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Sharp bounds for Seiffert and Neuman-Sándor means in terms of generalized logarithmic means

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Abstract

In this paper, we prove three sharp inequalities as follows: $P(a, b) > L_2(a, b)$, $T(a, b) > L_5(a, b)$ and $M(a, b) > L_4(a, b)$ for all $a, b > 0$ with $a \neq b$. Here, $L_r(a, b)$, $M(a, b)$, $P(a, b)$ and $T(a, b)$ are the r th generalized logarithmic, Neuman-Sándor, first and second Seiffert means of a and b , respectively.

MSC: 26E60

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1 Introduction

The Neuman-Sándor mean $M(a, b)$ [1] and the first and second Seiffert means $P(a, b)$ [2] and $T(a, b)$ [3] of two positive numbers a and b are defined by

$$M(a, b) = \begin{cases} \frac{a-b}{2 \operatorname{arcsinh}(\frac{a-b}{a+b})}, & a \neq b, \\ a, & a = b, \end{cases} \quad (1.1)$$

$$P(a, b) = \begin{cases} \frac{a-b}{4 \arctan(\sqrt{a/b}) - \pi}, & a \neq b, \\ a, & a = b \end{cases} \quad (1.2)$$

and

$$T(a, b) = \begin{cases} \frac{a-b}{2 \arctan(\frac{a-b}{a+b})}, & a \neq b, \\ a, & a = b, \end{cases} \quad (1.3)$$

respectively.

Recently, these means, M , P and T , have been the subject of intensive research. In particular, many remarkable inequalities for M , P and T can be found in the literature [1, 4–8].

The power mean $M_p(a, b)$ of order r of two positive numbers a and b is defined by

$$M_p(a, b) = \begin{cases} (\frac{a^p + b^p}{2})^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases}$$

The main properties for $M_p(a, b)$ are given in [9]. In particular, the function $p \mapsto M_p(a, b)$ ($a \neq b$) is continuous and strictly increasing on \mathbb{R} .

The arithmetic-geometric mean $AG(a, b)$ of two positive numbers a and b is defined as the common limit of sequences $\{a_n\}$ and $\{b_n\}$, which are given by

$$a_0 = a, \quad b_0 = b, \quad a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}.$$

Let $H(a, b) = 2ab/(a + b)$, $G(a, b) = \sqrt{ab}$, $L(a, b) = (b - a)/(\log b - \log a)$, $I(a, b) = 1/e(b^b/a^a)^{1/(b-a)}$, $A(a, b) = (a + b)/2$, and $S(a, b) = \sqrt{(a^2 + b^2)/2}$ be the harmonic, geometric, logarithmic, identric, arithmetic and root-square means of two positive numbers a and b with $a \neq b$, respectively. Then it is well known that the inequalities

$$\begin{aligned} H(a, b) = M_{-1}(a, b) < G(a, b) = M_0(a, b) < L(a, b) < AG(a, b) < I(a, b) \\ < A(a, b) = M_1(a, b) < M(a, b) < T(a, b) < S(a, b) = M_2(a, b) \end{aligned}$$

hold for all $a, b > 0$ with $a \neq b$.

For $r > 0$ the r th generalized logarithmic mean $L_r(a, b)$ of two positive numbers a and b is defined by

$$L_r(a, b) = L^{1/r}(a^r, b^r) = \begin{cases} \left[\frac{b^r - a^r}{r(\log b - \log a)} \right]^{1/r}, & a \neq b, \\ a, & a = b. \end{cases} \quad (1.4)$$

It is not difficult to verify that $L_r(a, b)$ is continuous and strictly increasing with respect to $r \in (0, +\infty)$ for fixed $a, b > 0$ with $a \neq b$.

In [2, 3] Seiffert proved that the double inequalities

$$L(a, b) < P(a, b) < I(a, b)$$

and

$$A(a, b) < T(a, b) < S(a, b)$$

hold for all $a, b > 0$ with $a \neq b$.

The following bounds for the first Seiffert mean $P(a, b)$ in terms of power mean were presented by Jagers in [10]:

$$M_{1/2} < P(a, b) < M_{2/3}(a, b)$$

for all $a, b > 0$ with $a \neq b$.

Hästö [11, 12] proved that the function $x \rightarrow T(1, x)/M_p(1, x)$ is increasing on $(0, +\infty)$ if $p \leq 1$ and found the sharp lower power mean bound for the Seiffert mean $P(a, b)$ as follows:

$$P(a, b) > M_{\log 2 / \log \pi}(a, b)$$

for all $a, b > 0$ with $a \neq b$.

In [13] the authors presented the following best possible Lehmer mean bounds for the Seiffert means $P(a, b)$ and $T(a, b)$:

$$\bar{L}_{-1/6}(a, b) < P(a, b) < \bar{L}_0(a, b) \quad \text{and} \quad \bar{L}_0(a, b) < T(a, b) < \bar{L}_{1/3}(a, b)$$

for all $a, b > 0$ with $a \neq b$. Here, $\bar{L}_p(a, b) = (a^{p+1} + b^{p+1}) / (a^p + b^p)$ is the Lehmer mean of a and b .

In [14, 15] the authors proved that the inequalities

$$\alpha_1 S(a, b) + (1 - \alpha_1)A(a, b) < T(a, b) < \beta_1 S(a, b) + (1 - \beta_1)A(a, b),$$

$$S^{\alpha_2}(a, b)A^{1-\alpha_2}(a, b) < T(a, b) < S^{\beta_2}(a, b)A^{1-\beta_2}(a, b)$$

and

$$\alpha_3 T(a, b) + (1 - \alpha_3)G(a, b) < A(a, b) < \beta_3 T(a, b) + (1 - \beta_3)G(a, b)$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq (4 - \pi) / [(\sqrt{2} - 1)\pi]$, $\beta_1 \geq 2/3$, $\alpha_2 \leq 2/3$, $\beta_2 \geq 4 - 2 \log \pi / \log 2$, $\alpha_3 \leq 3/5$ and $\beta_3 \geq \pi/4$.

For all $a, b > 0$ with $a \neq b$, the following inequalities can be found in [16, 17]:

$$L(a, b) = L_1(a, b) < AG(a, b) < L_{3/2}(a, b) < M_{1/2}(a, b).$$

Neuman and Sándor [1] established that

$$P(a, b) < A(a, b) < M(a, b) < T(a, b)$$

and

$$\frac{\pi}{2}P(a, b) > A(a, b) > \operatorname{arcsinh}(1)M(a, b) > \frac{\pi}{4}T(a, b)$$

for all $a, b > 0$ with $a \neq b$. In particular, the Ky Fan inequalities

$$\frac{G(a, b)}{G(a', b')} < \frac{L(a, b)}{L(a', b')} < \frac{P(a, b)}{P(a', b')} < \frac{A(a, b)}{A(a', b')} < \frac{M(a, b)}{M(a', b')} < \frac{T(a, b)}{T(a', b')}$$

hold for all $0 < a, b \leq 1/2$ with $a \neq b$, $a' = 1 - a$ and $b' = 1 - b$.

It is the aim of this paper to find the best possible generalized logarithmic mean bounds for the Neuman-Sándor and Seiffert means.

2 Lemmas

In order to establish our main results, we need three lemmas, which we present in this section.

Lemma 2.1 *The inequality*

$$\left(\frac{x^4 - 1}{4 \log x} \right)^{1/2} < \frac{x^2 - 1}{4 \arctan x - \pi} \tag{2.1}$$

holds for all $x > 1$.

Proof Let

$$f(x) = \frac{x^4 - 1}{4 \log x} - \left(\frac{x^2 - 1}{4 \arctan x - \pi} \right)^2. \tag{2.2}$$

Then $f(x)$ can be rewritten as

$$f(x) = \frac{(x^4 - 1)f_1(x)}{4(4 \arctan x - \pi)^2 \log x}, \tag{2.3}$$

where

$$f_1(x) = (4 \arctan x - \pi)^2 - \frac{4(x^2 - 1) \log x}{x^2 + 1}.$$

Simple computations lead to

$$f_1(1) = 0, \tag{2.4}$$

$$f_1'(x) = \frac{4}{1 + x^2} f_2(x), \tag{2.5}$$

where

$$f_2(x) = 8 \arctan x - \frac{4x \log x}{x^2 + 1} - x + \frac{1}{x} - 2\pi, \tag{2.6}$$

$$f_2(1) = 0,$$

$$f_2'(x) = \frac{f_3(x)}{(x^2 + 1)^2}, \tag{2.7}$$

where

$$f_3(x) = 4(x^2 - 1) \log x - x^4 + x^2 + 1 - x^{-2}, \tag{2.8}$$

$$f_3(1) = 0,$$

$$f_3'(x) = 8x \log x - 4x^3 + 6x - 4x^{-1} + 2x^{-3}, \tag{2.9}$$

$$f_3'(1) = 0,$$

$$f_3''(x) = 8 \log x - 12x^2 + 14 + 4x^{-2} - 6x^{-4}, \tag{2.10}$$

$$f_3''(1) = 0$$

and

$$f_3'''(x) = -\frac{8}{x^3}(x + 1)(x - 1)(3x^4 + 2x^2 + 3) < 0 \tag{2.11}$$

for $x > 1$.

Inequality (2.11) implies that $f_3''(x)$ is strictly decreasing in $[1, +\infty)$, then equation (2.10) leads to the conclusion that $f_3'(x)$ is strictly decreasing in $[1, +\infty)$.

From equations (2.4)-(2.9) and the monotonicity of $f_3'(x)$, we clearly see that

$$f_1(x) < 0 \tag{2.12}$$

for $x > 1$.

Therefore, inequality (2.1) follows from equations (2.2) and (2.3) together with inequality (2.12). \square

Lemma 2.2 *The inequality*

$$\left(\frac{x^5 - 1}{5 \log x}\right)^{1/5} < \frac{x - 1}{2 \arctan \frac{x-1}{x+1}} \tag{2.13}$$

holds for all $x > 1$.

Proof Let

$$g(x) = \frac{1}{5} \log\left(\frac{x^5 - 1}{5 \log x}\right) - \log\left(\frac{x - 1}{2 \arctan \frac{x-1}{x+1}}\right). \tag{2.14}$$

Then simple computations lead to

$$g(1) = 0, \tag{2.15}$$

$$g'(x) = \frac{[5x(x^3 + x^2 + x + 1) \log x + x^5 - 1]g_1(x)}{5x(x^5 - 1) \arctan\left(\frac{x-1}{x+1}\right) \log x}, \tag{2.16}$$

where

$$g_1(x) = \frac{5x(x^5 - 1) \log x}{(1 + x^2)[5x(x^3 + x^2 + x + 1) \log x + x^5 - 1]} - \arctan \frac{x - 1}{x + 1},$$

$$g_1(1) = 0, \tag{2.17}$$

$$g'_1(x) = \frac{x^3 g_2(x)}{(1 + x^2)^2 [5x(x^3 + x^2 + x + 1) \log x + x^5 - 1]^2}, \tag{2.18}$$

where

$$g_2(x) = 50(x^6 + x^5 - x^4 + 2x^3 - x^2 + x + 1) \log^2 x$$

$$- 5(x^9 + 2x^8 + x^7 + 4x^6 + 4x^5 - 4x - 4 - x^{-1} - 2x^{-2} - x^{-3}) \log x$$

$$+ 4(x^9 + x^7 - 2x^4 - 2x^2 + x^{-1} + x^{-3}),$$

$$g_2(1) = 0, \tag{2.19}$$

$$g'_2(x) = 50(6x^5 + 5x^4 - 4x^3 + 6x^2 - 2x + 1) \log^2 x$$

$$+ 5(-9x^8 - 16x^7 - 7x^6 - 4x^5 - 20x^3 + 40x^2 - 20x + 24 + 20x^{-1}$$

$$- x^{-2} - 4x^{-3} - 3x^{-4}) \log x + 31x^8 - 10x^7 + 23x^6 - 20x^5$$

$$- 20x^4 - 32x^3 - 16x + 20 + 20x^{-1} + x^{-2} + 10x^{-3} - 7x^{-4},$$

$$g'_2(1) = 0, \tag{2.20}$$

$$g''_2(x) = 100(15x^4 + 10x^3 - 6x^2 + 6x - 1) \log^2 x + 10(-36x^7 - 56x^6$$

$$- 21x^5 + 50x^4 + 50x^3 - 70x^2 + 100x - 30 + 10x^{-1} - 10x^{-2} + x^{-3}$$

$$+ 6x^{-4} + 6x^{-5}) \log x + 203x^7 - 150x^6 + 103x^5 - 120x^4 - 80x^3$$

$$- 196x^2 + 200x - 116 + 120x^{-1} + 80x^{-2} - 7x^{-3} - 50x^{-4} + 13x^{-5},$$

$$g_2''(1) = 0, \tag{2.21}$$

$$g_2'''(x) = 600(10x^3 + 5x^2 - 2x + 1) \log^2 x + 10(-252x^6 - 336x^5 - 105x^4 + 500x^3 + 350x^2 - 260x + 220 - 20x^{-1} - 10x^{-2} + 20x^{-3} - 3x^{-4} - 24x^{-5} - 30x^{-6}) \log x + 1,061x^6 - 1,460x^5 + 305x^4 + 20x^3 + 260x^2 - 1,092x + 1,200 - 300x^{-1} - 20x^{-2} - 260x^{-3} + 31x^{-4} + 260x^{-5} - 5x^{-6},$$

$$g_2'''(1) = 0, \tag{2.22}$$

$$g_2^{(4)}(x) = 2g_3(x), \tag{2.23}$$

where

$$g_3(x) = 600(15x^2 + 5x - 1) \log^2 x + 10(-756x^5 - 840x^4 - 210x^3 + 1,300x^2 + 650x - 250 + 6x^{-1} + 10x^{-2} + 10x^{-3} - 30x^{-4} + 6x^{-5} + 60x^{-6} + 90x^{-7}) \log x + 1,923x^5 - 5,330x^4 + 85x^3 + 2,530x^2 + 2,010x - 1,846 + 1,100x^{-1} + 50x^{-2} - 30x^{-3} + 490x^{-4} - 77x^{-5} - 770x^{-6} - 135x^{-7},$$

$$g_3(1) = 0, \tag{2.24}$$

$$g_3'(x) = 3,000(6x + 1) \log^2 x + 10(-3,780x^4 - 3,360x^3 - 630x^2 + 4,400x + 1,250 - 120x^{-1} - 6x^{-2} - 20x^{-3} - 30x^{-4} + 120x^{-5} - 30x^{-6} - 360x^{-7} - 630x^{-8}) \log x + 2,055x^4 - 29,720x^3 - 1,845x^2 + 18,060x + 8,510 - 2,500x^{-1} - 1,040x^{-2} + 190x^{-4} - 2,260x^{-5} + 445x^{-6} + 5,220x^{-7} + 1,845x^{-8} < 3,000(6x + 1)(x - 1) \log x + 10(-3,780x^4 - 3,360x^3 - 630x^2 + 4,400x + 1,250 - 120x^{-1} - 6x^{-2} - 20x^{-3} - 30x^{-4} + 120x^{-5} - 30x^{-6} - 360x^{-7} - 630x^{-8}) \log x + 2,055x^4 - 29,720x^3 - 1,845x^2 + 18,060x + 8,510 - 2,500x^{-1} - 1,040x^{-2} + 190x^{-4} - 2,260x^{-5} + 445x^{-6} + 5,220x^{-7} + 1,845x^{-8} = 10(-3,780x^4 - 3,360x^3 + 1,170x^2 + 2,900x + 950 - 120x^{-1} - 6x^{-2} - 20x^{-3} - 30x^{-4} + 120x^{-5} - 30x^{-6} - 360x^{-7} - 630x^{-8}) \log x + 2,055x^4 - 29,720x^3 - 1,845x^2 + 18,060x + 8,510 - 2,500x^{-1} - 1,040x^{-2} + 190x^{-4} - 2,260x^{-5} + 445x^{-6} + 5,220x^{-7} + 1,845x^{-8}. \tag{2.25}$$

Let

$$g_4(x) = 10(-3,780x^4 - 3,360x^3 + 1,170x^2 + 2,900x + 950 - 120x^{-1} - 6x^{-2} - 20x^{-3} - 30x^{-4} + 120x^{-5} - 30x^{-6} - 360x^{-7} - 630x^{-8}) \log x + 2,055x^4 - 29,720x^3 - 1,845x^2 + 18,060x + 8,510 - 2,500x^{-1} - 1,040x^{-2} + 190x^{-4} - 2,260x^{-5} + 445x^{-6} + 5,220x^{-7} + 1,845x^{-8}.$$

Then

$$g_4(1) = -1,040 < 0, \tag{2.26}$$

$$g'_4(x) = 10x^{-9}g_5(x), \tag{2.27}$$

where

$$\begin{aligned} g_5(x) &= (-15,120x^{12} - 10,080x^{11} + 2,340x^{10} + 2,900x^9 + 120x^7 + 12x^6 \\ &\quad + 60x^5 + 120x^4 - 600x^3 + 180x^2 + 2,520x + 5,040) \log x - 2,958x^{12} \\ &\quad - 12,276x^{11} + 801x^{10} + 4,706x^9 + 950x^8 + 130x^7 + 202x^6 - 20x^5 \\ &\quad - 106x^4 + 1,250x^3 - 297x^2 - 4,014x - 2,106 \\ &< (-15,120x^{11} - 10,080x^{11} + 2,340x^{11} + 2,900x^{11} + 120x^{11} + 12x^{11} \\ &\quad + 60x^{11} + 120x^{11} - 600x^3 + 180x^{11} + 2,520x^{11} + 5,040x^{11}) \log x \\ &\quad - 2,958x^{11} - 12,276x^{11} + 801x^{11} + 4,706x^{11} + 950x^{11} + 130x^{11} \\ &\quad + 202x^{11} - 20x^5 - 106x^4 + 1,250x^{11} - 297x^2 - 4,014x - 2,106 \\ &= -(11,908x^{11} + 600x^3) \log x - 7,195x^{11} - 20x^5 - 106x^4 - 297x^2 - 4,014x - 2,106 \\ &< 0 \end{aligned} \tag{2.28}$$

for $x > 1$.

Equation (2.27) and inequality (2.28) lead to the conclusion that $g_4(x)$ is strictly decreasing in $[1, +\infty)$. Then equation (2.26) implies that

$$g_4(x) < 0 \tag{2.29}$$

for $x > 1$.

Inequalities (2.25) and (2.29) imply that $g'_3(x) < 0$. Then equation (2.24) shows that

$$g_3(x) < 0 \tag{2.30}$$

for $x > 1$.

From equations (2.17)-(2.23) and inequality (2.30), we clearly see that

$$g_1(x) < 0 \tag{2.31}$$

for $x > 1$.

Therefore, inequality (2.13) follows easily from equations (2.14)-(2.16) and inequality (2.31). \square

Lemma 2.3 *The inequality*

$$\operatorname{arcsinh}^4\left(\frac{x-1}{x+1}\right) - \frac{(x-1)^4 \log x}{4(x^4-1)} < 0 \tag{2.32}$$

holds for all $x > 1$.

Proof Let

$$h(x) = \log \left[\operatorname{arcsinh}^4 \left(\frac{x-1}{x+1} \right) \right] - \log \left[\frac{(x-1)^4 \log x}{4(x^4-1)} \right]. \tag{2.33}$$

Then simple computations lead to

$$h(1) = 0, \tag{2.34}$$

$$h'(x) = \frac{[4x(x^2+x+1)\log x + x^4 - 1]h_1(x)}{x(x^4-1)\operatorname{arcsinh}\left(\frac{x-1}{x+1}\right)\log x}, \tag{2.35}$$

where

$$h_1(x) = \frac{4\sqrt{2}x(x^4-1)\log x}{(x+1)[4x(x^2+x+1)\log x + x^4 - 1]\sqrt{1+x^2}} - \operatorname{arcsinh}\left(\frac{x-1}{x+1}\right),$$

$$h_1(1) = 0, \tag{2.36}$$

$$h'_1(x) = \frac{\sqrt{2}x^2h_2(x)}{(x+1)(1+x^2)^{3/2}[4x(x^2+x+1)\log x + x^4 - 1]^2}, \tag{2.37}$$

where

$$h_2(x) = 16(x^6+x^5+2x^3+x+1)\log^2 x - 4(x^4-1)(x^4+x^3+3x^2+2x+3+x^{-1}+x^{-2})\log x + 3(x^4-1)^2(1+x^{-2}),$$

$$h_2(1) = 0, \tag{2.38}$$

$$h'_2(x) = 2h_3(x), \tag{2.39}$$

where

$$h_3(x) = 8(6x^5+5x^4+6x^2+1)\log^2 x - 2(8x^7+7x^6+10x^5+2x^4+8x^3-16x^2-4x-10-8x^{-1}+x^{-2}+2x^{-3})\log x + 10x^7-2x^6+3x^5-4x^4-16x^3-2x+4+6x^{-1}+2x^{-2}-x^{-3},$$

$$h_3(1) = 0, \tag{2.40}$$

$$h'_3(x) = xh_4(x), \tag{2.41}$$

$$h_4(x) = 16(15x^3+10x^2+6)\log^2 x - 4(28x^5+21x^4+x^3-16x^2+12x-40-2x^{-1}-4x^{-2}+4x^{-3}-x^{-4}-3x^{-5})\log x + 54x^5-26x^4-5x^3-20x^2-64x+32+6x^{-1}+20x^{-2}+10x^{-3}-6x^{-4}-x^{-5},$$

$$h_4(1) = 0, \tag{2.42}$$

$$h'_4(x) = xh_5(x), \tag{2.43}$$

where

$$\begin{aligned}
 h_5(x) &= 80(9x + 4) \log^2 x - 4(140x^3 + 84x^2 - 117x - 112 + 12x^{-1} - 48x^{-2} \\
 &\quad + 2x^{-3} + 8x^{-4} - 12x^{-5} + 4x^{-6} + 15x^{-7}) \log x + 158x^3 - 188x^2 - 19x \\
 &\quad + 24 - 112x^{-1} + 160x^{-2} + 2x^{-3} - 24x^{-4} - 46x^{-5} + 28x^{-6} + 17x^{-7}, \\
 h_5(1) &= 0,
 \end{aligned} \tag{2.44}$$

$$\begin{aligned}
 h'_5(x) &= 720 \log^2 x + 4(-420x^2 - 168x + 477 + 160x^{-1} + 12x^{-2} - 96x^{-3} \\
 &\quad + 6x^{-4} + 32x^{-5} - 60x^{-6} + 24x^{-7} + 105x^{-8}) \log x - 86x^2 - 712x \\
 &\quad + 449 + 448x^{-1} + 64x^{-2} - 128x^{-3} - 14x^{-4} + 64x^{-5} + 278x^{-6} \\
 &\quad - 184x^{-7} - 179x^{-8}, \\
 h'_5(1) &= 0,
 \end{aligned} \tag{2.45}$$

$$h''_5(x) = 4x^{-9}h_6(x), \tag{2.46}$$

where

$$\begin{aligned}
 h_6(x) &= -8(105x^{10} + 21x^9 - 45x^8 + 20x^7 + 3x^6 - 36x^5 + 3x^4 + 20x^3 \\
 &\quad - 45x^2 + 21x + 105) \log x - (x^2 - 1)(463x^8 + 346x^7 - 14x^6 \\
 &\quad + 298x^5 + 6x^4 + 298x^3 - 14x^2 + 346x + 463) \\
 &< 0
 \end{aligned} \tag{2.47}$$

for all $x > 1$.

Equations (2.45) and (2.46) together with inequality (2.47) imply that $h_5(x)$ is strictly decreasing in $[1, +\infty)$. Then equation (2.44) leads to

$$h_5(x) < 0 \tag{2.48}$$

for all $x > 1$.

From equations (2.36)-(2.43) and inequality (2.48), we clearly see that

$$h_1(x) < 0 \tag{2.49}$$

for all $x > 1$.

Therefore, inequality (2.32) follows from equations (2.33)-(2.35) and inequality (2.49). □

3 Main results

Theorem 3.1 *The inequality*

$$P(a, b) > L_2(a, b)$$

holds for all $a, b > 0$ with $a \neq b$, and $L_2(a, b)$ is the best possible lower generalized logarithmic mean bound for the first Seiffert mean $P(a, b)$.

Proof From (1.2) and (1.4), we clearly see that both $P(a, b)$ and $L_r(a, b)$ are symmetric and homogenous of degree one. Without loss of generality, we assume that $b = 1$ and $a = x^2 > 1$. Then (1.2) and (1.4) lead to

$$L_2(x^2, 1) - P(x^2, 1) = \left(\frac{x^4 - 1}{4 \log x}\right)^{1/2} - \frac{x^2 - 1}{4 \arctan x - \pi}. \tag{3.1}$$

Therefore, $P(x^2, 1) > L_2(x^2, 1)$ follows from Lemma 2.1 and equation (3.1).

Next, we prove that $L_2(a, b)$ is the best possible lower generalized logarithmic mean bound for the first Seiffert mean $P(a, b)$.

For any $\epsilon > 0$ and $x > 0$, from (1.2) and (1.4), one has

$$L_{2+\epsilon}(1+x, 1) - P(1+x, 1) = \left[\frac{(1+x)^{2+\epsilon} - 1}{(2+\epsilon) \log(1+x)}\right]^{1/(2+\epsilon)} - \frac{x}{4 \arctan \sqrt{1+x} - \pi}. \tag{3.2}$$

Letting $x \rightarrow 0$ and making use of Taylor expansion, we get

$$\begin{aligned} & \left[\frac{(1+x)^{2+\epsilon} - 1}{(2+\epsilon) \log(1+x)}\right]^{1/(2+\epsilon)} - \frac{x}{4 \arctan \sqrt{1+x} - \pi} \\ &= \left[1 + \frac{2+\epsilon}{2}x + \frac{(2+\epsilon)(1+2\epsilon)}{12}x^2 + o(x^2)\right]^{1/(2+\epsilon)} - \frac{x}{x - \frac{1}{2}x^2 + \frac{7}{24}x^3 + o(x^3)} \\ &= \left[1 + \frac{1}{2}x - \frac{1-\epsilon}{24}x^2 + o(x^2)\right] - \left[1 + \frac{1}{2}x - \frac{1}{24}x^2 + o(x^2)\right] \\ &= \frac{\epsilon}{24}x^2 + o(x^2). \end{aligned} \tag{3.3}$$

Equations (3.2) and (3.3) imply that for any $\epsilon > 0$, there exists $\delta_1 = \delta_1(\epsilon) > 0$ such that $L_{2+\epsilon}(1+x, 1) > P(1+x, 1)$ for $x \in (0, \delta_1)$. □

Remark 3.1 It follows from (1.2) and (1.4) that

$$\lim_{x \rightarrow +\infty} \frac{P(x, 1)}{L_\lambda(x, 1)} = \lim_{x \rightarrow +\infty} \frac{\lambda^{1/\lambda}(1-1/x) \log^{1/\lambda} x}{\pi(1-x^{-\lambda})^{1/\lambda}} = +\infty \tag{3.4}$$

for all $\lambda > 0$.

Equation (3.4) implies that $\lambda > 0$ such that $L_\lambda(a, b) > P(a, b)$ for all $a, b > 0$ does not exist.

Theorem 3.2 *The inequality*

$$T(a, b) > L_5(a, b)$$

holds for all $a, b > 0$ with $a \neq b$, and $L_5(a, b)$ is the best possible lower generalized logarithmic mean bound for the second Seiffert mean $T(a, b)$.

Proof Without loss of generality, we assume that $b = 1$ and $a = x > 1$. Then (1.3) and (1.4) lead to

$$L_5(x, 1) - T(x, 1) = \left(\frac{x^5 - 1}{5 \log x}\right)^{1/5} - \frac{x - 1}{2 \arctan \frac{x-1}{x+1}}. \tag{3.5}$$

Therefore, $T(x, 1) > L_5(x, 1)$ follows from Lemma 2.2 and equation (3.5).

Next, we prove that $L_5(a, b)$ is the best possible lower generalized logarithmic mean bound for the second Seiffert mean $T(a, b)$.

For any $\epsilon > 0$ and $x > 0$, from (1.3) and (1.4), one has

$$L_{5+\epsilon}(1+x, 1) - T(1+x, 1) = \left[\frac{(1+x)^{5+\epsilon} - 1}{(5+\epsilon)\log(1+x)} \right]^{1/(5+\epsilon)} - \frac{x}{2 \arctan \frac{x}{2+x}}. \tag{3.6}$$

Letting $x \rightarrow 0$ and making use of Taylor expansion, we have

$$\begin{aligned} & \left[\frac{(1+x)^{5+\epsilon} - 1}{(5+\epsilon)\log(1+x)} \right]^{1/(5+\epsilon)} - \frac{x}{2 \arctan \frac{x}{2+x}} \\ &= \left[1 + \frac{5+\epsilon}{2}x + \frac{(5+\epsilon)(7+2\epsilon)}{12}x^2 + o(x^2) \right]^{1/(5+\epsilon)} - \frac{x}{x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + o(x^3)} \\ &= \left[1 + \frac{1}{2}x + \frac{2+\epsilon}{24}x^2 + o(x^2) \right] - \left[1 + \frac{1}{2}x + \frac{1}{12}x^2 + o(x^2) \right] \\ &= \frac{\epsilon}{24}x^2 + o(x^2). \end{aligned} \tag{3.7}$$

Equations (3.6) and (3.7) imply that for any $\epsilon > 0$, there exists $\delta_2 = \delta_2(\epsilon) > 0$ such that $L_{5+\epsilon}(1+x, 1) > T(1+x, 1)$ for $x \in (0, \delta_2)$. \square

Remark 3.2 It follows from (1.3) and (1.4) that

$$\lim_{x \rightarrow +\infty} \frac{T(x, 1)}{L_\mu(x, 1)} = \lim_{x \rightarrow +\infty} \frac{2\mu^{1/\mu}(1-1/x)\log^{1/\mu} x}{\pi(1-x^{-\mu})^{1/\mu}} = +\infty \tag{3.8}$$

for all $\mu > 0$.

Equation (3.8) implies that $\mu > 0$ such that $L_\mu(a, b) > T(a, b)$ for all $a, b > 0$ does not exist.

Theorem 3.3 *The inequality*

$$M(a, b) > L_4(a, b)$$

holds for all $a, b > 0$ with $a \neq b$, and $L_4(a, b)$ is the best possible lower generalized logarithmic mean bound for the Neuman-Sándor mean $M(a, b)$.

Proof Without loss of generality, we assume that $b = 1$ and $a = x > 1$. Then (1.1) and (1.4) lead to

$$\begin{aligned} & L_4^4(x, 1) - M^4(x, 1) \\ &= \frac{x^4 - 1}{4 \log x} - \frac{(x-1)^4}{16 \operatorname{arcsinh}^4\left(\frac{x-1}{x+1}\right)} \\ &= \frac{x^4 - 1}{4 \operatorname{arcsinh}^4\left(\frac{x-1}{x+1}\right) \log x} \left[\operatorname{arcsinh}^4\left(\frac{x-1}{x+1}\right) - \frac{(x-1)^4 \log x}{4(x^4 - 1)} \right]. \end{aligned} \tag{3.9}$$

Therefore, $M(x, 1) > L_4(x, 1)$ follows from Lemma 2.3 and equation (3.9).

Next, we prove that $L_4(a, b)$ is the best possible lower generalized logarithmic mean bound for the Neuman-Sándor mean $M(a, b)$.

For any $\epsilon > 0$ and $x > 0$, from (1.1) and (1.4), one has

$$L_{4+\epsilon}(1+x, 1) - T(1+x, 1) = \left[\frac{(1+x)^{4+\epsilon} - 1}{(4+\epsilon)\log(1+x)} \right]^{1/(4+\epsilon)} - \frac{x}{2 \operatorname{arcsinh}(\frac{x}{2+x})}. \tag{3.10}$$

Letting $x \rightarrow 0$ and making use of Taylor expansion, we have

$$\begin{aligned} & \left[\frac{(1+x)^{4+\epsilon} - 1}{(4+\epsilon)\log(1+x)} \right]^{1/(4+\epsilon)} - \frac{x}{2 \operatorname{arcsinh}(\frac{x}{2+x})} \\ &= \left[1 + \frac{4+\epsilon}{2}x + \frac{(4+\epsilon)(5+2\epsilon)}{12}x^2 + o(x^2) \right]^{1/(4+\epsilon)} - \frac{x}{x - \frac{1}{2}x^2 + \frac{5}{24}x^3 + o(x^3)} \\ &= \left[1 + \frac{1}{2}x + \frac{1+\epsilon}{24}x^2 + o(x^2) \right] - \left[1 + \frac{1}{2}x + \frac{1}{24}x^2 + o(x^2) \right] \\ &= \frac{\epsilon}{24}x^2 + o(x^2). \end{aligned} \tag{3.11}$$

Equations (3.10) and (3.11) imply that for any $\epsilon > 0$, there exists $\delta_3 = \delta_3(\epsilon) > 0$ such that $L_{4+\epsilon}(1+x, 1) > M(1+x, 1)$ for $x \in (0, \delta_3)$. □

Remark 3.3 It follows from (1.1) and (1.4) that

$$\lim_{x \rightarrow +\infty} \frac{M(x, 1)}{L_\nu(x, 1)} = \lim_{x \rightarrow +\infty} \frac{\nu^{1/\nu}(1-1/x)\log^{1/\nu} x}{2 \operatorname{arcsinh}(1)(1-x^{-\nu})^{1/\nu}} = +\infty \tag{3.12}$$

for all $\nu > 0$.

Equation (3.12) implies that $\nu > 0$ such that $L_\nu(a, b) > M(a, b)$ for all $a, b > 0$ does not exist.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

YMC provided the main idea and carried out the proof of Remarks 3.1-3.3 in this article. BYL carried out the proof of Lemma 2.1 and Theorem 3.1 in this article. WMG carried out the proof of Lemma 2.2 and Theorem 3.2 in this article. YQS carried out the proof of Lemma 2.3 and Theorem 3.3 in this article. All authors read and approved the final manuscript.

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