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Damped projection method for split common fixed point problems

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Abstract
The paper deals with the split common fixed-point problem (SCFP) introduced by Censor and Segal. Motivated by Eicke's damped projection method, we propose a cyclic iterative scheme and prove its strong convergence to a solution of SCFP under some mild assumptions. An application of the proposed method to multiple-set split feasibility problems is also included.

1 Introduction

The split feasibility problem (SFP) [1] consists of finding an element $\widehat{x} \in \mathcal{H}$ satisfying

$$\widehat{x} \in C, \qquad A\widehat{x} \in Q, \tag{1}$$

where C and Q are closed convex subsets in Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Moreover, if C and Q are the intersections of finitely many closed convex subsets, then the problem is known as the multiple-set split feasibility problem (MSFP) [2]. Note that SFP and MSFP model image retrieval [3] and intensity-modulated radiation therapy [4], and they have recently been investigated by many researchers (see, *e.g.*, [5–11]). One method for solving SFP is Byrne's CQ algorithm [5]: For any initial guess $x_1 \in \mathcal{H}$, define $\{x_n\}$ recursively by

$$x_{n+1} = P_C\big(x_n - \lambda A^*(I - P_Q)Ax_n\big), \tag{2}$$

where P_C stands for the metric projection onto C, I is the identity operator on $\mathcal{K}$ and λ is the step-size satisfying $0 < \lambda < \frac{2}{\|A\|^2}$. By using Hundal's counterexample, Xu [12] showed the CQ algorithm does not converge strongly in infinite-dimensional spaces. Motivated by Byrne's CQ algorithm, Wang and Xu [13] proposed the following iterative method: For any initial guess $x_1 \in \mathcal{H}$, define $\{x_n\}$ recursively by

$$x_{n+1} = P_C\big[(1 - \alpha_n)\big(x_n - \lambda A^*(I - P_Q)Ax_n\big)\big], \tag{3}$$

where $\{\alpha_n\} \subset (0,1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$; $\sum_{n=1}^{\infty} \alpha_n = \infty$; either $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n|/\alpha_n = 0$. It is worth noting that this algorithm is in fact a generalization of Eicke's damped projection method [14] for solving convexly constrained linear inverse problems (see [15]). Motivated by Krasnosel'skii-Mann's iteration, Dang and Gao [16] proposed the following algorithm: For any initial guess $x_1 \in \mathcal{H}$, define $\{x_n\}$ recursively by

$$x_{n+1} = (1-\beta_n)x_n + \beta_n P_C\big[(1-\alpha_n)\big(x_n - \lambda A^*(I-P_Q)Ax_n\big)\big], \tag{4}$$

where $\{\alpha_n\} \subset (0,1)$ satisfies (i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$; (ii) $\lim_{n\to\infty}|\alpha_{n+1}-\alpha_n| = 0$; (iii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. It is clear that such an algorithm is an extension of (3). However, algorithm (4) fails to include the original one (3) because of condition (iii).

In the case where C and Q in (1) are the intersections of finitely many fixed-point sets of nonlinear operators, problem (1) is called by Censor and Segal [17] the split common fixed-point problem (SCFP). More precisely, SCFP requires to seek an element $\widehat{x} \in \mathcal{H}$ satisfying

$$\widehat{x} \in \bigcap_{i=1}^{p} \mathrm{Fix}(U_i), \qquad A\widehat{x} \in \bigcap_{j=1}^{s} \mathrm{Fix}(T_j), \tag{5}$$

where $p, s \in \mathbb{N}$, $\mathrm{Fix}(U_i)$ and $\mathrm{Fix}(T_j)$ denote the fixed point sets of two classes of nonlinear operators $U_i : \mathcal{H} \to \mathcal{H}$, $i = 1, \ldots, p$ and $T_j : \mathcal{K} \to \mathcal{K}$, $j = 1, \ldots, s$. In this situation, Byrne's CQ algorithm does not work because the metric projection onto fixed point sets is generally not easy to calculate. To solve the two-set SCFP, that is, $p = s = 1$ in (5), Censor and Segal [17] proposed the following iterative method: For any initial guess $x_1 \in \mathcal{H}$, define $\{x_n\}$ recursively by

$$x_{n+1} = U\big(x_n - \lambda A^*(I-T)Ax_n\big), \tag{6}$$

where $\lambda > 0$ is known as the step-size. They proved that if U and T in (6) are directed operators, then λ should be chosen in $(0, \frac{2}{\|A\|^2})$. Some further generations of this algorithm were studied by Moudafi [18] for demicontractive operators and by Wang-Xu [19] for finitely many directed operators.

We note that the existing algorithms for SCFP have only weak convergence in the framework of infinite-dimensional spaces (see [18, 19]). However, as pointed by Bauschke and Combettes [20], norm convergence of the algorithm is much more desirable than weak convergence in some applied sciences. It is therefore of interest to seek modifications of these algorithms so that strong convergence is guaranteed. Following the damped projection method, we propose in this paper a new iterative scheme and prove its strong convergence to a solution of SCFP. An application of our method to multiple-set split feasibility problems is also included. This enables us to cover some recent results on split feasibility problems.

2 Preliminary and notation

Throughout this paper, I denotes the identity operator on $\mathcal{H}$, $\mathrm{Fix}(T)$ the set of fixed points of an operator T, '$\to$' strong convergence, and '$\rightharpoonup$' weak convergence. Given a positive integer p, denote by $[n] := (n \bmod p)$ the mod function taking values in $\{1, 2, \ldots, p\}$.

Definition 1 An operator $T : \mathcal{H} \to \mathcal{H}$ is called *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$, $\forall x, y \in \mathcal{H}$; *firmly nonexpansive* if $\|Tx - Ty\|^2 \le \|x - y\|^2 - \|(I-T)x - (I-T)y\|^2$, $\forall x, y \in \mathcal{H}$.

Definition 2 Assume that $T : \mathcal{H} \to \mathcal{H}$ is a nonlinear operator. Then $I - T$ is said to be *demiclosed at zero*, if, for any $\{x_n\}$ in $\mathcal{H}$, the following implication holds:

$$\left.\begin{array}{l} x_n \rightharpoonup x, \\ (I-T)x_n \to 0 \end{array}\right] \quad \Rightarrow \quad x \in \operatorname{Fix}(T).$$

Clearly, firm nonexpansiveness implies nonexpansiveness. It is well known that nonexpansive operators are demiclosed at zero (*cf.* [21]).

Definition 3 Let $T : \mathcal{H} \to \mathcal{H}$ be an operator with $\operatorname{Fix}(T) \neq \emptyset$. Then T is called *directed* if $\langle z - Tx, x - Tx\rangle \le 0$, $\forall z \in \operatorname{Fix}(T)$, $x \in \mathcal{H}$; ν*-demicontractive* with $\nu \in (-\infty, 1)$ if $\|Tx - z\|^2 \le \|x - z\|^2 + \nu\|(I - T)x\|^2$, $\forall z \in \operatorname{Fix}(T)$, $x \in \mathcal{H}$.

Lemma 1 (Bauschke-Combettes [20]) *An operator $T : \mathcal{H} \to \mathcal{H}$ is directed if and only if one of following inequalities holds for all $z \in \operatorname{Fix}(T)$ and $x \in \mathcal{H}$:*

$$\|Tx - z\|^2 \le \|x - z\|^2 - \big\|(I - T)x\big\|^2; \tag{7}$$

$$\big\langle (I - T)x, x - z\big\rangle \ge \big\|(I - T)x\big\|^2. \tag{8}$$

It is clear that demicontractive operators include directed operators, while the latter include firmly nonexpansive operators with nonempty fixed-point sets. The concept of directed operators was introduced by Bauschke and Combettes [20]. Such a class of operators is important because they include many types of nonlinear operators arising in applied mathematics. For instance, the metric projections onto a closed convex subset. Recall that the metric projection, denoted by $P_C : \mathcal{H} \to C$, is defined by

$$P_C x = \arg\min_{y\in C} \|x - y\|, \quad x \in \mathcal{H}.$$

It is well known that $P_C x$ is characterized by the variational inequality

$$\langle x - P_C x, P_C x - z\rangle \ge 0, \quad \forall z \in C. \tag{9}$$

Lemma 2 (Wang-Xu [19]) *Assume that $A : \mathcal{H} \to \mathcal{K}$ is a bounded linear operator and $T : \mathcal{K} \to \mathcal{K}$ is a directed operator. Let $V_\lambda = I - \lambda A^*(I - T)A$ with $\lambda > 0$. Then*

$$\operatorname{Fix}(V_\lambda) = A^{-1}\big(\operatorname{Fix}(T)\big),$$

whenever $A^{-1}(\operatorname{Fix}(T)) := \{x \in \mathcal{H} : Ax \in \operatorname{Fix}(T)\}$ is nonempty.

Lemma 3 *Assume that $A : \mathcal{H} \to \mathcal{K}$ is a bounded linear operator and $T : \mathcal{K} \to \mathcal{K}$ is a directed operator. Let $V_\lambda = I - \lambda A^*(I - T)A$ with $0 < \lambda < \frac{2}{\|A\|^2}$. If $A^{-1}(\operatorname{Fix}(T))$ is nonempty, then*

$$\|V_\lambda x - z\|^2 \le \|x - z\|^2 - \frac{2 - \lambda\|A\|^2}{\lambda\|A\|^2}\|V_\lambda x - x\|^2, \tag{10}$$

for all $z \in A^{-1}(\operatorname{Fix}(T))$ and $x \in \mathcal{H}$.

Proof Since $Az \in \mathrm{Fix}(T)$, it follows from (8) that

$$\begin{aligned}\langle (I - V_\lambda)x, x - z\rangle &= \lambda\langle (I - T)Ax, Ax - Az\rangle \\ &\geq \lambda \|(I - T)Ax\|^2 \\ &\geq \frac{1}{\lambda\|A\|^2}\|(I - V_\lambda)x\|^2.\end{aligned}$$

Consequently,

$$\begin{aligned}\|V_\lambda x - z\|^2 &= \|(x - z) + (V_\lambda x - x)\|^2 \\ &= \|x - z\|^2 + \|V_\lambda x - x\|^2 + 2\langle x - z, V_\lambda x - x\rangle \\ &\leq \|x - z\|^2 - \frac{2 - \lambda\|A\|^2}{\lambda\|A\|^2}\|V_\lambda x - x\|^2.\end{aligned}$$

Hence the proof is complete. □

We end this section by a useful lemma.

Lemma 4 (Xu [22]) *Let $\{a_n\}$ be a nonnegative real sequence satisfying*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n,$$

where $\{\alpha_n\} \subset (0,1)$ and $\{b_n\}$ are real sequences. Then $a_n \to 0$ provided that

(i) $\sum_n \alpha_n = \infty$, $\lim_n \alpha_n = 0$,
(ii) $\overline{\lim}_n b_n \leq 0$ *or* $\sum \alpha_n |b_n| < \infty$.

3 Algorithm and its convergence analysis

In this section, we consider the following problem.

Problem 1 Find an element $\widehat{x} \in \mathcal{H}$ satisfying

$$\widehat{x} \in \bigcap_{i=1}^{p} \mathrm{Fix}(U_i), \qquad A\widehat{x} \in \bigcap_{i=1}^{p} \mathrm{Fix}(T_i), \tag{11}$$

where p is a positive integer and $(U_i)_{i=1}^p$, $(T_i)_{i=1}^p$ are two classes of directed operators such that $U_i - I$ and $T_i - I$ are demiclosed at zero for every $i = 1, 2, \ldots, p$.

We remark here that problem (11) is a special case of (5). However, this is not restrictive. Indeed, following an idea in [19], one can easily extend the results to the general case. We now present our algorithm for SCFP: Take $x_1 \in \mathcal{H}$ and define a sequence $\{x_n\}$ by the iterative procedure:

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n U_n\big[(1 - \alpha_n)\big(x_n - \lambda_n A^*(I - T_n)Ax_n\big)\big], \tag{12}$$

where $U_n := U_{[n]}$, $T_n := T_{[n]}$ and $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subseteq [0,1]$, $\{\lambda_n\} \subseteq \mathbb{R}^+$ are properly chosen real sequences.

Theorem 1 *Assume that the following conditions hold*:

(i) $\liminf_{n\to\infty}\beta_n > 0$,

(ii) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(iii) $0 < \underline{\lambda} \le \lambda_n \le \overline{\lambda} < \frac{2}{\|A\|^2}$.

If the solution set of problem (11) *denoted by* Ω *is nonempty, then the sequence* $\{x_n\}$ *generated by* (12) *converges strongly to* $P_\Omega(0)$.

Proof We first show the boundedness of $\{x_n\}$. To see this, let $z = P_\Omega(0)$ and set $V_n = I - \lambda_n A^*(I - T_n)A$, $y_n = (1-\alpha_n)V_n x_n$. Hence

$$\begin{aligned}\|y_n - z\| &= \big\|(1-\alpha_n)(V_n x_n - z) - \alpha_n z\big\| \\ &\le (1-\alpha_n)\|x_n - z\| + \alpha_n\|z\|.\end{aligned}$$

Since U_n is directed, it follows that

$$\begin{aligned}\|x_{n+1} - z\| &= \big\|(1-\beta_n)(x_n - z) + \beta_n(U_n y_n - z)\big\| \\ &\le (1-\beta_n)\|x_n - z\| + \beta_n\|y_n - z\|.\end{aligned}$$

Adding up these inequalities, we have

$$\|x_{n+1} - z\| \le (1-\alpha_n\beta_n)\|x_n - z\| + \alpha_n\beta_n\|z\|.$$

By induction, the sequence $\{x_n\}$ is bounded, and so is $\{y_n\}$.

Next we show the following key inequality:

$$s_{n+1} \le (1-\alpha_n\beta_n)s_n + 2\alpha_n\beta_n\langle z - y_n, z\rangle - c_n, \tag{13}$$

where $s_n = \|x_n - z\|^2$ and

$$c_n = \beta_n\left[\frac{(1-\alpha_n)(2-\overline{\lambda}\|A\|^2)}{\overline{\lambda}\|A\|^2}\big\|(I - V_n)x_n\big\|^2 + \big\|(I - U_n)y_n\big\|^2\right].$$

Indeed, in view of Lemma 3, we arrive at

$$\begin{aligned}\|U_n y_n - z\|^2 &\le \|y_n - z\|^2 - \big\|(I - U_n)y_n\big\|^2, \\ \|V_n x_n - z\|^2 &\le \|x_n - z\|^2 - \frac{2-\lambda_n\|A\|^2}{\lambda_n\|A\|^2}\big\|(I - V_n)x_n\big\|^2 \\ &\le \|x_n - z\|^2 - \frac{2-\overline{\lambda}\|A\|^2}{\overline{\lambda}\|A\|^2}\big\|(I - V_n)x_n\big\|^2.\end{aligned} \tag{14}$$

On the other hand, we deduce that

$$\begin{aligned}\|y_n - z\|^2 &= \big\|(1-\alpha_n)(V_n x_n - z) - \alpha_n z\big\|^2 \\ &\le (1-\alpha_n)\|V_n x_n - z\|^2 + 2\alpha_n\langle z - y_n, z\rangle,\end{aligned} \tag{15}$$

where we use the subdifferential inequality, and also that

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \big\|(1 - \beta_n)(x_n - z) + \beta_n(U_n y_n - z)\big\|^2 \\ &\le (1 - \beta_n)\|x_n - z\|^2 + \beta_n\|U_n y_n - z\|^2. \end{aligned} \tag{16}$$

Adding up (14)-(16), we thus get inequality (13).

Finally, we prove $s_n \to 0$. To see this, let $\{s_{n_k}\}$ be a subsequence such that it includes all elements in $\{s_n\}$ with the property: each of them is less than or equal to the term after it. Following an idea developed by Maingé [23], we consider two possible cases on such a sequence.

Case 1. Assume that $\{s_{n_k}\}$ is finite. Then there exists $N \in \mathbb{N}$ such that $s_n > s_{n+1}$ for all $n \ge N$, and therefore $\{s_n\}$ must be convergent. It follows from (13) that

$$c_n \le M\alpha_n\beta_n + (s_n - s_{n+1}),$$

where $M > 0$ is a sufficiently large real number. Consequently, both $\|(I - V_n)x_n\|$ and $\|(I - U_n)y_n\|$ converge to zero. We have

$$\begin{aligned}\|y_n - x_n\| &\le \|y_n - V_n x_n\| + \|V_n x_n - x_n\| \\ &= \alpha_n\|V_n x_n\| + \|V_n x_n - x_n\| \to 0,\end{aligned}$$

which implies

$$\|x_{n+1} - x_n\| \le \|U_n y_n - y_n\| + \|y_n - x_n\| \to 0.$$

Take a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ so that

$$\limsup_{n\to\infty}\langle z, z - y_n\rangle = \lim_{k\to\infty}\langle z, z - y_{n_k}\rangle.$$

Without loss of generality, we assume that $\{y_{n_k}\}$ weakly converges to an element y'. Let an index $i \in \{1, 2, \dots, p\}$ be fixed. Noticing that the pool of indexes is finite, we can find a subsequence $\{y_{m_k}\}$ of $\{y_n\}$ such that $y_{m_k} \rightharpoonup y'$ and $[m_k] = i$ for all k. Since $\|(I - U_i)y_{m_k}\| = \|(I - U_{m_k})y_{m_k}\| \to 0$, we thus use the demiclosedness of $I - U_i$ at zero to conclude that $y' \in \mathrm{Fix}(U_i)$. On the other hand, we deduce from (8) that

$$\begin{aligned}\big\|(I - T_i)Ax_{m_k}\big\|^2 &\le \big\langle (I - T_i)Ax_{m_k}, Ax_{m_k} - Az\big\rangle \\ &= \big\langle A^*(I - T_i)Ax_{m_k}, x_{m_k} - z\big\rangle \\ &\le \frac{1}{\underline{\lambda}}\big\|(I - V_i)x_{m_k}\big\|\|x_{m_k} - z\| \\ &\le M\big\|(I - V_i)x_{m_k}\big\| \to 0.\end{aligned}$$

As $x_{m_k} - y_{m_k} \to 0$, the weak continuity of A yields that $Ax_{m_k} \rightharpoonup Ay'$, which together with the demiclosedness of $I - T_i$ at zero enables us to deduce $Ay' \in \mathrm{Fix}(T_i)$. Since the index i is arbitrary, we therefore conclude $y' \in \Omega$. Consequently,

$$\limsup_{n\to\infty}\langle z, z - y_n\rangle = \big\langle z, z - y'\big\rangle \le 0,$$

where the inequality uses (9). It then follows from (13) that

$$s_{n+1} \le (1-\alpha_n\beta_n)s_n + 2\alpha_n\beta_n\langle z-y_n, z\rangle.$$

We therefore apply Lemma 4 to conclude $s_n \to 0$.

Case 2. Assume now that $\{s_{n_k}\}$ is infinite. Let $n \in \mathbb{N}$ be fixed. Then there exists $k \in \mathbb{N}$ such that $n_k \le n \le n_{k+1}$. By the choice of $\{s_{n_k}\}$, we see that s_{n_k+1} is the largest one among $\{s_{n_k}, s_{n_k+1}, \dots, s_{n_{k+1}}\}$; in particular,

$$s_{n_k} \le s_{n_k+1} \quad \text{and} \quad s_n \le s_{n_k+1}. \tag{17}$$

Then we deduce from (13) that $c_{n_k} \le M\alpha_{n_k}$ so that

$$\|(I-V_{n_k})x_{n_k}\| + \|(I-U_{n_k})y_{n_k}\| \to 0. \tag{18}$$

In a similar way to case 1, we deduce $\|x_{n_k+1} - x_{n_k}\| \to 0$ and

$$\limsup_{n\to\infty}\langle u-z, y_{n_k}-z\rangle \le 0.$$

Since by (17) $s_{n_k} \le s_{n_k+1}$, it follows from (13) that

$$s_{n_k} \le 2\langle z, z-y_{n_k}\rangle. \tag{19}$$

Hence $\overline{\lim}_{k\to\infty} s_{n_k} \le 0$ so that $s_{n_k} \to 0$. Moreover,

$$\begin{aligned}|s_{n_k+1} - s_{n_k}| &= \bigl|\|x_{n_k+1}-z\|^2 - \|x_{n_k}-z\|^2\bigr| \\ &\le \|x_{n_k+1}-x_{n_k}\|\bigl(\|x_{n_k+1}-z\| + \|x_{n_k}-z\|\bigr) \to 0,\end{aligned}$$

which immediately implies $s_{n_k+1} \to 0$. Consequently, $s_n \to 0$ follows from (17) and the proof is complete. □

We next use our algorithm to approximate a solution to the two-set SCFP: Find an element $\widehat{x} \in \mathcal{H}$ such that

$$\widehat{x} \in \operatorname{Fix}(U), \qquad A\widehat{x} \in \operatorname{Fix}(T), \tag{20}$$

where $U : \mathcal{H} \to \mathcal{H}$ and $T : \mathcal{K} \to \mathcal{K}$ are directed operators so that $U - I$ and $T - I$ are demiclosed at zero.

Corollary 2 *Suppose that the following conditions hold*:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $0 < \underline{\lambda} \le \lambda_n \le \overline{\lambda} < \frac{2}{\|A\|^2}$.

Then the sequence $\{x_n\}$, *generated by*

$$x_{n+1} = U\bigl[(1-\alpha_n)\bigl(x_n - \lambda_n A^*(I-T)Ax_n\bigr)\bigr], \tag{21}$$

converges strongly to $P_\Omega(0)$, *whenever such point exists*.

4 Some applications

In this section, we extend our result to SCFP for demicontractive operators recently considered by Moudafi [18].

Problem 2 Find an element $\widehat{x} \in \mathcal{H}$ satisfying

$$\widehat{x} \in \bigcap_{i=1}^{p} \operatorname{Fix}(U_i), \qquad A\widehat{x} \in \bigcap_{i=1}^{p} \operatorname{Fix}(T_i), \tag{22}$$

where p is a positive integer and $(U_i)_{i=1}^{p}$, $(T_i)_{i=1}^{p}$ are respectively ν_i-demicontractive and κ_i-demicontractive operator so that $U_i - I$ and $T_i - I$ are demiclosed at zero for every $i = 1, 2, \ldots, p$.

The following lemma states a relation between directed and demicontractive operators.

Lemma 5 *Let $\nu \in (-\infty, 1)$ and $\tau \in (0, \frac{1-\nu}{2}]$. If T is ν-demicontractive, then $T_\tau := (1-\tau)I + \tau T$ is directed.*

Proof For $\forall z \in \operatorname{Fix}(T)$, we deduce that

$$\begin{aligned}
\|T_\tau x - z\|^2 &= \big\|(1-\tau)(x-z) + \tau(Tx - z)\big\|^2 \\
&= (1-\tau)\|x-z\|^2 + \tau\|Tx - z\|^2 - \tau(1-\tau)\big\|(I-T)x\big\|^2 \\
&\le \|x-z\|^2 - \tau(1-\nu-\tau)\big\|(I-T)x\big\|^2 \\
&= \|x-z\|^2 - \frac{1-\tau/(1-\nu)}{\tau/(1-\nu)}\big\|(I-T_\tau)x\big\|^2 \\
&\le \|x-z\|^2 - \big\|(I-T_\tau)x\big\|^2.
\end{aligned}$$

Then the result follows from Lemma 1. □

We now propose an algorithm to solve problem (22). Take $x_1 \in \mathcal{H}$ and define a sequence $\{x_n\}$ by the iterative procedure

$$x_{n+1} = U_{\tau_n}\big[(1-\alpha_n)\big(x_n - \lambda A^*(I - T_{\gamma_n})Ax_n\big)\big], \tag{23}$$

where $\{\alpha_n\} \subset (0,1)$, $U_{\tau_n} = (1-\tau_{[n]})I + \tau_{[n]}U_{[n]}$ and $T_{\gamma_n} = (1-\gamma_{[n]})I + \gamma_{[n]}T_{[n]}$. By using the previous lemma, we can easily extend our result to demicontractive operators.

Theorem 3 *Let $0 < \tau_i \le \frac{1-\nu_i}{2}$ and $0 < \gamma_i \le \frac{1-\kappa_i}{2}$ for every $i = 1, 2, \ldots, p$. Assume that the following conditions hold*:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \lambda < \min\{\frac{1-\kappa_i}{2\gamma_i\|A\|^2} : 1 \le i \le p\}$.

If the solution set of problem (22) *denoted by Ω is nonempty, then the sequence $\{x_n\}$ generated by* (23) *converges strongly to $P_\Omega(0)$.*

Remark 1 Theorem 3 also holds true if we relax hypothesis (ii) above as $0 < \lambda < \min\{\frac{1-\kappa_i}{\gamma_i\|A\|^2} : 1 \le i \le p\}$.

We next consider the multiple-set split feasibility problem (MSFP): Find an element $\widehat{x} \in \mathcal{H}$ satisfying

$$\widehat{x} \in \bigcap_{i=1}^{p} C_i, \qquad A\widehat{x} \in \bigcap_{i=1}^{p} Q_i, \tag{24}$$

where $\{C_i\}_{i=1}^p$ and $\{Q_i\}_{i=1}^p$ are closed convex subsets in $\mathcal{H}$ and $\mathcal{K}$, respectively. Take $x_1 \in \mathcal{H}$ and define a sequence $\{x_n\}$ by the iterative procedure

$$x_{n+1} = (1-\beta_n)x_n + \beta_n P_{C_n}\big[(1-\alpha_n)\big(x_n - \lambda_n A^*(I - P_{Q_n})Ax_n\big)\big], \tag{25}$$

where $C_n := C_{[n]}$, $Q_n := Q_{[n]}$, and $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subseteq [0,1]$, $\{\lambda_n\} \subseteq \mathbb{R}^+$ are properly chosen real sequences.

Theorem 4 *Assume that the following conditions hold*:

(i) $\liminf_{n\to\infty} \beta_n > 0$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$,

(iii) $0 < \underline{\lambda} \le \lambda_n \le \overline{\lambda} < \frac{2}{\|A\|^2}$.

If the solution set of MSFP denoted by Ω *is nonempty, then the sequence* $\{x_n\}$ *generated by* (25) *converges strongly to* $P_\Omega(0)$.

Proof We note that the metric projection P_C is firmly nonexpansive, which implies P_C is directed and $I - P_C$ is demiclosed at zero. Hence, by using Theorem 1, one can immediately get the desired result. □

Remark 2 Theorem 4 covers [16, Theorem 3.1], and we relax the condition on $\{\beta_n\}$ as $\liminf_{n\to\infty} \beta_n > 0$. Moreover, the choice of variable $\{\lambda_n\}$ is more flexible than the fixed one. Also, we cover the result of [19] and remove one condition posed on $\{\alpha_n\}$: either $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n|/\alpha_n = 0$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly to writing this manuscript. All authors read and approved the manuscript.

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