sequence $\{x_n\}\subset X$, if $x_n\rightharpoonup x\in X$ and $\|x_n\|\to\|x\|$, then $x_n\to x\in X$.

Next we assume that X is a smooth, strictly convex, and reflexive Banach space and D is a nonempty closed convex subset of X. In the sequel, we always use $\phi:X\times X\to R^+$ to denote the Lyapunov bifunction defined by

$$\phi(x,y)=\|x\|^2-2\langle x,Jy\rangle+\|y\|^2,\quad x,y\in X. \tag{1.2}$$

It is obvious from the definition of the function ϕ that

$$\big(\|x\|-\|y\|\big)^2\le\phi(x,y)\le\big(\|x\|+\|y\|\big)^2, \tag{1.3}$$

$$\phi(y,x)=\phi(y,z)+\phi(z,x)+2\langle z-y,Jx-Jz\rangle,\quad x,y,z\in X, \tag{1.4}$$

and

$$\phi\big(x,J^{-1}\big(\alpha Jy+(1-\alpha)Jz\big)\big)\le\alpha\phi(x,y)+(1-\alpha)\phi(x,z) \tag{1.5}$$

for all $\alpha\in[0,1]$ and $x,y,z\in X$.

Following Alber [2], the generalized projection $\Pi_D:X\to D$ is defined by

$$\Pi_D(x)=\arg\inf_{y\in D}\phi(y,x),\quad\forall x\in X.$$

Many problems in nonlinear analysis can be reformulated as a problem of finding a fixed point of a nonexpansive mapping.

Remark 1.2 (see [3]) Let Π_D be the generalized projection from a smooth, reflexive and strictly convex Banach space X onto a nonempty closed convex subset D of X, then Π_D is a closed and quasi-ϕ-nonexpansive from X onto D.

In 1953, Mann [4] introduced the following iterative sequence $\{x_n\}$:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n,$$

where the initial guess $x_1 \in D$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0,1]$. It is known that under appropriate settings the sequence $\{x_n\}$ converges weakly to a fixed point of T. However, even in a Hilbert space, the Mann iteration may fail to converge strongly [5]. Some attempts to construct an iteration method guaranteeing the strong convergence have been made. For example, Halpern [6] proposed the following so-called Halpern iteration:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n,$$

where $u, x_1 \in D$ are arbitrarily given and $\{\alpha_n\}$ is a real sequence in $[0,1]$. Another approach was proposed by Nakajo and Takahashi [7]. They generated a sequence as follows:

$$\begin{cases} x_1 \in X \quad \text{is arbitrary;} \\ y_n = \alpha_n u + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in D : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in D : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1 \quad (n = 1, 2, \ldots), \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$ and P_K denotes the metric projection from a Hilbert space H onto a closed convex subset K of H. It should be noted here that the iteration above works only in the Hilbert space setting. To extend this iteration to a Banach space, the concept of relatively nonexpansive mappings and quasi-ϕ-nonexpansive mappings have been introduced by Aoyama *et al.* [8], Chang *et al.* [9, 10], Chidume *et al.* [11], Matsushita *et al.* [12–14], Qin *et al.* [15], Song *et al.* [16], Wang *et al.* [17] and others.

Inspired by the work of Matsushita and Takahashi, in this paper, we introduce modifying Halpern-Mann iterations sequence for finding a fixed point of a multi-valued mapping $T : D \to \mathrm{CB}(D)$ and prove some strong convergence theorems. The results presented in the paper improve and extend the corresponding results in [9].

2 Preliminaries

In the sequel, we denote the strong convergence and weak convergence of the sequence $\{x_n\}$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Lemma 2.1 (see [2]) *Let X be a smooth, strictly convex and reflexive Banach space, and let D be a nonempty closed convex subset of X. Then the following conclusions hold*:

(a) $\phi(x,y) = 0$ *if and only if* $x = y$.
(b) $\phi(x,\Pi_D y) + \phi(\Pi_D y, y) \le \phi(x,y)$, $\forall x,y \in D$.
(c) *If* $x \in X$ *and* $z \in D$, *then* $z = \Pi_D x$ *if and only if* $\langle z - y, Jx - Jz\rangle \ge 0$, $\forall y \in D$.

Lemma 2.2 (see [9]) *Let X be a real uniformly smooth and strictly convex Banach space with the Kadec-Klee property, and let D be a nonempty closed convex subset of X. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in D such that $x_n \to p$ and $\phi(x_n,y_n) \to 0$, where ϕ is the function defined by* (1.2), *then $y_n \to p$.*

Definition 2.1 A point $p \in D$ is said to be an asymptotic fixed point of a multi-valued mapping $T : D \to \mathrm{CB}(D)$ if there exists a sequence $\{x_n\} \subset D$ such that $x_n \rightharpoonup x \in X$ and $d(x_n, T(x_n)) \to 0$. Denote the set of all asymptotic fixed points of T by $\hat{F}(T)$.

Definition 2.2
(1) A multi-valued mapping $T : D \to \mathrm{CB}(D)$ is said to be relatively nonexpansive if $F(T) \neq \Phi$, $\hat{F}(T) = F(T)$ and $\phi(p,z) \le \phi(p,x)$, $\forall x \in D, p \in F(T), z \in T(x)$.
(2) A multi-valued mapping $T : D \to \mathrm{CB}(D)$ is said to be closed if for any sequence $\{x_n\} \subset D$ with $x_n \to x \in X$ and $d(y, T(x_n)) \to 0$, then $d(y, T(x)) = 0$.

Remark 2.1 If H is a real Hilbert space, then $\phi(x,y) = \|x-y\|^2$ and Π_D is the metric projection P_D of H onto D.

Next, we present an example of a relatively nonexpansive multi-valued mapping.

Example 2.1 (see [18]) Let X be a smooth, strictly convex and reflexive Banach space, let D be a nonempty closed and convex subset of X, and let $f : D \times D \to R$ be a bifunction satisfying the conditions: (A1) $f(x,x) = 0$, $\forall x \in D$; (A2) $f(x,y) + f(y,x) \le 0$, $\forall x,y \in D$; (A3) for each $x,y,z \in D$, $\lim_{t\to 0} f(tz + (1-t)x, y) \le f(x,y)$; (A4) for each given $x \in D$, the function $y \longmapsto f(x,y)$ is convex and lower semicontinuous. The so-called equilibrium problem for f is to find an $x^* \in D$ such that $f(x^*,y) \ge 0$, $\forall y \in D$. The set of its solutions is denoted by $\mathrm{EP}(f)$.

Let $r > 0$, $x \in D$ and define a multi-valued mapping $T_r : D \to N(D)$ as follows:

$$T_r(x) = \left\{ z \in D, f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \forall y \in D \right\}, \quad \forall x \in D, \tag{2.1}$$

then (1) T_r is single-valued, and so $\{z\} = T_r(x)$; (2) T_r is a relatively nonexpansive mapping, therefore, it is a closed quasi-ϕ-nonexpansive mapping; (3) $F(T_r) = \mathrm{EP}(f)$.

Definition 2.3
(1) A multi-valued mapping $T : D \to \mathrm{CB}(D)$ is said to be quasi-ϕ-nonexpansive if $F(T) \neq \Phi$ and $\phi(p,z) \le \phi(p,x)$, $\forall x \in D, p \in F(T), z \in Tx$.
(2) A multi-valued mapping $T : D \to \mathrm{CB}(D)$ is said to be quasi-ϕ-asymptotically nonexpansive if $F(T) \neq \Phi$ and there exists a real sequence $k_n \subset [1,+\infty)$, $k_n \to 1$, such that

$$\phi(p,z_n) \le k_n \phi(p,x), \quad \forall x \in D, p \in F(T), z_n \in T^n x. \tag{2.2}$$

(3) A multi-valued mapping $T: D \to \mathrm{CB}(D)$ is said to be totally quasi-ϕ-asymptotically nonexpansive if $F(T) \neq \Phi$ and there exist nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n, \mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta: R^+ \to R^+$ with $\zeta(0) = 0$ such that

$$\begin{aligned} &\phi(p,z_n) \le \phi(p,x) + \nu_n\zeta\big[\phi(p,x)\big] + \mu_n, \\ &\quad \forall x \in D, \forall n \ge 1, p \in F(T), z_n \in T^n x. \end{aligned} \tag{2.3}$$

Remark 2.2 From the definitions, it is obvious that a relatively nonexpansive multi-valued mapping is a quasi-ϕ-nonexpansive multi-valued mapping, and a quasi-ϕ-nonexpansive multi-valued mapping is a quasi-ϕ-asymptotically nonexpansive multi-valued mapping, and a quasi-ϕ-asymptotically nonexpansive multi-valued mapping is a total quasi-ϕ-asymptotically nonexpansive multi-valued mapping, but the converse is not true.

Lemma 2.3 *Let X and D be as in Lemma* 2.2. *Let $T: D \to \mathrm{CB}(D)$ be a closed and totally quasi-ϕ-asymptotically nonexpansive multi-valued mapping with nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ and a strictly increasing continuous function $\zeta: R^+ \to R^+$ with $\zeta(0) = 0$. If $\nu_n, \mu_n \to 0$ (as $n \to \infty$) and $\mu_1 = 0$, then $F(T)$ is a closed and convex subset of D.*

Proof Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \to x^*$. Since T is a totally quasi-ϕ-asymptotically nonexpansive multi-valued mapping, we have

$$\phi(x_n, z) \le \phi\big(x_n, x^*\big) + \nu_1\zeta\big[\phi\big(x_n, x^*\big)\big]$$

for all $z \in Tx^*$ and for all $n \in N$. Therefore,

$$\phi\big(x^*, z\big) = \lim_{n\to\infty} \phi(x_n, z) \le \lim_{n\to\infty} \big\{\phi\big(x_n, x^*\big) + \nu_1\zeta\big[\phi\big(x_n, x^*\big)\big]\big\} = \phi\big(x^*, x^*\big) = 0.$$

By Lemma 2.1(a), we obtain $z = x^*$. Hence, $Tx^* = \{x^*\}$. So, we have $x^* \in F(T)$. This implies $F(T)$ is closed.

Let $p, q \in F(T)$ and $t \in (0,1)$, and put $w = tp + (1-t)q$. Next we prove that $w \in F(T)$. Indeed, in view of the definition of ϕ, letting $z_n \in T^n w$, we have

$$\begin{aligned} \phi(w, z_n) &= \|w\|^2 - 2\langle w, Jz_n\rangle + \|z_n\|^2 \\ &= \|w\|^2 - 2\big\langle tp + (1-t)q, Jz_n\big\rangle + \|z_n\|^2 \\ &= \|w\|^2 + t\phi(p, z_n) + (1-t)\phi(q, z_n) - t\|p\|^2 - (1-t)\|q\|^2. \end{aligned} \tag{2.4}$$

Since

$$\begin{aligned} &t\phi(p, z_n) + (1-t)\phi(q, z_n) \\ &\quad \le t\big[\phi(p,w) + \nu_n\zeta\big[\phi(p,w)\big] + \mu_n\big] + (1-t)\big[\phi(q,w) + \nu_n\zeta\big[\phi(q,w)\big] + \mu_n\big] \\ &\quad = t\big\{\|p\|^2 - 2\langle p, Jw\rangle + \|w\|^2 + \nu_n\zeta\big[\phi(p,w)\big] + \mu_n\big\} \\ &\qquad + (1-t)\big\{\|q\|^2 - 2\langle q, Jw\rangle + \|w\|^2 + \nu_n\zeta\big[\phi(q,w)\big] + \mu_n\big\} \\ &\quad = t\|p\|^2 + (1-t)\|q\|^2 - \|w\|^2 + t\nu_n\zeta\big[\phi(p,w)\big] + (1-t)\nu_n\zeta\big[\phi(q,w)\big] + \mu_n. \end{aligned} \tag{2.5}$$

Substituting (2.4) into (2.5) and simplifying it, we have

$$\phi(w,z_n) \le t\nu_n\zeta\big[\phi(p,w)\big] + (1-t)\nu_n\zeta\big[\phi(q,w)\big] + \mu_n \to 0 \quad (\text{as } n\to\infty).$$

By Lemma 2.2, we have $z_n \to w$. This implies that z_{n+1} ($\in TT^n w$) $\to w$. Since T is closed, we have $Tw = \{w\}$, *i.e.*, $w \in F(T)$. This completes the proof of Lemma 2.3. □

Definition 2.4 A mapping $T : D \to \mathrm{CB}(D)$ is said to be uniformly L-Lipschitz continuous if there exists a constant $L > 0$ such that $\|x_n - y_n\| \le L\|x - y\|$, where $x, y \in D$, $x_n \in T^n x$, $y_n \in T^n y$.

3 Main results

Theorem 3.1 *Let X be a real uniformly smooth and strictly convex Banach space with the Kadec-Klee property, let D be a nonempty closed convex subset of X, and let $T : D \to \mathrm{CB}(D)$ be a closed and uniformly L-Lipschitz continuous totally quasi-ϕ-asymptotically nonexpansive multi-valued mapping with nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$, $\nu_n, \mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta : R^+ \to R^+$ with $\zeta(0) = 0$ satisfying condition* (2.3). *Let $\{\alpha_n\}$ be a sequence in* $[0,1]$ *such that $\alpha_n \to 0$. If $\{x_n\}$ is the sequence generated by*

$$\begin{cases} x_1 \in X \quad \text{is arbitrary}; \qquad D_1 = D, \\ y_n = J^{-1}[\alpha_n Jx_1 + (1-\alpha_n)Jz_n], \quad z_n \in T^n x_n, \\ D_{n+1} = \{z \in D_n : \phi(z,y_n) \le \alpha_n\phi(z,x_1) + (1-\alpha_n)\phi(z,x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{D_{n+1}} x_1 \quad (n = 1,2,\ldots), \end{cases} \tag{3.1}$$

where $\xi_n = \nu_n \sup_{p\in F(T)} \zeta[\phi(p,x_n)] + \mu_n$, $F(T)$ is the fixed point set of T, and $\Pi_{D_{n+1}}$ is the generalized projection of X onto D_{n+1}. If $F(T)$ is nonempty and $\mu_1 = 0$, then $\lim_{n\to\infty} x_n = \Pi_{F(T)}x_1$.

Proof (I) First, we prove that D_n is a closed and convex subset in D.

By the assumption, $D_1 = D$ is closed and convex. Suppose that D_n is closed and convex for some $n \ge 1$. In view of the definition of ϕ, we have

$$\begin{aligned} D_{n+1} &= \big\{z \in D_n : \phi(z,y_n) \le \alpha_n\phi(z,x_1) + (1-\alpha_n)\phi(z,x_n) + \xi_n\big\} \\ &= \big\{z \in D : \phi(z,y_n) \le \alpha_n\phi(z,x_1) + (1-\alpha_n)\phi(z,x_n) + \xi_n\big\} \cap D_n \\ &= \big\{z \in D : 2\alpha_n\langle z, Jx_1\rangle + 2(1-\alpha_n)\langle z, Jx_n\rangle - 2\langle z, Jy_n\rangle \\ &\quad \le \alpha_n\|x_1\|^2 + (1-\alpha_n)\|x_n\|^2 - \|y_n\|^2\big\} \cap D_n. \end{aligned}$$

This shows that D_{n+1} is closed and convex. The conclusions are proved.

(II) Next, we prove that $F(T) \subset D_n$ for all $n \ge 1$.

In fact, it is obvious that $F(T) \subset D_1$. Suppose that $F(T) \subset D_n$. Hence, for any $u \in F(T) \subset D_n$, by (1.5), we have

$$\begin{aligned} \phi(u,y_n) &= \phi\big(u, J^{-1}\big(\alpha_n Jx_1 + (1-\alpha_n)Jz_n\big)\big) \\ &\le \alpha_n\phi(u,x_1) + (1-\alpha_n)\phi(u,z_n) \end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n\phi(u,x_1) + (1-\alpha_n)\bigl\{\phi(u,x_n) + \nu_n\zeta\bigl[\phi(u,x_n)\bigr] + \mu_n\bigr\} \\
&\leq \alpha_n\phi(u,x_1) + (1-\alpha_n)\Bigl\{\phi(u,x_n) + \nu_n \sup_{p\in F(T)}\zeta\bigl[\phi(p,x_n)\bigr] + \mu_n\Bigr\} \\
&= \alpha_n\phi(u,x_1) + (1-\alpha_n)\phi(u,x_n) + \xi_n.
\end{aligned} \tag{3.2}$$

This shows that $u \in F(T) \subset D_{n+1}$, and so $F(T) \subset D_n$.

(III) Now we prove that $\{x_n\}$ converges strongly to some point p^*.

In fact, since $x_n = \Pi_{D_n}x_1$, from Lemma 2.1(c), we have

$$\langle x_n - y, Jx_1 - Jx_n\rangle \geq 0, \quad \forall y \in D_n.$$

Again since $F(T) \subset D_n$, we have

$$\langle x_n - u, Jx_1 - Jx_n\rangle \geq 0, \quad \forall u \in F(T).$$

It follows from Lemma 2.1(b) that for each $u \in F(T)$ and for each $n \geq 1$,

$$\phi(x_n,x_1) = \phi(\Pi_{D_n}x_1,x_1) \leq \phi(u,x_1) - \phi(u,x_n) \leq \phi(u,x_1). \tag{3.3}$$

Therefore, $\{\phi(x_n,x_1)\}$ is bounded and so is $\{x_n\}$. Since $x_n = \Pi_{D_n}x_1$ and $x_{n+1} = \Pi_{D_{n+1}}x_1 \in D_{n+1} \subset D_n$, we have $\phi(x_n,x_1) \leq \phi(x_{n+1},x_1)$. This implies that $\{\phi(x_n,x_1)\}$ is nondecreasing. Hence $\lim_{n\to\infty}\phi(x_n,x_1)$ exists. Since X is reflexive, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup p^*$ (some point in $D = D_1$). Since D_n is closed and convex and $D_{n+1} \subset D_n$. This implies that D_n is weakly closed and $p^* \in D_n$ for each $n \geq 1$. In view of $x_{n_i} = \Pi_{D_{n_i}}x_1$, we have

$$\phi(x_{n_i},x_1) \leq \phi\bigl(p^*,x_1\bigr), \quad \forall n_i \geq 1.$$

Since the norm $\|\cdot\|$ is weakly lower semi-continuous, we have

$$\begin{aligned}
\lim_{n_i\to\infty}\inf\phi(x_n,x_1) &= \lim_{n_i\to\infty}\inf\bigl(\|x_{n_i}\|^2 - 2\langle x_{n_i},Jx_1\rangle + \|x_1\|^2\bigr) \\
&\geq \bigl\|p^*\bigr\|^2 - 2\bigl\langle p^*,Jx_1\bigr\rangle + \|x_1\|^2 \\
&= \phi\bigl(p^*,x_1\bigr),
\end{aligned}$$

and so

$$\begin{aligned}
\phi\bigl(p^*,x_1\bigr) &\leq \lim_{n_i\to\infty}\inf\phi(x_n,x_1) \\
&\leq \lim_{n_i\to\infty}\sup\phi(x_n,x_1) = \phi\bigl(p^*,x_1\bigr).
\end{aligned}$$

This shows that $\lim_{n_i\to\infty}\phi(x_{n_i},x_1) = \phi(p^*,x_1)$, and we have $\|x_{n_i}\| \to \|p^*\|$. Since $x_{n_i} \rightharpoonup p^*$, by virtue of the Kadec-Klee property of X, we obtain that $x_{n_i} \to p^*$. Since $\{\phi(x_n,x_1)\}$ is convergent, this together with $\lim_{n_i\to\infty}\phi(x_{n_i},x_1) = \phi(p^*,x_1)$ shows that $\lim_{n_i\to\infty}\phi(x_n,x_1) =$

$\phi(p^*, x_1)$. If there exists some subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \to q$, then from Lemma 2.1 we have

$$\begin{aligned}\phi(p^*, q) &= \lim_{n_i, n_j \to \infty} \phi(x_{n_i}, x_{n_j}) = \lim_{n_i, n_j \to \infty} \phi(x_{n_i}, \Pi_{D_{n_j}} x_1) \\ &\le \lim_{n_i, n_j \to \infty} \left[\phi(x_{n_i}, x_1) - \phi(\Pi_{D_{n_j}} x_1, x_1)\right] = \lim_{n_i, n_j \to \infty} \left[\phi(x_{n_i}, x_1) - \phi(x_{n_j}, x_1)\right] \\ &= \phi(p^*, x_1) - \phi(p^*, x_1) = 0,\end{aligned}$$

i.e., $p^* = q$, and hence

$$x_n \to p^*. \tag{3.4}$$

By the way, from (3.4), it is easy to see that

$$\xi_n = \nu_n \sup_{p \in F(T)} \zeta\left[\phi(p, x_n)\right] + \mu_n \to 0. \tag{3.5}$$

(IV) Now we prove that $p^* \in F(T)$.

In fact, since $x_{n+1} \in D_{n+1}$, from (3.1), (3.4) and (3.5), we have

$$\phi(x_{n+1}, y_n) \le \alpha_n \phi(x_{n+1}, x_1) + (1 - \alpha_n)\phi(x_{n+1}, x_n) + \xi_n \to 0. \tag{3.6}$$

Since $x_n \to p^*$, it follows from (3.6) and Lemma 2.2 that

$$y_n \to p^*. \tag{3.7}$$

Since $\{x_n\}$ is bounded and T is a totally quasi-ϕ-asymptotically nonexpansive multi-valued mapping, $T^n x_n$ is bounded. In view of $\alpha_n \to 0$, from (3.1), we have

$$\lim_{n\to\infty} \|Jy_n - Jz_n\| = \lim_{n\to\infty} \alpha_n \|Jx_1 - Jz_n\| = 0. \tag{3.8}$$

Since $Jy_n \to Jp^*$, this implies $Jz_n \to Jp^*$. From Remark 1.1, it yields that

$$z_n \rightharpoonup p^*. \tag{3.9}$$

Again, since

$$\|z_n\| - \|p^*\| = \|Jz_n\| - \|Jp^*\| \le \|Jz_n - Jp^*\| \to 0, \tag{3.10}$$

this together with (3.9) and the Kadec-Klee-property of X shows that

$$z_n \to p^*. \tag{3.11}$$

On the other hand, by the assumption that T is L-Lipschitz continuous, we have

$$\begin{aligned}d(Tz_n, z_n) &\le d(Tz_n, z_{n+1}) + \|z_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - z_n\| \\ &\le (L+1)\|x_{n+1} - x_n\| + \|z_{n+1} - x_{n+1}\| + \|x_n - z_n\|.\end{aligned} \tag{3.12}$$

From (3.11) and $x_n \to p^*$, we have that $d(Tz_n, z_n) \to 0$. In view of the closedness of T, it yields that $T(p^*) = \{p^*\}$, which implies that $p^* \in F(T)$.

(V) Finally, we prove that $p^* = \Pi_{F(T)}x_1$ and so $x_n \to \Pi_{F(T)}x_1$.

Let $w = \Pi_{F(T)}x_1$. Since $w \in F(T) \subset D_n$, we have $\phi(p^*, x_1) \le \phi(w, x_1)$. This implies that

$$\phi(p^*, x_1) = \lim_{n\to\infty} \phi(x_n, x_1) \le \phi(w, x_1), \tag{3.13}$$

which yields that $p^* = w = \Pi_{F(T)}x_1$. Therefore, $x_n \to \Pi_{F(T)}x_1$. The proof of Theorem 3.1 is completed. □

By Remark 2.2, the following corollaries are obtained.

Corollary 3.1 *Let X and D be as in Theorem* 3.1, *and let $T : D \to \mathrm{CB}(D)$ be a closed and uniformly L-Lipschitz continuous relatively nonexpansive multi-valued mapping. Let $\{\alpha_n\}$ in* $(0,1)$ *with* $\lim_{n\to\infty} \alpha_n = 0$. *Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 \in X \quad \textit{is arbitrary}; \qquad D_1 = D, \\ y_n = J^{-1}[\alpha_n Jx_1 + (1-\alpha_n)Jz_n], \quad z_n \in Tx_n, \\ D_{n+1} = \{z \in D_n : \phi(z, y_n) \le \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \Pi_{D_{n+1}}x_1 \quad (n = 1, 2, \ldots), \end{cases} \tag{3.14}$$

where $F(T)$ is the set of fixed points of T, and $\Pi_{D_{n+1}}$ is the generalized projection of X onto D_{n+1}, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_1$.

Corollary 3.2 *Let X and D be as in Theorem* 3.1, *and $T : D \to \mathrm{CB}(D)$ be a closed and uniformly L-Lipschitz continuous quasi-ϕ-nonexpansive multi-valued mapping. Let $\{\alpha_n\}$ be a sequence of real numbers such that $\alpha_n \in (0,1)$ for all $n \in N$ and satisfy* $\lim_{n\to\infty} \alpha_n = 0$. *Let $\{x_n\}$ be the sequence generated by* (3.14). *Then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_1$.*

Corollary 3.3 *Let X be a real uniformly smooth and strictly convex Banach space with the Kadec-Klee property, let D be a nonempty closed convex subset of X, and let $T : D \to \mathrm{CB}(D)$ be a closed and uniformly L-Lipschitz continuous quasi-ϕ-asymptotically nonexpansive multi-valued mapping with nonnegative real sequences $\{k_n\} \subset [1, +\infty)$ and $k_n \to 1$ satisfying condition* (2.2). *Let $\{\alpha_n\}$ be a sequence in* $(0,1)$ *and satisfy* $\lim_{n\to\infty} \alpha_n = 0$. *If $\{x_n\}$ is the sequence generated by*

$$\begin{cases} x_1 \in X \quad \textit{is arbitrary}; \qquad D_1 = D, \\ y_n = J^{-1}[\alpha_n Jx_1 + (1-\alpha_n)Jz_n], \quad z_n \in T^n x_n, \\ D_{n+1} = \{z \in D_n : \phi(z, y_n) \le \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{D_{n+1}}x_1 \quad (n = 1, 2, \ldots), \end{cases} \tag{3.15}$$

where $\xi_n = (k_n - 1)\sup_{p\in F(T)} \phi(p, x_n)$, $F(T)$ is the fixed point set of T, and $\Pi_{D_{n+1}}$ is the generalized projection of X onto D_{n+1}, if $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_1$.

4 Application

We utilize Corollary 3.2 to study a modified Halpern iterative algorithm for a system of equilibrium problems.

Theorem 4.1 *Let D, X and $\{\alpha_n\}$ be the same as in Theorem* 3.1. *Let $f : D \times D \to R$ be a bifunction satisfying conditions* (A1)-(A4) *as given in Example* 2.1. *Let $T_r : X \to D$ be a mapping defined by* (2.1), *i.e.*,

$$T_r(x) = \left\{ x \in D, f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in D \right\}, \quad \forall x \in X.$$

Let $\{x_n\}$ be the sequence generated by

$$\begin{cases} x_1 \in X \quad \textit{is arbitrary}; \qquad D_1 = D, \\ f(u_n, y) + \frac{1}{r}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0, \quad \forall y \in D, r > 0, u_n \in T_r x_n, \\ y_n = J^{-1}[\alpha_n Jx_1 + (1-\alpha_n)Ju_n], \\ D_{n+1} = \{z \in D_n : \phi(z, y_n) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \prod_{D_{n+1}} x_1 \quad (n = 1, 2, \ldots). \end{cases} \tag{4.1}$$

If $F(T_r) \neq \Phi$, then $\{x_n\}$ converges strongly to $\prod_{F(T)} x_1$, which is a common solution of the system of equilibrium problems for f.

Proof In Example 2.1, we have pointed out that $u_n = T_r(x_n)$, $F(T_r) = \mathrm{EP}(f)$ and T_r is a closed quasi-ϕ-nonexpansive mapping. Hence (4.1) can be rewritten as follows:

$$\begin{cases} x_1 \in X \quad \text{is arbitrary}; \qquad D_1 = D, \\ y_n = J^{-1}[\alpha_n Jx_1 + (1-\alpha_n)Ju_n], \quad u_n \in T_r x_n, \\ D_{n+1} = \{z \in D_n : \phi(z, y_n) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \prod_{D_{n+1}} x_1 \quad (n = 1, 2, \ldots). \end{cases} \tag{4.2}$$

Therefore the conclusion of Theorem 4.1 can be obtained from Corollary 3.2. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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