Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH | Open Access

C^*-ternary 3-derivations on C^*-ternary algebras

Mehdi Dehghanian[1], Seyed Mohammad Sadegh Modarres Mosadegh[2], Choonkil Park[3] and Dong Yun Shin[4*]

*Correspondence: dyshin@uos.ac.kr
[4]Department of Mathematics, University of Seoul, Seoul, 130-743, Korea
Full list of author information is available at the end of the article

Abstract

In this paper, we prove the Hyers-Ulam stability of C^*-ternary 3-derivations and of C^*-ternary 3-homomorphisms for the functional equation

$$f(x_1 + x_2, y_1 + y_2, z_1 + z_2) = \sum_{1\leq i,j,k\leq 2} f(x_i, y_j, z_k)$$

in C^*-ternary algebras.

MSC: Primary 17A40; 39B52; 46Lxx; 46K70; 46L05; 46B99

Keywords: Hyers-Ulam stability; 3-additive mapping; C^*-ternary algebra; C^*-ternary 3-derivation; C^*-ternary 3-homomorphism

1 Introduction and preliminaries

Ternary algebraic operations were considered in the nineteenth century by several mathematicians such as Cayley [1] who introduced the notion of a cubic matrix, which in turn was generalized by Kapranov, Gelfand and Zelevinskii [2]. The simplest example of such non-trivial ternary operation is given by the following composition rule:

$$\{a, b, c\}_{ijk} = \sum_{1\leq l,m,n\leq N} a_{nil} b_{ljm} c_{mkn} \quad (i, j, k = 1, 2, \ldots, N).$$

Ternary structures and their generalization, the so-called n-ary structures, raise certain hopes in view of their applications in physics. Some significant physical applications are as follows (see [3, 4]).

(1) The algebra of nonions generated by two matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \omega^2 & 0 & 0 \end{pmatrix} \quad \left(\omega = e^{\frac{2\pi i}{3}}\right)$$

was introduced by Sylvester as a ternary analog of Hamiltons quaternions (*cf.* [5]).

(2) The quark model inspired a particular brand of ternary algebraic systems. The so-called Nambu mechanics is based on such structures (see [6]).

There are also some applications, although still hypothetical, in the fractional quantum Hall effect, the nonstandard statistics, supersymmetric theory, and Yang-Baxter equation (*cf.* [3, 5, 7]).

A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product $(x,y,z) \to [x,y,z]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[x,y,[z,w,v]] = [x,[w,z,y],v] = [[x,y,z],w,v]$, and satisfies $\|[x,y,z]\| \leq \|x\| \cdot \|y\| \cdot \|z\|$ and $\|[x,x,x]\| = \|x\|^3$ (see [8]). Every left Hilbert C^*-module is a C^*-ternary algebra via the ternary product $[x,y,z] := \langle x,y\rangle z$.

If a C^*-ternary algebra $(A,[\cdot,\cdot,\cdot])$ has an identity, *i.e.*, an element $e \in A$ such that $x = [x,e,e] = [e,e,x]$ for all $x \in A$, then it is routine to verify that A, endowed with $x \circ y := [x,e,y]$ and $x^* := [e,x,e]$, is a unital C^*-algebra. Conversely, if $(A,\circ)$ is a unital C^*-algebra, then $[x,y,z] := x \circ y^* \circ z$ makes A into a C^*-ternary algebra.

Throughout this paper, assume that C^*-ternary algebras A and B are induced by unital C^*-algebras with units e and e', respectively.

A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *C^*-ternary homomorphism* if $H([x,y,z]) = [H(x),H(y),H(z)]$ for all $x,y,z \in A$. If, in addition, the mapping H is bijective, then the mapping $H : A \to B$ is called a C^*-ternary algebra isomorphism. A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *C^*-ternary derivation* if

$$\delta\big([x,y,z]\big) = \big[\delta(x),y,z\big] + \big[x,\delta(y),z\big] + \big[x,y,\delta(z)\big]$$

for all $x,y,z \in A$ (see [9]).

In 1940, Ulam [10] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms:

We are given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy),f(x)f(y)) < \delta$ for all $x,y \in G$, then a homomorphism $h : G \to G$ exists with $\rho(f(x),h(x)) < \epsilon$ for all $x \in G$?

In 1941, Hyers [11] gave the first partial solution to Ulam's question for the case of approximate additive mappings under the assumption that G and G' are Banach spaces. Then, Aoki [12] and Bourgin [13] considered the stability problem with unbounded Cauchy differences. In 1978, Rassias [14] generalized the theorem of Hyers [11] by considering the stability problem with unbounded Cauchy differences. In 1991, Gajda [15], following the same approach as that by Rassias [14], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [15] as well as by Rassias and Šemrl [16], that one cannot prove a Rassias-type theorem when $p = 1$. Găvruta [17] obtained the generalized result of the Rassias theorem which allows the Cauchy difference to be controlled by a general unbounded function. During the last two decades, a number of articles and research monographs have been published on various generalizations and applications of the Hyers-Ulam stability to a number of functional equations and mappings, for example, Cauchy-Jensen mappings, k-additive mappings, invariant means, multiplicative mappings, bounded nth differences, convex functions, generalized orthogonality mappings, Euler-Lagrange functional equations, differential equations, and Navier-Stokes equations. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results, containing ternary homomorphisms and ternary derivations, concerning this problem (see [18–31]).

Let X and Y be complex vector spaces. A mapping $f : X \times X \times X \to Y$ is called a 3-*additive mapping* if f is additive for each variable, and a mapping $f : X \times X \times X \to Y$ is called a 3-$\mathbb{C}$-*linear mapping* if f is $\mathbb{C}$-linear for each variable.

A 3-$\mathbb{C}$-linear mapping $H : A \times A \times A \to B$ is called a C^*-*ternary* 3-*homomorphism* if it satisfies

$$H\big([x_1,y_1,z_1],[x_2,y_2,z_2],[x_3,y_3,z_3]\big) = \big[H(x_1,x_2,x_3),H(y_1,y_2,y_3),H(z_1,z_2,z_3)\big]$$

for all $x_1,y_1,z_1,x_2,y_2,z_2,x_3,y_3,z_3 \in A$.

For a given mapping $f : A^3 \to B$, we define

$$\begin{aligned} &D_{\lambda,\mu,\nu}f(x_1,x_2,y_1,y_2,z_1,z_2) \\ &\quad := f(\lambda x_1 + \lambda x_2, \mu y_1 + \mu y_2, \nu z_1 + \nu z_2) - \lambda\mu\nu \sum_{1\le i,j,k\le 2} f(x_i,y_j,z_k) \end{aligned}$$

for all $\lambda,\mu,\nu \in S^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x_1,x_2,y_1,y_2,z_1,z_2 \in A$.

Bae and Park [32] proved the Hyers-Ulam stability of 3-homomorphisms in C^*-ternary algebras for the functional equation

$$D_{\lambda,\mu,\nu}f(x_1,x_2,y_1,y_2,z_1,z_2) = 0.$$

Lemma 1.1 [32] *Let X and Y be complex vector spaces, and let $f : X \times X \times X \to Y$ be a 3-additive mapping such that $f(\lambda x,\mu y,\nu z) = \lambda\mu\nu f(x,y,z)$ for all $\lambda,\mu,\nu \in S^1$ and all $x,y,z \in X$. Then f is 3-$\mathbb{C}$-linear.*

Theorem 1.2 [32] *Let $p,q,r \in (0,\infty)$ with $p+q+r < 3$ and $\theta \in (0,\infty)$, and let $f : A^3 \to B$ be a mapping such that*

$$\begin{aligned} &\big\|D_{\lambda,\mu,\nu}f(x_1,x_2,y_1,y_2,z_1,z_2)\big\| \\ &\quad \le \theta \cdot \max\big\{\|x_1\|,\|x_2\|\big\}^p \cdot \max\big\{\|y_1\|,\|y_2\|\big\}^q \cdot \max\big\{\|z_1\|,\|z_2\|\big\}^r \end{aligned} \tag{1}$$

and

$$\begin{aligned} &\big\|f\big([x_1,y_1,z_1],[x_2,y_2,z_2],[x_3,y_3,z_3]\big) - \big[f(x_1,x_2,x_3),f(y_1,y_2,y_3),f(z_1,z_2,z_3)\big]\big\| \\ &\quad \le \theta \sum_{i=1}^{3} \|x_i\|^p \cdot \|y_i\|^q \cdot \|z_i\|^r \end{aligned} \tag{2}$$

for all $\lambda,\mu,\nu \in S^1$ and all $x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3 \in A$. Then there exists a unique C^*-ternary 3-homomorphism $H : A^3 \to B$ such that

$$\big\|f(x,y,z) - H(x,y,z)\big\| \le \frac{\theta}{2^3 - 2^{p+q+r}} \|x\|^p \cdot \|y\|^q \cdot \|z\|^r \tag{3}$$

for all $x,y,z \in A$.

2 C^*-ternary 3-homomorphisms in C^*-ternary algebras

Theorem 2.1 *Let $p,q,r \in (0,\infty)$ with $p+q+r<3$ and $\theta \in (0,\infty)$, and let $f : A^3 \to B$ be a mapping satisfying* (1) *and* (2). *If there exists an $(x_0,y_0,z_0) \in A^3$ such that $\lim_{n\to\infty} \frac{1}{8^n} f(2^n x_0, 2^n y_0, 2^n z_0) = e'$, then the mapping f is a C^*-ternary 3-homomorphism.*

Proof By Theorem 1.2, there exists a unique C^*-ternary 3-homomorphism $H : A^3 \to B$ satisfying (3). Note that

$$H(x,y,z) := \lim_{n\to\infty} \frac{1}{8^n} f\bigl(2^n x, 2^n y, 2^n z\bigr)$$

for all $x,y,z \in A$. By the assumption, we get that

$$H(x_0,y_0,z_0) = \lim_{n\to\infty} \frac{1}{8^n} f\bigl(2^n x_0, 2^n y_0, 2^n z_0\bigr) = e'.$$

It follows from (2) that

$$\begin{aligned}
&\bigl\|\bigl[H(x_1,x_2,x_3),H(y_1,y_2,y_3),H(z_1,z_2,z_3)\bigr] - \bigl[H(x_1,x_2,x_3),H(y_1,y_2,y_3),f(z_1,z_2,z_3)\bigr]\bigr\| \\
&\quad = \bigl\|H\bigl([x_1,y_1,z_1],[x_2,y_2,z_2],[x_3,y_3,z_3]\bigr) \\
&\qquad - \bigl[H(x_1,x_2,x_3),H(y_1,y_2,y_3),f(z_1,z_2,z_3)\bigr]\bigr\| \\
&\quad = \lim_{n\to\infty} \frac{1}{8^{2n}} \bigl\|f\bigl(\bigl[2^n x_1, 2^n y_1, z_1\bigr],\bigl[2^n x_2, 2^n y_2, z_2\bigr],\bigl[2^n x_3, 2^n y_3, z_3\bigr]\bigr) \\
&\qquad - \bigl[f\bigl(2^n x_1, 2^n x_2, 2^n x_3\bigr), f\bigl(2^n y_1, 2^n y_2, 2^n y_3\bigr), f(z_1,z_2,z_3)\bigr]\bigr\| \\
&\quad \le \lim_{n\to\infty} \frac{\theta 2^{n(p+q)}}{8^{2n}} \sum_{i=1}^{3} \|x_i\|^p \cdot \|y_i\|^q \cdot \|z_i\|^r = 0
\end{aligned}$$

for all $x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3 \in A$. So,

$$\bigl[H(x_1,x_2,x_3),H(y_1,y_2,y_3),H(z_1,z_2,z_3)\bigr] = \bigl[H(x_1,x_2,x_3),H(y_1,y_2,y_3),f(z_1,z_2,z_3)\bigr]$$

for all $x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3 \in A$. Letting $x_1 = y_1 = x_0$, $x_2 = y_2 = y_0$ and $x_3 = y_3 = z_0$ in the last equality, we get $f(z_1,z_2,z_3) = H(z_1,z_2,z_3)$ for all $z_1,z_2,z_3 \in A$. Therefore, the mapping f is a C^*-ternary 3-homomorphism. □

Theorem 2.2 *Let $p_i,q_i,r_i \in (0,\infty)$ $(i=1,2,3)$ such that $p_i \neq 1$ or $q_i \neq 1$ or $r_i \neq 1$ for some $1 \le i \le 3$ and $\theta,\eta \in (0,\infty)$, and let $f : A^3 \to B$ be a mapping such that*

$$\begin{aligned}
&\bigl\|D_{\lambda,\mu,\nu} f(x_1,x_2,y_1,y_2,z_1,z_2)\bigr\| \\
&\quad \le \theta\bigl(\|x_1\|^{p_1} \cdot \|x_2\|^{p_2} \cdot \|y_1\|^{q_1} \cdot \|y_2\|^{q_2} \\
&\qquad + \|y_1\|^{q_1} \cdot \|y_2\|^{q_2} \cdot \|z_1\|^{r_1} \cdot \|z_2\|^{r_2} + \|x_1\|^{p_1} \cdot \|x_2\|^{p_2} \cdot \|z_1\|^{r_1} \cdot \|z_2\|^{r_2}\bigr)
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
&\bigl\|f\bigl([x_1,y_1,z_1],[x_2,y_2,z_2],[x_3,y_3,z_3]\bigr) - \bigl[f(x_1,x_2,x_3),f(y_1,y_2,y_3),f(z_1,z_2,z_3)\bigr]\bigr\| \\
&\quad \le \eta \|x_1\|^{p_1} \cdot \|x_2\|^{p_2} \cdot \|x_3\|^{p_3} \cdot \|y_1\|^{q_1} \cdot \|y_2\|^{q_2} \cdot \|y_3\|^{q_3} \cdot \|z_1\|^{r_1} \cdot \|z_2\|^{r_2} \cdot \|z_3\|^{r_3}
\end{aligned} \tag{5}$$

for all $\lambda, \mu, \nu \in S^1$ *and all* $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. *Then the mapping* $f : A^3 \to B$ *is a* C^*-*ternary* 3-*homomorphism*.

Proof Letting $x_i = y_j = z_k = 0$ $(i, j, k = 1, 2)$ in (4), we get $f(0, 0, 0) = 0$. Putting $\lambda = \mu = \nu = 1$, $x_2 = 0$ and $y_j = z_k = 0$ $(j, k = 1, 2)$ in (4), we have $f(x_1, 0, 0) = 0$ for all $x_1 \in A$. Similarly, we get $f(0, y_1, 0) = f(0, 0, z_1) = 0$ for all $y_1, z_1 \in A$. Setting $\lambda = \mu = \nu = 1$, $x_2 = 0$, $y_2 = 0$ and $z_1 = z_2 = 0$, we have $f(x_1, y_1, 0) = 0$ for all $x_1, y_1 \in A$. Similarly, we get $f(x_1, 0, z_1) = f(0, y_1, z_1) = 0$ for all $x_1, y_1, z_1 \in A$. Now letting $\lambda = \mu = \nu = 1$ and $y_2 = z_2 = 0$ in (4), we have

$$f(x_1 + x_2, y_1, z_1) = f(x_1, y_1, z_1) + f(x_2, y_1, z_1)$$

for all $x_1, x_2, y_1, z_1 \in A$.

Similarly, one can show that the other equations hold. So, f is 3-additive.

Letting $x_2 = y_2 = z_2 = 0$ in (4), we get $f(\lambda x_1, \mu y_1, \nu z_1) = \lambda\mu\nu f(x_1, y_1, z_1)$ for all $\lambda, \mu, \nu \in S^1$ and all $x_1, y_1, z_1 \in A$. So, by Lemma 1.1, the mapping f is 3-$\mathbb{C}$-linear.

Without any loss of generality, we may suppose that $p_1 \neq 1$.

Let $p_1 < 1$. It follows from (5) that

$$\begin{aligned}
&\bigl\|f\bigl([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]\bigr) - \bigl[f(x_1, x_2, x_3), f(y_1, y_2, y_3), f(z_1, z_2, z_3)\bigr]\bigr\| \\
&\quad = \lim_{n\to\infty} \frac{1}{3^n} \bigl\|f\bigl(\bigl[3^n x_1, y_1, z_1\bigr], [x_2, y_2, z_2], [x_3, y_3, z_3]\bigr) \\
&\qquad - \bigl[f\bigl(3^n x_1, x_2, x_3\bigr), f(y_1, y_2, y_3), f(z_1, z_2, z_3)\bigr]\bigr\| \\
&\quad \le \eta \lim_{n\to\infty} \frac{3^{np_1}}{3^n} \\
&\qquad \times \bigl(\|x_1\|^{p_1} \cdot \|x_2\|^{p_2} \cdot \|x_3\|^{p_3} \cdot \|y_1\|^{q_1} \cdot \|y_2\|^{q_2} \cdot \|y_3\|^{q_3} \cdot \|z_1\|^{r_1} \cdot \|z_2\|^{r_2} \cdot \|z_3\|^{r_3}\bigr) \\
&\quad = 0
\end{aligned}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$.

Let $p_1 > 1$. It follows from (5) that

$$\begin{aligned}
&\bigl\|f\bigl([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]\bigr) - \bigl[f(x_1, x_2, x_3), f(y_1, y_2, y_3), f(z_1, z_2, z_3)\bigr]\bigr\| \\
&\quad = \lim_{n\to\infty} 3^n \left\|f\left(\left[\frac{1}{3^n} x_1, y_1, z_1\right], [x_2, y_2, z_2], [x_3, y_3, z_3]\right)\right. \\
&\qquad \left. - \left[f\left(\frac{1}{3^n} x_1, x_2, x_3\right), f(y_1, y_2, y_3), f(z_1, z_2, z_3)\right]\right\| \\
&\quad \le \eta \lim_{n\to\infty} \frac{3^n}{3^{np_1}} \\
&\qquad \times \bigl(\|x_1\|^{p_1} \cdot \|x_2\|^{p_2} \cdot \|x_3\|^{p_3} \cdot \|y_1\|^{q_1} \cdot \|y_2\|^{q_2} \cdot \|y_3\|^{q_3} \cdot \|z_1\|^{r_1} \cdot \|z_2\|^{r_2} \cdot \|z_3\|^{r_3}\bigr) \\
&\quad = 0
\end{aligned}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. Therefore,

$$f\bigl([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]\bigr) = \bigl[f(x_1, x_2, x_3), f(y_1, y_2, y_3), f(z_1, z_2, z_3)\bigr]$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. So, the mapping $f : A^3 \to B$ is a C^*-ternary 3-homomorphism. □

Theorem 2.3 *Let $\varphi : A^6 \to [0,\infty)$ and $\psi : A^9 \to [0,\infty)$ be functions such that*

$$\varphi(x_1, \dots, x_6) = 0$$

if $x_i = 0$ for some $1 \le i \le 6$ and

$$\frac{1}{3^n}\psi\left(x_1, \dots, 3^n x_i, \dots, x_9\right) = 0 \quad or \quad 3^n \psi\left(x_1, \dots, \frac{1}{3^n} x_i, \dots, x_9\right) = 0.$$

Suppose that $f : A^3 \to B$ is a mapping satisfying

$$\left\| D_{\lambda,\mu,\nu} f(x_1, x_2, y_1, y_2, z_1, z_2) \right\| \le \varphi(x_1, x_2, y_1, y_2, z_1, z_2)$$

and

$$\begin{aligned} &\left\| f\left([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]\right) - \left[f(x_1, x_2, x_3), f(y_1, y_2, y_3), f(z_1, z_2, z_3)\right] \right\| \\ &\quad \le \psi(x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3) \end{aligned}$$

for all $\lambda, \mu, \nu \in S^1$ and all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. Then the mapping f is a C^*-ternary 3-homomorphism.

Proof The proof is similar to the proof of Theorem 2.2. □

Corollary 2.4 *Let $p_i, q_i, r_i \in (0,\infty)$ $(i = 1, 2, 3)$ such that $p_i \neq 1$ or $q_i \neq 1$ or $r_i \neq 1$ for some $1 \le i \le 3$ and $\theta, \eta \in (0,\infty)$, and let $f : A^3 \to B$ be a mapping such that*

$$\left\| D_{\lambda,\mu,\nu} f(x_1, x_2, y_1, y_2, z_1, z_2) \right\| \le \theta \|x_1\|^{p_1} \cdot \|x_2\|^{p_2} \cdot \|y_1\|^{q_1} \cdot \|y_2\|^{q_2} \cdot \|z_1\|^{r_1} \cdot \|z_2\|^{r_2}$$

and

$$\begin{aligned} &\left\| f\left([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]\right) - \left[f(x_1, x_2, x_3), f(y_1, y_2, y_3), f(z_1, z_2, z_3)\right] \right\| \\ &\quad \le \eta \|x_1\|^{p_1} \cdot \|x_2\|^{p_2} \cdot \|x_3\|^{p_3} \cdot \|y_1\|^{q_1} \cdot \|y_2\|^{q_2} \cdot \|y_3\|^{q_3} \cdot \|z_1\|^{r_1} \cdot \|z_2\|^{r_2} \cdot \|z_3\|^{r_3} \end{aligned}$$

for all $\lambda, \mu, \nu \in S^1$ and all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. Then the mapping $f : A^3 \to B$ is a C^*-ternary 3-homomorphism.

3 C^*-ternary 3-derivations on C^*-ternary algebras

Definition 3.1 A 3-$\mathbb{C}$-linear mapping $D : A^3 \to A$ is called a C^*-*ternary* 3-*derivation* if it satisfies

$$\begin{aligned} D\left([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]\right) &= \left[D(x_1, x_2, x_3), \left[y_1, y_2^*, y_3\right], \left[z_1, z_2^*, z_3\right]\right] \\ &\quad + \left[\left[x_1, x_2^*, x_3\right], D(y_1, y_2, y_3), \left[z_1, z_2^*, z_3\right]\right] \\ &\quad + \left[\left[x_1, x_2^*, x_3\right], \left[y_1, y_2^*, y_3\right], D(z_1, z_2, z_3)\right] \end{aligned}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$.

Theorem 3.2 *Let $p,q,r \in (0,\infty)$ with $p+q+r<3$ and $\theta \in (0,\infty)$, and let $f : A^3 \to A$ be a mapping such that*

$$\begin{aligned}&\|D_{\lambda,\mu,\nu}f(x_1,x_2,y_1,y_2,z_1,z_2)\| \\ &\quad\le \theta \cdot \max\{\|x_1\|,\|x_2\|\}^p \cdot \max\{\|y_1\|,\|y_2\|\}^q \cdot \max\{\|z_1\|,\|z_2\|\}^r\end{aligned} \tag{6}$$

and

$$\begin{aligned}&\|f([x_1,y_1,z_1],[x_2,y_2,z_2],[x_3,y_3,z_3]) - [f(x_1,x_2,x_3),[y_1,y_2^*,y_3],[z_1,z_2^*,z_3]] \\ &\quad - [[x_1,x_2^*,x_3],f(y_1,y_2,y_3),[z_1,z_2^*,z_3]] - [[x_1,x_2^*,x_3],[y_1,y_2^*,y_3],f(z_1,z_2,z_3)]\| \\ &\le \theta\sum_{i=1}^{3}\|x_i\|^p \cdot \|y_i\|^q \cdot \|z_i\|^r\end{aligned} \tag{7}$$

for all $\lambda,\mu,\nu \in S^1$ and all $x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3 \in A$. Then there exists a unique C^*-ternary 3-derivation $\delta : A^3 \to A$ such that

$$\|f(x,y,z) - \delta(x,y,z)\| \le \frac{\theta}{2^3 - 2^{p+q+r}}\|x\|^p \cdot \|y\|^q \cdot \|z\|^r \tag{8}$$

for all $x,y,z \in A$.

Proof By the same method as in the proof of [32, Theorem 1.2], we obtain a 3-$\mathbb{C}$-linear mapping $\delta : A^3 \to A$ satisfying (8). The mapping $\delta(x,y,z) := \lim_{j\to\infty}\frac{1}{8^j}f(2^jx,2^jy,2^jz)$ for all $x,y,z \in A$.

It follows from (7) that

$$\begin{aligned}&\|\delta([x_1,y_1,z_1],[x_2,y_2,z_2],[x_3,y_3,z_3]) - [\delta(x_1,x_2,x_3),[y_1,y_2^*,y_3],[z_1,z_2^*,z_3]] \\ &\quad - [[x_1,x_2^*,x_3],\delta(y_1,y_2,y_3),[z_1,z_2^*,z_3]] - [[x_1,x_2^*,x_3],[y_1,y_2^*,y_3],\delta(z_1,z_2,z_3)]\| \\ &= \lim_{n\to\infty}\frac{1}{8^{3n}}\|f(2^{3n}[x_1,y_1,z_1],2^{3n}[x_2,y_2,z_2],2^{3n}[x_3,y_3,z_3]) \\ &\quad - [f(2^nx_1,2^nx_2,2^nx_3),[2^ny_1,2^ny_2^*,2^ny_3],[2^nz_1,2^nz_2^*,2^nz_3]] \\ &\quad - [[2^nx_1,2^nx_2^*,2^nx_3],f(2^ny_1,2^ny_2,2^ny_3),[2^nz_1,2^nz_2^*,2^nz_3]] \\ &\quad - [[2^nx_1,2^nx_2^*,2^nx_3],[2^ny_1,2^ny_2^*,2^ny_3],f(2^nz_1,2^nz_2,2^nz_3)]\| \\ &\le \lim_{n\to\infty}\frac{\theta 2^{n(p+q+r)}}{8^{3n}}\sum_{i=1}^{3}\|x_i\|^p \cdot \|y_i\|^q \cdot \|z_i\|^r = 0\end{aligned}$$

for all $x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3 \in A$.

Now, let $T : A^3 \to A$ be another 3-derivation satisfying (8). Then we have

$$\begin{aligned}\|\delta(x,y,z) - T(x,y,z)\| &= \frac{1}{8^n}\|\delta(2^nx,2^ny,2^nz) - T(2^nx,2^ny,2^nz)\| \\ &\le \frac{1}{8^n}\|\delta(2^nx,2^ny,2^nz) - f(2^nx,2^ny,2^nz)\|\end{aligned}$$

$$+\frac{1}{8^n}\left\|f\left(2^n x, 2^n y, 2^n z\right) - T\left(2^n x, 2^n y, 2^n z\right)\right\|$$
$$\leq \frac{\theta 2^{(p+q+r-3)n+1}}{2^3 - 2^{p+q+r}}\|x\|^p \cdot \|y\|^q \cdot \|z\|^r,$$

which tends to zero as $n \to \infty$ for all $x, y, z \in A$. So, we can conclude that $\delta(x, y, z) = T(x, y, z)$ for all $x, y, z \in A$. This proves the uniqueness of δ.

Therefore, the mapping $\delta : A^3 \to A$ is a unique C^*-ternary 3-derivation satisfying (8). □

Corollary 3.3 *Let $\epsilon \in (0, \infty)$, and let $f : A^3 \to A$ be a mapping satisfying*

$$\left\|D_{\lambda,\mu,\nu} f(x_1, x_2, y_1, y_2, z_1, z_2)\right\| \leq \epsilon$$

and

$$\begin{aligned}&\left\|f\left([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]\right) - \left[f(x_1, x_2, x_3), \left[y_1, y_2^*, y_3\right], \left[z_1, z_2^*, z_3\right]\right]\right.\\ &\quad \left. - \left[\left[x_1, x_2^*, x_3\right], f(y_1, y_2, y_3), \left[z_1, z_2^*, z_3\right]\right] - \left[\left[x_1, x_2^*, x_3\right], \left[y_1, y_2^*, y_3\right], f(z_1, z_2, z_3)\right]\right\| \leq 3\epsilon\end{aligned}$$

for all $\lambda, \mu, \nu \in S^1$ and all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. Then there exists a unique C^*-ternary 3-derivation $\delta : A^3 \to A$ such that

$$\left\|f(x, y, z) - \delta(x, y, z)\right\| \leq \frac{\epsilon}{7}$$

for all $x, y, z \in A$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Author details
[1]Department of Mathematics, Sirjan University of Technology, Sirjan, Iran. [2]Department of Mathematics, University of Yazd, P.O. Box 89195-741, Yazd, Iran. [3]Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul, 133-791, Korea. [4]Department of Mathematics, University of Seoul, Seoul, 130-743, Korea.

Acknowledgements
CP was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2012R1A1A2004299) and DYS was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2010-0021792).

Received: 31 October 2012 Accepted: 5 March 2013 Published: 25 March 2013

References
1. Cayley, A: On the 34 concomitants of the ternary cubic. Am. J. Math. **4**, 1-15 (1881)
2. Kapranov, M, Gelfand, IM, Zelevinskii, A: Discriminants, Resultants and Multidimensional Determinants. Birkhäuser, Berlin (1994)
3. Kerner, R: The cubic chessboard. Geometry and physics. Class. Quantum Gravity **14**(1A), A203-A225 (1997)
4. Kerner, R: Ternary algebraic structures and their applications in physics. Preprint, Univ. P. and M. Curie, Paris (2000)
5. Abramov, V, Kerner, R, Le Roy, B: Hypersymmetry: a Z_3-graded generalization of supersymmetry. J. Math. Phys. **38**, 1650-1669 (1997)
6. Daletskii, YL, Takhtajan, L: Leibniz and Lie algebra structures for Nambu algebras. Lett. Math. Phys. **39**, 127-141 (1997)
7. Vainerman, L, Kerner, R: On special classes of n-algebras. J. Math. Phys. **37**, 2553-2565 (1996)

8. Zettl, H: A characterization of ternary rings of operators. Adv. Math. **48**, 117-143 (1983)
9. Moslehian, MS: Almost derivations on C^*-ternary rings. Bull. Belg. Math. Soc. Simon Stevin **14**, 135-142 (2007)
10. Ulam, SM: A Collection of the Mathematical Problems. Interscience, New York (1960)
11. Hyers, DH: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. USA **27**, 222-224 (1941)
12. Aoki, T: On the stability of the linear transformation in Banach spaces. J. Math. Soc. Jpn. **2**, 64-66 (1950)
13. Bourgin, DG: Classes of transformations and bordering transformations. Bull. Am. Math. Soc. **57**, 223-237 (1951)
14. Rassias, TM: On the stability of the linear mapping in Banach spaces. Proc. Am. Math. Soc. **72**, 297-300 (1978)
15. Gajda, Z: On stability of additive mappings. Int. J. Math. Math. Sci. **14**, 431-434 (1991)
16. Rassias, TM, Šemrl, P: On the behaviour of mappings which do not satisfy Hyers-Ulam stability. Proc. Am. Math. Soc. **114**, 989-993 (1992)
17. Gǎvruta, P: A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. J. Math. Anal. Appl. **184**, 431-436 (1994)
18. Bae, J, Park, W: Approximate bi-homomorphisms and bi-derivations in C^*-ternary algebras. Bull. Korean Math. Soc. **47**, 195-209 (2010)
19. Bavand Savadkouhi, M, Eshaghi Gordji, M, Rassias, JM, Ghobadipour, N: Approximate ternary Jordan derivations on Banach ternary algebras. J. Math. Phys. **50**, Article ID 042303 (2009)
20. Ebadian, A, Nikoufar, I, Eshaghi Gordji, M: Nearly $(\theta_1, \theta_2, \theta_3, \phi)$-derivations on C^*-modules. Int. J. Geom. Methods Mod. Phys. **9**(3), Article ID 1250019 (2012)
21. Eshaghi Gordji, M, Ghaemi, MB, Kaboli Gharetapeh, S, Shams, S, Ebadian, A: On the stability of J^*-derivations. J. Geom. Phys. **60**, 454-459 (2010)
22. Eshaghi Gordji, M, Ghobadipour, N: Stability of (α, β, γ)-derivations on Lie C^*-algebras. Int. J. Geom. Methods Mod. Phys. **7**(7), 1093-1102 (2010)
23. Eshaghi Gordji, M, Najati, A: Approximately J^*-homomorphisms: a fixed point approach. J. Geom. Phys. **60**, 809-814 (2010)
24. Park, C: Isomorphisms between C^*-ternary algebras. J. Math. Phys. **47**, Article ID 103512 (2006)
25. Park, C, Eshaghi Gordji, M: Comment on 'Approximate ternary Jordan derivations on Banach ternary algebras' [Bavand Savadkouhi et al., J. Math. Phys. **50**, 042303 (2009)]. J. Math. Phys. **51**, Article ID 044102 (2010)
26. Rassias, JM, Kim, H: Approximate homomorphisms and derivations between C^*-ternary algebras. J. Math. Phys. **49**, Article ID 063507 (2008)
27. Najati, A, Ranjbari, A: Stability of homomorphisms for 3D Cauchy-Jensen type functional equation on C^*-ternary algebras. J. Math. Anal. Appl. **341**, 62-79 (2008)
28. Najati, A, Park, C, Lee, JR: Homomorphisms and derivations in C^*-ternary algebras. Abstr. Appl. Anal. **2009**, Article ID 612392 (2009)
29. Moradlou, F, Najati, A, Vaezi, H: Stability of homomorphisms and derivations on C^*-ternary rings associated to an Euler-Lagrange type additive mapping. Results Math. **55**, 469-486 (2009)
30. Najati, A: On generalized Jordan derivations of Lie triple systems. Czechoslov. Math. J. **60**(135), 541-547 (2010)
31. Najati, A, Ranjbari, A: On homomorphisms between C^*-ternary algebras. J. Math. Inequal. **1**, 387-407 (2007)
32. Bae, J, Park, W: Generalized Ulam-Hyers stability of C^*-ternary algebra 3-homomorphisms for a functional equation. J. Chungcheong Math. Soc. **24**, 147-162 (2011)

doi:10.1186/1029-242X-2013-124
Cite this article as: Dehghanian et al.: C^***-ternary** 3**-derivations on** C^***-ternary algebras.** *Journal of Inequalities and Applications* 2013 **2013**:124.