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A priori error estimates for higher order variational discretization and mixed finite element methods of optimal control problems

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Abstract

In this article, we investigate *a priori* error estimates for the optimal control problems governed by elliptic equations using higher order variational discretization and mixed finite element methods. The state and the co-state are approximated by the order *k* Raviart-Thomas mixed finite element spaces and the control is not discreted. *A priori* error estimates for the higher order variational discretization and mixed finite element approximation of control problems are obtained. Finally, we present some numerical examples which confirm our theoretical results.

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1. Introduction

Optimal control problems governed by elliptic equations are important problems in engineering applications. Efficient numerical methods are critical for successful applications of optimal control problems in such cases. Recently, the finite element method of optimal control problems plays an important role in numerical methods for these problems. Systematic introduction of the finite element method for optimal control problems can be found in, for example, [1]. The finite element approximation of optimal control problem by piecewise constant functions is well investigated by Falk [2] and Geveci [3]. Arada et al. [4] discussed the discretization for semilinear elliptic optimal control problems. In [5], Malanowski discussed a constrained parabolic optimal control problems.

In many control problems, the objective functional contains gradient of the state variables. Thus accuracy of gradient is important in numerical approximation of the state equations. In the finite element community, mixed finite element methods should be used for discretization of the state equations in such cases. In computational optimal control problems, mixed finite element methods are not widely used in engineering simulations. In particular there doesn't seem to exist much work on theoretical analysis of mixed finite element approximation of optimal control problems in the literature. More recently, we have done some preliminary work on sharp a *posteriori* error estimates, error estimates and superconvergence of mixed finite element methods for optimal control problems (see, for example, [6-13]).

In [14], the author first presents the variational discretization concept for optimal control problems with control constraints, which implicitly utilizes the first order optimality conditions and the discretization of the state and adjoint equations for the discretization of the control instead of discretizing the space of admissible controls.

For $1 \le p < \infty$ and m, any nonnegative integer, let $W^{m,p}(\Omega) = \{v \in L^p(\Omega); D^\beta v \in L^p(\Omega)$ if $|\beta| \le m\}$ denote the Sobolev spaces endowed with the norm $\|v\|_{m,p}^p = \sum_{|\beta|\le m} \|D^\beta v\|_{L^p(\Omega)}^p$, and the semi-norm $|v|_{m,p}^p = \sum_{|\beta|=m} \|D^\beta v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$.

In this article we derive *a priori* error estimates for higher order variational discretization and mixed finite element methods of quadratic optimal control problems.

We consider the following quadratic optimal control problems:

$$\min_{u\in K\subset U} \left\{ \frac{1}{2}\|\boldsymbol{p} - \boldsymbol{p}_d\|^2 + \frac{1}{2}\|y - y_d\|^2 + \frac{1}{2}\|u\|^2 \right\} \tag{1.1}$$

subject to the state equation

$$\operatorname{div}\boldsymbol{p} = f + Bu, \quad x \in \Omega, \tag{1.2}$$

$$\boldsymbol{p} = -A\nabla y, \quad x \in \Omega, \tag{1.3}$$

$$y = 0, \quad x \in \partial\Omega, \tag{1.4}$$

where the bounded open set $\Omega \subset \mathbb{R}^2$, is a convex domain with the boundary $\partial\Omega$. We shall assume that $\boldsymbol{p}_d \in L^2(\Omega)^2$, $y_d \in L^2(\Omega)$, $f \in H^1(\Omega)$, and B is a continuous linear operator from $U = L^2(\Omega)$ to $H^1(\Omega)$. Furthermore, we assume the coefficient matrix $A(x) = (a_{i,j}(x))_{2\times 2} \in (W^{1,\infty}(\Omega))^{2\times 2}$ is a symmetric 2×2-matrix and there is a constant $c > 0$ satisfying for any vector $\mathrm{X} \in \mathbb{R}^2, \mathrm{X}'A\mathrm{X} \ge c\|\mathrm{X}\|_{\mathbb{R}^2}^2$. Here, K denotes the admissible set of the control variable, defined by

$$K = \left\{ u \in U = L^2(\Omega) : \int_\Omega u \ge 0 \right\}. \tag{1.5}$$

Optimal control problems have been so widely met in all kinds of practical problems. Now we mention their application of the optimal control problems (1.1)-(1.4). Let us recall the static temperature control problem. Let y be the temperature distribution in the body Ω, which satisfies the elliptic Equations (1.2) and (1.3), where A is conductivity matrix and Bu represents heat resource density inside the body, which can be controlled by u. For example, often $(Bu)(x) = c(x)u(x)$, where $c(x)$ is the density factor. Assume that the body's surface temperature is fixed, say, zero. The aim of the control is to make the temperature distribution y as close as to the desirable distribution y_d, e.g., $y_d = 10$. Of course there could be many controls to achieve this objective, but we wish to achieve this using as small as possible energy, which can be represented by the formula $\int_\Omega u^2$. Then we give the temperature control model for the problem (1.1)-(1.4).

Now, we introduce the co-state elliptic equation

$$-\mathrm{div}(A(\nabla z + \boldsymbol{p} - \boldsymbol{p}_d)) = y - y_d, \quad x \in \Omega, \tag{1.6}$$

with the boundary condition

$$z = 0, \quad x \in \partial\Omega. \tag{1.7}$$

The outline of this article is as follows. In Section 2, we construct the higher order variational discretization and mixed finite element approximation for optimal control problems governed by elliptic equations. Furthermore, we briefly state the definitions and properties of some interpolation operators. In Section 3, we derive *a priori* error estimates for the higher order variational discretization and mixed finite element solutions of the optimal control problems. Numerical examples are presented in Section 4. Finally, we analyze the conclusion and the future studies in Section 5.

2. Variational discretization and mixed finite element methods

We shall now describe the variational discretization and mixed finite element approximation of the optimal control problems (1.1)-(1.4). Let

$$\boldsymbol{V} = H(\mathrm{div}; \Omega) = \{\boldsymbol{v} \in (L^2(\Omega))^2, \mathrm{div}\boldsymbol{v} \in L^2(\Omega)\}, \quad W = L^2(\Omega).$$

The Hilbert space $\boldsymbol{V}$ is equipped with the following norm:

$$\|\boldsymbol{v}\|_{\mathrm{div}} = \|\boldsymbol{v}\|_{H(\mathrm{div};\Omega)} = \left(\|\boldsymbol{v}\|_{0,\Omega}^2 + \|\mathrm{div}\boldsymbol{v}\|_{0,\Omega}^2\right)^{1/2}.$$

We recast (1.1)-(1.4) as the following weak form: find $(\boldsymbol{p}, y, u) \in \boldsymbol{V} \times W \times U$ such that

$$\min_{u \in K \subset U} \left\{ \frac{1}{2}\|\boldsymbol{p} - \boldsymbol{p}_d\|^2 + \frac{1}{2}\|y - y_d\|^2 + \frac{1}{2}\|u\|^2 \right\} \tag{2.1}$$

$$(A^{-1}\boldsymbol{p}, \boldsymbol{v}) - (y, \mathrm{div}\boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{2.2}$$

$$(\mathrm{div}\boldsymbol{p}, w) = (f + Bu, w), \quad \forall w \in W. \tag{2.3}$$

It is well known (see e.g., [15,16]) that the optimal control problem (2.1)-(2.3) has a solution $(\boldsymbol{p}, y, u)$, and that a triplet $(\boldsymbol{p}, y, u)$ is the solution of (2.1)-(2.3) if and only if there is a co-state $(\boldsymbol{q}, z) \in \boldsymbol{V} \times W$ such that $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ satisfies the following optimality conditions:

$$(A^{-1}\boldsymbol{p}, \boldsymbol{v}) - (y, \mathrm{div}\boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{2.4}$$

$$(\mathrm{div}\boldsymbol{p}, w) = (f + Bu, w), \quad \forall w \in W, \tag{2.5}$$

$$(A^{-1}\boldsymbol{q}, \boldsymbol{v}) - (z, \mathrm{div}\boldsymbol{v}) = -(\boldsymbol{p} - \boldsymbol{p}_d, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{2.6}$$

$$(\mathrm{div}\boldsymbol{q}, w) = (y - y_d, w), \quad \forall w \in W, \tag{2.7}$$

$$(u + B^*z, \tilde{u} - u)_U \geq 0, \quad \forall \tilde{u} \in K, \tag{2.8}$$

where $(\cdot, \cdot)_U$ is the inner product of U, B^* is the adjoint operator of B. In the rest of the article, we shall simply write the product as $(\cdot, \cdot)$ whenever no confusion should be caused.

Now, we will show that the control variable of the optimal control problem (2.4)-(2.8) can be infinitely smooth if the special constraint set K defined as (1.5).

Lemma 2.1. *Let* $(\boldsymbol{p}, y, \boldsymbol{q}, z, u) \in (\boldsymbol{V} \times W)^2 \times K$ *be the solution of (2.4)-(2.8). Then we have*

$$u = \max\left(0, \overline{B^*z}\right) - B^*z, \tag{2.9}$$

where $\overline{B^*z} = \int_\Omega B^*z / |\Omega|$ *denotes the integral average on* Ω *of the function* z.

Proof. If $\overline{B^*z} > 0$, then $u = \overline{B^*z} - B^*z$ and for any $v \in K$

$$\begin{aligned}(u + B^*z, v - u) &= \int_\Omega (u + B^*z)(v - u) \\ &= \int_\Omega \overline{B^*z}\left(v - \overline{B^*z} + B^*z\right) \\ &= \overline{B^*z} \int_\Omega v \geq 0.\end{aligned}$$

If $\overline{B^*z} \leq 0$, then $u = -B^* z$ and $(u + B^* z, v - u) = 0$. Now, for the costate solution z, since the solution of (2.8) is unique, then we have proved the Lemma.

From Lemma 3.1, we obtain the following regularity result for the control variable.

Lemma 2.2. *Let* $(\boldsymbol{p}, y, \boldsymbol{q}, z, u) \in (\boldsymbol{V} \times W)^2 \times K$ *be the solution of* (2.4)-(2.8). *Assume that the data functions* f, y_d, $\boldsymbol{p}_d$, *and the domain* Ω *are infinitely smooth. Then the control function* $u \in C^\infty(\bar{\Omega})$.

Proof. By applying the regularity argument of elliptic problem (1.2)-(1.3), it is clear that $y \in H^2(\Omega)$, so that $\boldsymbol{p} \in H^1(\Omega)$. It follows from the costate Equation (1.6) and the assumption of y_d, $\boldsymbol{p}_d$, we can obtain that $z \in H^2(\Omega)$. Using the relationship between the control and the costate $u = \max\left(0, \overline{B^*z}\right) - B^*z$, then $u \in H^2(\Omega)$. Thus $y \in H^4(\Omega)$, $\boldsymbol{p} \in H^3(\Omega)$. By repeating the above process, we can conclude that $u \in C^\infty(\bar{\Omega})$.

Let $\mathcal{T}_h$ be regular triangulation of Ω. They are assumed to satisassociated with the triangulationfy the angle condition which means that there is a positive constant C such that for all $T \in \mathcal{T}_h, C^{-1}h_T^2 \leq |T| \leq Ch_T^2$, where $|T|$ is the area of T, h_T is the diameter of T and $h = \max h_T$. In addition C or c denotes a general positive constant independent of h.

Let $\boldsymbol{V}_h \times W_h \subset \boldsymbol{V} \times W$ denotes the Raviart-Thomas space [17] of the lowest order associated with the triangulation $\mathcal{T}_h$ of Ω. P_k denotes the space of polynomials of total degree at most k. Let $\boldsymbol{V}(T) = \{\boldsymbol{v} \in P_k^2(T) + x \cdot P_k(T)\}$, $W(T) = P_k(T)$. We define

$$\begin{aligned}\boldsymbol{V}_h &:= \{\boldsymbol{v}_h \in \boldsymbol{V} : \forall T \in \mathcal{T}_h, \boldsymbol{v}_h\big|_T \in \boldsymbol{V}(T)\}, \\ W_h &:= \{w_h \in W : \forall T \in \mathcal{T}_h, w_h\big|_T \in W(T)\}.\end{aligned}$$

By the definition of finite element subspace, the mixed finite element discretization of (2.1)-(2.3) is as follows: compute $(\boldsymbol{p}_h, y_h, u_h) \in \boldsymbol{V}_h \times W_h \times K$ such that

$$\min_{u_h \in K} \left\{ \frac{1}{2} \left\| \boldsymbol{p}_h - \boldsymbol{p}_d \right\|^2 + \frac{1}{2} \left\| y_h - y_d \right\|^2 + \frac{1}{2} \| u_h \|^2 \right\} \tag{2.10}$$

$$(A^{-1}\boldsymbol{p}_h, \boldsymbol{v}_h) - (y_h, \operatorname{div}\boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.11}$$

$$(\operatorname{div}\boldsymbol{p}_h, w_h) = (f + Bu_h, w_h), \qquad \forall w_h \in W_h. \tag{2.12}$$

It is well known that the optimal control problem (2.10)-(2.12) again has a solution $(\boldsymbol{p}_h, y_h, u_h)$, and that a triplet $(\boldsymbol{p}_h, y_h, u_h)$ is the solution of (2.10)-(2.12) if and only if there is a co-state $(\boldsymbol{q}_h, z_h) \in \boldsymbol{V}_h \times W_h$ such that $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h)$ satisfies the following optimality conditions:

$$(A^{-1}\boldsymbol{p}_h, \boldsymbol{v}_h) - (y_h, \operatorname{div}\boldsymbol{v}_h) = 0, \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.13}$$

$$(\operatorname{div}\boldsymbol{p}_h, w_h) = (f + Bu_h, w_h), \qquad \forall w_h \in W_h, \tag{2.14}$$

$$(A^{-1}\boldsymbol{q}_h, \boldsymbol{v}_h) - (z_h, \operatorname{div}\boldsymbol{v}_h) = -(\boldsymbol{p}_h - \boldsymbol{p}_d, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.15}$$

$$(\operatorname{div}\boldsymbol{q}_h, w_h) = (y_h - y_d, w_h), \qquad \forall w_h \in W_h, \tag{2.16}$$

$$(u_h + B^* z_h, \tilde{u} - u_h) \geq 0, \qquad \forall \tilde{u} \in K. \tag{2.17}$$

Let $P_h : W \to W_h$ *be* the orthogonal $L^2(\Omega)$-projection into W_h define by [18]:

$$(P_h w - w, \mathcal{X}) = 0, \quad w \in W, \quad \mathcal{X} \in W_h, \tag{2.18}$$

which satisfies

$$\| P_h w - w \|_{0,q} \leq C \| w \|_{t,q} h^t, \quad 0 \leq t \leq k+1, \quad \text{if } w \in W \cap W^{t,q}(\Omega), \tag{2.19}$$

$$\| P_h w - w \|_{-r} \leq C \| w \|_t h^{r+t}, \quad 0 \leq r, \quad t \leq k+1, \quad \text{if } w \in H^t(\Omega), \tag{2.20}$$

$$(\operatorname{div}\boldsymbol{v}_h, w - P_h w) = 0, \quad w \in W, \quad \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{2.21}$$

Let $\pi_h : \boldsymbol{V} \to \boldsymbol{V}_h$ be the Raviart-Thomas projection [19], which satisfies

$$(\operatorname{div}(\pi_h \boldsymbol{v} - \boldsymbol{v}), w_h) = 0, \quad \boldsymbol{v} \in \boldsymbol{V}, \quad w \in W_h, \tag{2.22}$$

$$\| \pi_h \boldsymbol{v} - \boldsymbol{v} \|_{0,q} \leq C \| \boldsymbol{v} \|_{t,q} h^t, \quad 1/q < t \leq k+1, \quad \text{if } \boldsymbol{v} \in \boldsymbol{V} \cap W^{t,q}(\Omega)^2, \tag{2.23}$$

$$\left\| \operatorname{div}(\pi_h \boldsymbol{v} - \boldsymbol{v}) \right\|_0 \leq C \left\| \operatorname{div}\boldsymbol{v} \right\|_t h^t, \quad 0 \leq t \leq k+1, \quad \text{if } \boldsymbol{v} \in \boldsymbol{V} \cap H^t(\operatorname{div}; \Omega). \tag{2.24}$$

We have the commuting diagram property [20]

$$\operatorname{div} \circ \pi_h = P_h \circ \operatorname{div} : \boldsymbol{V} \to W_h \quad \text{and} \quad \operatorname{div}(I - \pi_h)\boldsymbol{V} \perp W_h, \tag{2.25}$$

where and after, I denotes identity matrix.

3. *A priori* error estimates

In the rest of the article, we shall use some intermediate variables. For any control function $\tilde{u} \in K$, we first define the state solution $(\boldsymbol{p}(\tilde{u}), y(\tilde{u}), \boldsymbol{q}(\tilde{u}), z(\tilde{u}))$ associated with $\tilde{u}$ that satisfies

$$(A^{-1}\boldsymbol{p}(\tilde{u}), \boldsymbol{v}) - (y(\tilde{u}), \operatorname{div}\boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{3.1}$$

$$(\operatorname{div}\boldsymbol{p}(\tilde{u}), w) = (f + B\tilde{u}, w), \quad \forall w \in W, \tag{3.2}$$

$$(A^{-1}\boldsymbol{q}(\tilde{u}), v) - (z(\tilde{u}), \operatorname{div}v) = -(\boldsymbol{p}(\tilde{u}) - \boldsymbol{p}_d, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{3.3}$$

$$(\operatorname{div}\boldsymbol{q}(\tilde{u}), w) = (y(\tilde{u}) - y_d, w), \quad \forall w \in W. \tag{3.4}$$

Correspondingly, we define the discrete state solution $(\boldsymbol{p}_h(\tilde{u}), y_h(\tilde{u}), \boldsymbol{q}_h(\tilde{u}), z_h(\tilde{u}))$ associated with $\tilde{u} \in K$ that satisfies

$$(A^{-1}\boldsymbol{p}_h(\tilde{u}), \boldsymbol{v}_h) - (y_h(\tilde{u}), \operatorname{div}\boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.5}$$

$$(\operatorname{div}\boldsymbol{p}_h(\tilde{u}), w_h) = (f + B\tilde{u}, w_h), \quad \forall w_h \in W_h, \tag{3.6}$$

$$(A^{-1}\boldsymbol{q}_h(\tilde{u}), \boldsymbol{v}_h) - (z_h(\tilde{u}), \operatorname{div}\boldsymbol{v}_h) = -(\boldsymbol{p}_h(\tilde{u}) - \boldsymbol{p}_d, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.7}$$

$$(\operatorname{div}\boldsymbol{q}_h(\tilde{u}), w_h) = (y_h(\tilde{u}) - y_d, w_h), \quad \forall w_h \in W_h. \tag{3.8}$$

We define another discrete state solution $(\hat{\boldsymbol{p}}_h(u), \hat{z}_h(u))$ that satisfies

$$(A^{-1}\hat{\boldsymbol{q}}_h(\tilde{u}), \boldsymbol{v}_h) - (\hat{z}_h(\tilde{u}), \operatorname{div}\boldsymbol{v}_h) = -(\boldsymbol{p} - \boldsymbol{p}_d, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.9}$$

$$(\operatorname{div}\hat{\boldsymbol{q}}_h(\tilde{u}), w_h) = (y - y_d, w_h), \quad \forall w_h \in W_h. \tag{3.10}$$

Thus, as we defined, the exact solution and its approximation can be written in the following way:

$$\begin{aligned}(\boldsymbol{p}, y, \boldsymbol{q}, z) &= (\boldsymbol{p}(u), y(u), \boldsymbol{q}(u), z(u)),\\ (\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h) &= (\boldsymbol{p}_h(u_h), y_h(u_h), \boldsymbol{q}_h(u_h), z_h(u_h)).\end{aligned}$$

Combining Lemma 2.1 in [19] and (2.19), we obtain the following technical results:

Lemma 3.1. *Let* $\boldsymbol{\omega} \in \boldsymbol{V}$, $\phi \in L^2(\Omega)^2$, *and* $\psi \in L^2(\Omega)$. *If* $\tau \in W_h$ *satisfies*

$$\begin{cases}(A^{-1}\boldsymbol{\omega}, \boldsymbol{v}_h) - (\tau, \operatorname{div}\boldsymbol{v}_h) = (\boldsymbol{\varphi}, v_h), & \forall v_h \in \boldsymbol{V}_h,\\ (\operatorname{div}\boldsymbol{\omega}, w_h) = (\psi, w_h), & \forall w_h \in W_h,\end{cases}$$

then, there exists a constant C such that

$$\|\tau\|_0 \le C\left(h\|\boldsymbol{\omega}\|_0 + h\left\|\operatorname{div}\boldsymbol{\omega}\right\|_0 + \|\boldsymbol{\varphi}\|_0 + \|\psi\|_0\right), \tag{3.11}$$

for h sufficiently small.

Now we chose $\tilde{u} = u$ in (3.5)-(3.8), then we set some intermediate errors:

$$\varepsilon_1 := \boldsymbol{p} - \boldsymbol{p}_h(u) \quad \text{and} \quad e_1 := y - y_h(u). \tag{3.12}$$

To analyze the intermediate errors, let us first note the following error equations from (2.4), (2.5), (3.5) and (3.6) with the choice $\tilde{u} = u$:

$$(A^{-1}\varepsilon_1, \boldsymbol{v}_h) - (e_1, \mathrm{div}\boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.13}$$

$$(\mathrm{div}\varepsilon_1, w_h) = 0, \quad \forall w_h \in W_h. \tag{3.14}$$

By (2.18)-(2.24) and Lemma 3.1, we can establish the following error estimates:

Theorem 3.1. *Assume that* $y \in H^{k+3}(\Omega)$. *If h is sufficiently small, there is a positive constant C independent of h such that*

$$\|y - y_h(u)\|_0 \le Ch^{k+1}, \tag{3.15}$$

$$\|\boldsymbol{p} - \boldsymbol{p}_h(u)\|_0 \le Ch^{k+1}, \tag{3.16}$$

$$\|\boldsymbol{p} - \boldsymbol{p}_h(u)\|_{\mathrm{div}} \le Ch^{k+1}. \tag{3.17}$$

Proof. Let $\tau = P_h y - y_h(u)$ and $\sigma = \pi_h \boldsymbol{p} - \boldsymbol{p}_h(u)$. Rewrite (3.13) and (3.14) in the form

$$(A^{-1}\varepsilon_1, \boldsymbol{v}_h) - (\tau, \mathrm{div}\boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.18}$$

$$(\mathrm{div}\varepsilon_1, w_h) = 0, \quad \forall w_h \in W_h. \tag{3.19}$$

It follows from Lemma 3.1 that

$$\|\tau\|_0 \le C\left(h\|\varepsilon_1\|_0 + h\|\mathrm{div}\varepsilon_1\|_0\right). \tag{3.20}$$

By using (2.19) that

$$\begin{aligned} \|e_1\|_0 &= \|y - y_h(u)\|_0 \\ &= \|P_h y - y\|_0 + \|\tau\|_0 \\ &\le C\left(h\|\varepsilon_1\|_0 + h\|\mathrm{div}\varepsilon_1\|_0 + h^{k+1}\|y\|_{k+1}\right). \end{aligned} \tag{3.21}$$

If we now again rewrite (3.13) and (3.14) as

$$(A^{-1}\sigma, \boldsymbol{v}_h) - (\tau, \mathrm{div}\boldsymbol{v}_h) = (A^{-1}(\pi_h \boldsymbol{p} - \boldsymbol{p}), \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.22}$$

$$(\mathrm{div}\sigma, w_h) = 0, \quad \forall w_h \in W_h. \tag{3.23}$$

Using the standard stability results of mixed finite element methods in [21], we establish the following results:

$$\begin{aligned} \|\sigma\|_{\mathrm{div}} &\le C\left(\|\pi_h \boldsymbol{p} - \boldsymbol{p}\|_0 + \|e_1\|_0\right) \\ &\le C\left(h^{k+1}\|y\|_{k+2} + \|e_1\|_0\right). \end{aligned} \tag{3.24}$$

From (3.24), (2.23), and the commuting diagram property (2.25) we now obtain the bounds

$$\begin{aligned}\|\varepsilon_1\|_0 &\le C\left(\left\|\pi_h \boldsymbol{p} - \boldsymbol{p}\right\|_0 + \|\sigma\|_0\right)\\ &\le C\left(h^{k+1}\left\|\gamma\right\|_{k+2} + \|e_1\|_0\right),\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}\left\|\mathrm{div}\varepsilon_1\right\|_0 &\le C\left(\left\|\mathrm{div}(\pi_h \boldsymbol{p} - \boldsymbol{p})\right\|_0 + \left\|\mathrm{div}\sigma\right\|_0\right)\\ &= C\left(\left\|P_h \circ \mathrm{div}\boldsymbol{p} - \mathrm{div}\boldsymbol{p}\right\|_0 + \left\|\mathrm{div}\sigma\right\|_0\right)\\ &\le C\left(h^{k+1}\left\|\gamma\right\|_{k+3} + \|e_1\|\,0\right),\end{aligned} \tag{3.26}$$

when substituted into (3.21), which yields the estimate

$$\|e_1\|_0 \le C\left(h\|e_1\|_0 + h^{k+1}\left\|\gamma\right\|_{k+1}\right). \tag{3.27}$$

Then (3.27) implies (3.15) holds if h is small enough. Applying (3.27) to (3.25) and (3.26) shows that (3.16) and (3.17) also hold.

With the intermediate errors, we decompose the errors as follows

$$\boldsymbol{p} - \boldsymbol{p}_h = \boldsymbol{p} - \boldsymbol{p}_h(u) + \boldsymbol{p}_h(u) - \boldsymbol{p}_h := \varepsilon_1 + \epsilon_1, \tag{3.28}$$

$$\gamma - \gamma_h = \gamma - \gamma_h(u) + \gamma_h(u) - \gamma_h := e_1 + r_1. \tag{3.29}$$

From (2.13), (2.14), (3.5) and (3.6), we have

$$(A^{-1}\epsilon_1, \boldsymbol{v}_h) - (r_1, \mathrm{div}\boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.30}$$

$$(\mathrm{div}\epsilon_1, w_h) = (B(u - u_h), w_h), \quad \forall w_h \in W_h. \tag{3.31}$$

The assumption that $A \in L^{\infty}(\Omega;\mathbb{R}^{2\times 2})$ implies that it is bounded that the inverse operator of the map $\{\epsilon_1, r_1\}$: $\mathbb{R}^3 \to \boldsymbol{V} \times W$ defined by the above saddle-point problem [21]:

$$\|\epsilon_1\|_{\mathrm{div}} + \|r_1\|_0 \le C\|u - u_h\|_{0,} \tag{3.32}$$

where the continuity of the linear operator B has been used.

Now we are able to derive our main results.

Theorem 3.2. *Let* $(\boldsymbol{p}, y, \boldsymbol{q}, z, u) \in (\boldsymbol{V} \times W)^2 \times K$ *and* $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h) \in (\boldsymbol{V}_h \times W_h)^2 \times K$ *be the solutions of*(2.4)-(2.8) *and* (2.13)-(2.17), *respectively. We assume thaty,* $z \in H^{k+3}(\Omega)$.

Then, we have

$$\|u - u_h\|\,0 \le Ch^{k+1}, \tag{3.33}$$

$$\left\|\boldsymbol{p} - \boldsymbol{p}_h\right\|_{\mathrm{div}} + \left\|\gamma - \gamma_h\right\|_0 \le Ch^{k+1}, \tag{3.34}$$

$$\left\|\boldsymbol{q} - \boldsymbol{q}_h\right\|_{\mathrm{div}} + \|z - z_h\|_0 \le Ch^{k+1}. \tag{3.35}$$

Proof. We choose $\tilde{u} = u_h$ in (2.8) and $\tilde{u} = u$ in (2.17) to get that

$$(u + B^*z, u_h - u) \ge 0, \tag{3.36}$$

and

$$(u_h + B^* z_h, u - u_h) \geq 0. \tag{3.37}$$

Then we have

$$\begin{aligned} \|u - u_h\|^2 &\leq (B^* z_h - B^* z, u - u_h) \\ &= (B^* \hat{z}_h(u) - B^* z, u - u_h) + (B^* z_h - B^* \hat{z}_h(u), u - u_h). \end{aligned} \tag{3.38}$$

Moreover, the δ-Caunchy inequality leads that

$$\begin{aligned} (B^* \hat{z}_h(u) - B^* z, u - u_h) &\leq C \left\| \hat{z}_h(u) - z \right\| \cdot \|u - u_h\| \\ &\leq C \left\| \hat{z}_h(u) - z \right\|^2 + C\delta \|u - u_h\|^2, \end{aligned} \tag{3.39}$$

where δ is an arbitrary positive number. From (3.30)-(3.31), (2.15)-(2.16) and (3.9)-(3.10), we obtain the following equations:

$$(A^{-1}(\boldsymbol{p}_h - \boldsymbol{p}_h(u)), \boldsymbol{v}_h) - (\gamma_h - \gamma_h(u), \operatorname{div} \boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.40}$$

$$(\operatorname{div}(\boldsymbol{p}_h - \boldsymbol{p}_h(u)), w_h) = (B(u - u_h), w_h), \qquad \forall w_h \in W_h, \tag{3.41}$$

$$(A^{-1}(\hat{\boldsymbol{q}}_h(u) - \boldsymbol{q}_h), \boldsymbol{v}_h) - (\hat{z}_h(u) - z_h, \operatorname{div} \boldsymbol{v}_h) = -(\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.42}$$

$$(\operatorname{div}(\hat{\boldsymbol{q}}_h(u) - \boldsymbol{q}_h), w_h) = (\gamma - \gamma_h, w_h), \qquad \forall w_h \in W_h. \tag{3.43}$$

By applying the above error equations, we obtain

$$\begin{aligned} &(B^* z_h - B^* \hat{z}_h(u), u - u_h) \\ &= (z_h - \hat{z}_h(u), B(u - u_h)) \\ &= (\operatorname{div}(\boldsymbol{p}_h - \boldsymbol{p}_h(u)), z_h - \hat{z}_h(u)) \\ &= (A^{-1}(\hat{\boldsymbol{q}}_h(u) - \boldsymbol{q}_h), \boldsymbol{p}_h - \boldsymbol{p}_h(u)) + (\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{p}_h - \boldsymbol{p}_h(u)) \\ &= (\gamma_h - \gamma_h(u), \operatorname{div}(\hat{\boldsymbol{q}}_h(u) - \boldsymbol{q}_h) + (\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{p}_h - \boldsymbol{p}_h(u)) \\ &= (\gamma - \gamma_h, \gamma_h - \gamma_h(u)) + (\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{p}_h - \boldsymbol{p}_h(u)) \\ &= (\gamma - \gamma_h, \gamma - \gamma_h(u)) + (\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{p} - \boldsymbol{p}_h(u)) - \left\| \gamma - \gamma_h \right\|^2 - \left\| \boldsymbol{p} - \boldsymbol{p}_h \right\|^2 \\ &= \left\| \gamma - \gamma_h \right\| \cdot \left\| \gamma - \gamma_h(u) \right\| + \left\| \boldsymbol{p} - \boldsymbol{p}_h \right\| \cdot \left\| \boldsymbol{p} - \boldsymbol{p}_h(u) \right\| - \left\| \gamma - \gamma_h \right\|^2 - \left\| \boldsymbol{p} - \boldsymbol{p}_h \right\|^2 \\ &\leq C \left\| \gamma - \gamma_h(u) \right\|^2 + C \left\| \boldsymbol{p} - \boldsymbol{p}_h(u) \right\|^2 - \frac{1}{2} \left\| \gamma - \gamma_h \right\|^2 - \frac{1}{2} \left\| \boldsymbol{p} - \boldsymbol{p}_h \right\|^2. \end{aligned} \tag{3.44}$$

From (3.38), (3.39), and (3.44), we derive that

$$\begin{aligned} &\|u - u_h\|^2 + \left\| \gamma - \gamma_h \right\|^2 + \left\| \boldsymbol{p} - \boldsymbol{p}_h \right\|^2 \\ &\leq C \left\| \gamma - \gamma_h(u) \right\|^2 + C \left\| \boldsymbol{p} - \boldsymbol{p}_h(u) \right\|^2 + C \left\| z - \hat{z}_h(u) \right\|^2. \end{aligned} \tag{3.45}$$

Note that $\hat{z}_h(u), \hat{\boldsymbol{q}}_h(u)$ are the mixed finite element approximation of z, $\boldsymbol{q}$, using the results of [22], we have

$$\left\| \boldsymbol{q} - \hat{\boldsymbol{q}}_h(u) \right\|_{\operatorname{div}} + \left\| z - \hat{z}_h(u) \right\| \leq C h^{k+1}. \tag{3.46}$$

From the Theorem 3.1, (3.45), and (3.46), we obtain

$$\|u - u_h\| + \|\gamma - \gamma_h\| + \|\boldsymbol{p} - \boldsymbol{p}_h\| \le Ch^{k+1}, \tag{3.47}$$

then we derive (3.33). By using (3.42) and (3.43) and the stability results of mixed finite element methods [21], we have

$$\|\hat{\boldsymbol{q}}_h(u) - \boldsymbol{q}_h\|_{\mathrm{div}} + \|\hat{z}_h(u) - z_h\| \le C\|\boldsymbol{p} - \boldsymbol{p}_h\| + C\|\gamma - \gamma_h\|. \tag{3.48}$$

Combining (3.46)-(3.48), we derive the following result

$$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{\mathrm{div}} + \|z - z_h\| \le Ch^{k+1}. \tag{3.49}$$

From (3.17), (3.32), and (3.47), it is easy to see that

$$\|\boldsymbol{p} - \boldsymbol{p}_h\|_{\mathrm{div}} \le Ch^{k+1}. \tag{3.50}$$

Then we derive the results (3.34) and (3.35).

4. Numerical tests

In this section, we are going to validate the *a priori* error estimates for the error in the control, state, and co-state numerically. The optimization problems were dealt numerically with codes developed based on AFEPACK. The package is freely available and the details can be found at [23].

In our numerical examples, we consider the following optimal control problems:

$$\min_{u\in K}\left\{\frac{1}{2}\|\boldsymbol{p} - \boldsymbol{p}_d\|^2 + \frac{1}{2}\|\gamma - \gamma_d\|^2 + \frac{1}{2}\|u\|^2\right\} \tag{4.1}$$

$$\mathrm{div}\boldsymbol{p} = Bu + f, \quad \boldsymbol{p} = -A\nabla\gamma, \quad x \in \Omega, \quad \gamma|_{\partial\Omega} = 0, \tag{4.2}$$

$$\mathrm{div}\boldsymbol{q} = \gamma - \gamma_d, \quad \boldsymbol{q} = -A(\nabla z + \boldsymbol{p} - \boldsymbol{p}_d), \quad x \in \Omega, \quad z|_{\partial\Omega} = 0. \tag{4.3}$$

In our examples, we choose the domain $\Omega = [0, 1] \times [0, 1]$ and $A = B = I$. We present below two examples to illustrate the theoretical results of the optimal control problems. The convergence order is computed by the following formula: order $\simeq \dfrac{\log(E_i/E_{i+1})}{\log(h_i/h_{i+1})}$, where i responds to the spatial partition, and E_i denotes the for the state, costate and control approximation.

Example 1. In this example we set the other known functions as follows:

$$\begin{aligned}
&\gamma = 2\sin\pi x_1 \sin\pi x_2,\\
&z = -\sin\pi x_1 \sin\pi x_2,\\
&\boldsymbol{q} = (\pi\cos\pi x_1 \sin\pi x_2, \pi\cos\pi x_2 \sin\pi x_1),\\
&\gamma_d = (1+\pi^2)\gamma, \quad \boldsymbol{p} = \boldsymbol{p}_d = -2\boldsymbol{q},\\
&f = 2\pi^2\gamma - u,\\
&u = \max(\bar{z}, 0) - z.
\end{aligned}$$

In this numerical implementation, the error $\|u - u_h\|_0$, $\|\boldsymbol{p} - \boldsymbol{p}_h\|_{\mathrm{div}}$, $\|y - y_h\|_0$, $\|\boldsymbol{q} - \boldsymbol{q}_h\|_{\mathrm{div}}$, and $\|z - z_h\|_0$ obtained on RT0 mixed finite element approximation and RT1 mixed finite element approximation for state function are presented in Tables 1, 2, 3

Table 1 The numerical errors on RT0 mixed finite element for state function

h	Errors									
	$\|u - u_h\|$	Rate	$\|p - p_h\|$	Rate	$\|y - y_h\|$	Rate	$\|q - q_h\|$	Rate	$\|z - z_h\|$	Rate
1/16	3.63e - 02	-	3.25e - 01	-	7.24e - 02	-	1.63e - 01	-	3.63e - 02	-
1/32	1.80e - 02	1.01	1.62e - 01	1.00	3.63e - 02	0.99	8.12e - 02	1.01	1.80e - 02	1.01
1/64	9.06e - 03	0.99	8.13e - 02	0.99	1.78e - 02	1.03	4.07e - 02	1.00	9.06e - 03	0.99
1/128	4.48e - 03	1.01	4.03e - 02	1.01	8.87e - 03	1.01	2.05e - 02	0.99	4.48e - 03	1.01

and 4. The theoretical results can be observed clearly from the data. The profile of the

Table 2 The numerical errors on RT1 mixed finite element for state function

h	Errors									
	$\|u - u_h\|$	Rate	$\|p - p_h\|$	Rate	$\|y - y_h\|$	Rate	$\|q - q_h\|$	Rate	$\|z - z_h\|$	Rate
1/16	1.25e - 03	-	7.13e - 03	-	2.50e - 03	-	3.53e - 03	-	1.25e - 03	-
1/32	3.13e - 04	2.00	1.76e - 03	2.02	6.24e - 04	2.00	8.82e - 04	2.00	3.13e - 04	2.00
1/64	7.68e - 05	2.02	4.43e - 04	1.99	1.56e - 04	2.00	2.23e - 04	1.98	7.68e - 05	2.02
1/128	1.90e - 05	2.01	1.12e - 04	1.98	3.89e - 05	2.00	5.52e - 05	2.01	1.90e - 05	2.01

Table 3 The numerical error on RT0 mixed finite element for state function

h	Errors									
	$\|u - u_h\|$	Rate	$\|p - p_h\|$	Rate	$\|y - y_h\|$	Rate	$\|q - q_h\|$	Rate	$\|z - z_h\|$	Rate
1/16	3.66e - 02	-	1.63e - 01	-	3.67e - 02	-	1.63e - 01	-	3.66e - 02	-
1/32	1.79e - 02	1.03	8.14e - 02	1.00	1.79e - 02	1.04	8.14e - 02	1.00	1.79e - 02	1.03
1/64	8.94e - 03	0.99	4.09e - 02	0.99	8.94e - 03	0.99	4.09e - 02	0.99	8.94e - 03	0.99
1/128	4.50e - 03	0.99	2.04e - 02	1.00	4.46e - 03	1.00	2.04e - 02	1.00	4.50e - 03	0.99

numerical solution is plotted in Figures 1 and 2.

Example 2. In this example we set the other known functions as follows:

$$
\begin{aligned}
&y = (x_1 + x_2)\sin \pi x_1 \sin \pi x_2,\\
&z = -(x_1 + x_2)\sin \pi x_1 \sin \pi x_2,\\
&u = \max(\bar{z}, 0) - z, \quad \boldsymbol{p} = \boldsymbol{p}_d = -\boldsymbol{q},\\
&f = 2\pi^2 \sin \pi x_1 \sin \pi x_2 (x_1 + x_2) + \cos \pi x_1 \sin \pi x_2 - u,\\
&\boldsymbol{q} = (\pi \cos \pi x_1 \sin \pi x_2 + \sin \pi x_1 \sin \pi x_2, \pi \cos \pi x_2 \sin \pi x_1 + \sin \pi x_1 \sin \pi x_2),\\
&y_d = y + 2\pi^2 \sin \pi x_1 \cos \pi x_2 (x_1 + x_2) - 2\pi \cos \pi x_1 \sin \pi x_2 - 2\pi \sin \pi x_1 \cos \pi x_2.
\end{aligned}
$$

The example obviously indicates that the error estimates remains in output data. From the error data in the two examples, it can be seen that the priori error estimates that we have mentioned is exact.

Table 4 The numerical error on RT1 mixed finite element for state function

h	Errors									
	$\|u - u_h\|$	Rate	$\|p - p_h\|$	Rate	$\|y - y_h\|$	Rate	$\|q - q_h\|$	Rate	$\|z - z_h\|$	Rate
1/16	1.55e - 03	-	5.68e - 03	-	1.55e - 03	-	5.68e - 03	-	1.55e - 03	-
1/32	3.88e - 04	2.00	1.43e - 03	2.00	3.88e - 04	2.00	1.43e - 03	2.00	3.88e - 04	2.00
1/64	9.69e - 05	2.00	3.58e - 04	1.99	9.69e - 05	2.00	3.58e - 04	1.99	9.66e - 05	2.00
1/128	2.43e - 05	1.99	8.96e - 05	2.00	2.43e - 05	1.99	8.97e - 05	2.00	2.42e - 05	1.99

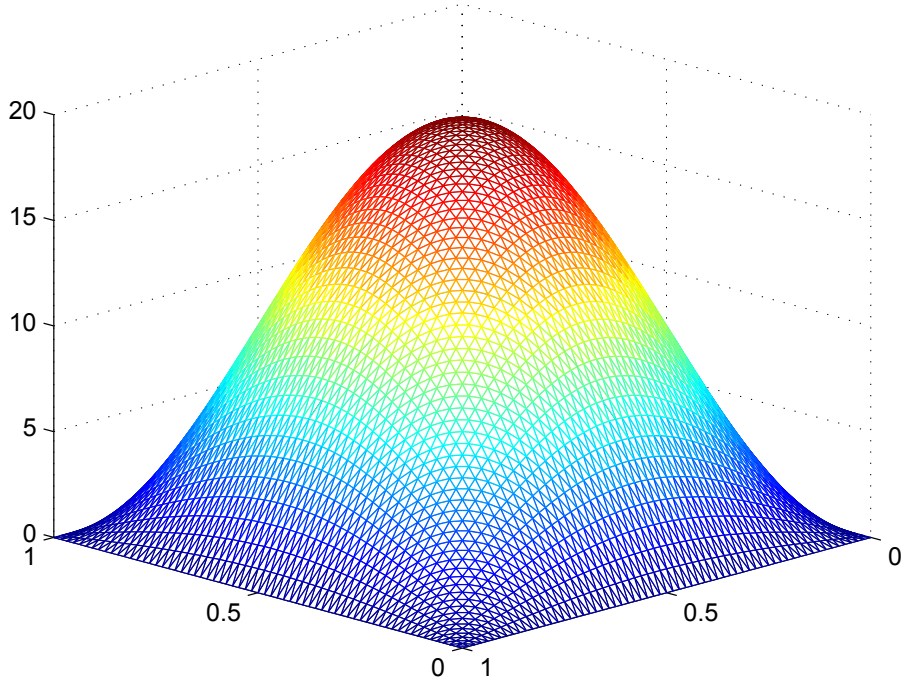

Figure 1 The profile of the control solution u on the 64 × 64 mesh grids.

5. Conclusion and future works

The present article discussed the higher order variational discretization and mixed finite element methods for the optimal control problems governed by elliptic equations. We have obtained some error estimate results for both the state, the co-state and the control approximation with convergence order h^{k+1}. The priori error estimates

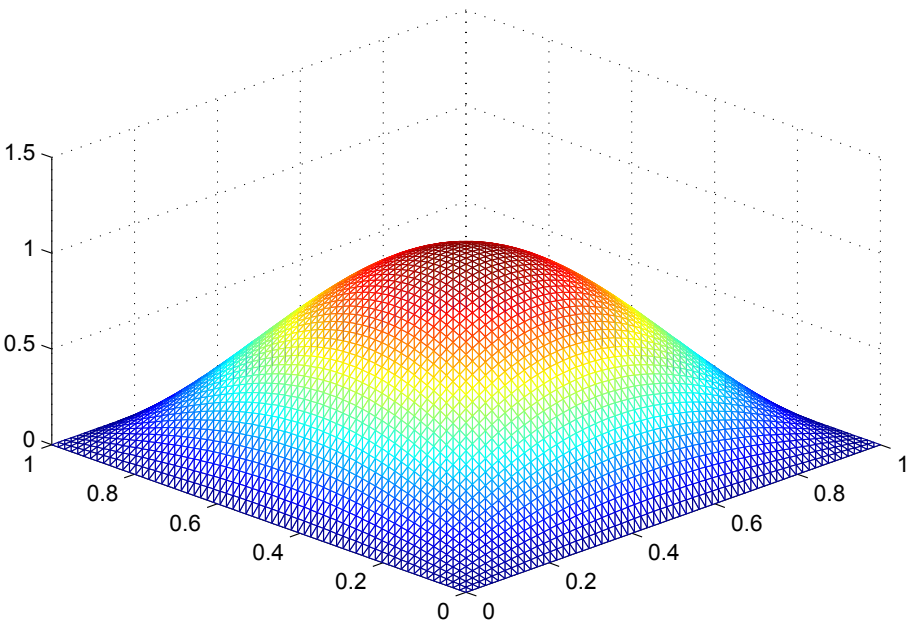

Figure 2 The profile of the control solution u on the 64 × 64 mesh grids.

for the elliptic optimal control problems by variational discretization and mixed finite element methods seem to be new.

In our future study, we shall use the variational discretization and mixed finite element method to deal with the optimal control problems governed by nonlinear parabolic equations and convex boundary control problems. Furthermore, we shall consider *a priori* error estimates and superconvergence of optimal control problems governed by nonlinear parabolic equations or convex boundary control problems.

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Authors' contributions
ZL carried out the molecular genetic studies, participated in the sequence alignment and drafted the manuscript. YC participated in the design of the study and performed the statistical analysis. YH conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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