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Generalized Weyl's theorem for algebraically quasi-paranormal operators

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Abstract

Let T or T^* be an algebraically quasi-paranormal operator acting on a Hilbert space. We prove: (i) generalized Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$; (ii) generalized a-Browder's theorem holds for $f(S)$ for every $S \prec T$ and $f \in H(\sigma(S))$; (iii) the spectral mapping theorem holds for the B-Weyl spectrum of T. Moreover, we show that if T is an algebraically quasi-paranormal operator, then $T + F$ satisfies generalized Weyl's theorem for every algebraic operator F which commutes with T.
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1. Introduction

Throughout this article, we assume that $\mathcal{H}$ is an infinite dimensional separable Hilbert space. Let $B(\mathcal{H})$ and $B_0(\mathcal{H})$ denote, respectively, the algebra of bounded linear operators and the ideal of compact operators acting on $\mathcal{H}$. If $T \in B(\mathcal{H})$ we shall write $N(T)$ and $R(T)$ for the null space and range of T. Also, let $\alpha(T) := \dim N(T)$, $\beta(T) := \dim N(T^*)$, and let $\sigma(T)$, $\sigma_a(T)$, $\sigma_p(T)$, $\pi(T)$, $E(T)$ denote the spectrum, approximate point spectrum, point spectrum of T, the set of poles of the resolvent of T, the set of all eigenvalues of T which are isolated in $\sigma(T)$, respectively. An operator $T \in B(\mathcal{H})$ is called *upper semi-Fredholm* if it has closed range and finite dimensional null space and is called *lower semi-Fredholm* if it has closed range and its range has finite co-dimension. If $T \in B(\mathcal{H})$ is either upper or lower semi-Fredholm, then T is called *semi-Fredholm*, and *index of a semi-Fredholm operator* $T \in B(\mathcal{H})$ is defined by

$$i(T) := \alpha(T) - \beta(T).$$

If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called *Fredholm*. $T \in B(\mathcal{H})$ is called *Weyl* if it is Fredholm of index zero. For $T \in B(\mathcal{H})$ and a nonnegative integer n define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_0 = T$). If for some integer n the range $R(T^n)$ is closed and T_n is upper (resp. lower) semi-Fredholm, then T is called *upper* (resp. *lower*) *semi-B-Fredholm*. Moreover, if T_n is Fredholm, then T is called *B-Fredholm*. T is called *semi-B-Fredholm* if it is upper or lower semi-B-Fredholm. Let T be semi-B-Fredholm and let d be the degree of stable iteration of T. It follows from [1, Proposition 2.1] that T_m is semi-Fredholm and $i(T_m) = i(T_d)$ for each m

$\geq d$. This enables us to define the *index of semi-B-Fredholm T* as the index of semi-Fredholm T_d. Let $BF(\mathcal{H})$ be the class of all B-Fredholm operators. In [2], they studied this class of operators and they proved [2, Theorem 2.7] that an operator $T \in B(\mathcal{H})$ is B-Fredholm if and only if $T = T_1 \oplus T_2$, where T_1 is Fredholm and T_2 is nilpotent. It appears that the concept of Drazin invertibility plays an important role for the class of B-Fredholm operators. Let $\mathcal{A}$ be a unital algebra. We say that an element $x \in \mathcal{A}$ is *Drazin invertible of degree k* if there exists an element $a \in \mathcal{A}$ such that

$$x^k ax = x^k, \ axa = a, \ \text{and } xa = ax.$$

Let $a \in \mathcal{A}$. Then the *Drazin spectrum* is defined by

$$\sigma_D(a) := \{\lambda \in \mathbb{C} : a - \lambda \text{ is not Drazin invertible}\}.$$

For $T \in B(\mathcal{H})$, the smallest nonnegative integer p such that $N(T^p) = N(Tp+^1)$ is called the *ascent* of T and denoted by $p(T)$. If no such integer exists, we set $p(T) = \infty$. The smallest nonnegative integer q such that $R(T^q) = R(T^{q+1})$ is called the *descent* of T and denoted by $q(T)$. If no such integer exists, we set $q(T) = \infty$. It is well known that T is Drazin invertible if and only if it has finite ascent and descent, which is also equivalent to the fact that

$$T = T_1 \oplus T_2, \ \text{where } T_1 \text{ is invertible and } T_2 \text{ is nilpotent.}$$

An operator $T \in B(\mathcal{H})$ is called *B-Weyl* if it is B-Fredholm of index 0. The *B-Fredholm spectrum* $\sigma_{BF}(T)$ and *B-Weyl spectrum* $\sigma_{BW}(T)$ of T are defined by

$$\sigma_{BF}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } B - \text{Fredholm}\},$$

$$\sigma_{BW}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } B - \text{Weyl}\}.$$

Now, we consider the following sets:

$$BF_+(\mathcal{H}) := \{T \in B(\mathcal{H}) : T \text{ is upper semi - } B \text{ - Ferdholm}\},$$
$$BF_+^-(\mathcal{H}) := \{T \in B(\mathcal{H}) : T \in BF_+(\mathcal{H}) \text{ and } i(T) \leq 0\},$$
$$LD(\mathcal{H}) := \{T \in B(\mathcal{H}) : p(T) < \infty \text{ and } R(T^{p(T)+1}) \text{ is closed}\}.$$

By definition,

$$\sigma_{Bea}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin BF_+^-(\mathcal{H})\},$$

is the upper semi-B-essential approximate point spectrum and

$$\sigma_{LD}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin LD(\mathcal{H})\}$$

is the left Drazin spectrum. It is well known that

$$\sigma_{Bea}(T) \subseteq \sigma_{LD}(T) = \sigma_{Bea}(T) \cup \text{acc } \sigma_a(T) \subseteq \sigma_D(T),$$

where we write acc K for the accumulation points of $K \subseteq \mathbb{C}$. If we write iso $K: = K \setminus$ *acc* K then we let

$$p_0^a(T) := \{\lambda \in \sigma_\alpha(T) : T - \lambda \in LD(\mathcal{H})\},$$
$$\pi_0^a(T) := \{\lambda \in \text{ iso } \sigma_a(T) : \lambda \in \sigma_p(T)\}.$$

We say that an operator T has the *single valued extension property at* λ (abbreviated SVEP at λ) if for every open set U containing λ the only analytic function $f : U \to \mathcal{H}$ which satisfies the equation

$$(T - \lambda)f(\lambda) = 0$$

is the constant function $f \equiv 0$ on U. T has SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

Definition 1.1. Let $T \in B(\mathcal{H})$.

(1) *Generalized Weyl's theorem holds for T* (in symbols, $T \in g\mathcal{W}$) if

$$\sigma(T)\backslash\sigma_{BW}(T) = E(T).$$

(2) *Generalized Browder's theorem holds for T* (in symbols, $T \in g\mathcal{B}$) if

$$\sigma(T)\backslash\sigma_{BW}(T) = \pi(T).$$

(3) *Generalized a-Weyl's theorem holds for T* (in symbols, $T \in ga\mathcal{W}$) if

$$\sigma_a(T)\backslash\sigma_{Bea}(T) = \pi_0^a(T).$$

(4) *Generalized a-Browder's theorem holds for T* (in symbols, $T \in ga\mathcal{B}$) if

$$\sigma_a(T)\backslash\sigma_{Bea}(T) = p_0^a(T).$$

It is known ([3]) that the following set inclusions hold:

$$\begin{array}{ccc} ga - \text{Weyl's theorem} & \Rightarrow & ga - \text{Browder's theorem} \\ \Downarrow & & \Downarrow \\ g - \text{Weyl's theorem} & \Rightarrow & g - \text{Browder's theorem} \end{array}$$

Recently, Han and Na introduced a new operator class which contains the classes of paranormal operators and quasi-class A operators [4]. In [5], it was shown that generalized Weyl's theorem holds for algebraically paranormal operators. In this article, we extend this result to algebraically quasi-paranormal operators using the local spectral theory

2. Generalized Weyl's theorem for algebraically quasi-paranormal operators

Definition 2.1. (1) An operator $T \in B(\mathcal{H})$ is said to be *class A* if

$$|T|^2 \leq |T^2|.$$

(2) T is called a *quasi-class A* operator if

$$T^*|T|^2T \leq T^*|T^2|T.$$

(3) An operator $T \in B(\mathcal{H})$ is said to be *paranormal* if

$$\|Tx\|^2 \leq \|T^2x\| \, \|x\| \quad \text{for all } x \in \mathcal{H}.$$

Recently, we introduced a new operator class which is a common generalization of paranormal operators and quasi-class A operators [4].

Definition 2.2. An operator $T \in B(\mathcal{H})$ is called quasi-paranormal if

$$\|T^2x\|^2 \leq \|T^3x\| \, \|Tx\| \quad \text{for all } x \in \mathcal{H}.$$

We say that $T \in B(\mathcal{H})$ is an *algebraically quasi-paranormal* operator if there exists a non-constant complex polynomial h such that $h(T)$ is quasi-paranormal.

In general, the following implications hold:

class $A \Rightarrow$ quasi-class $A \Rightarrow$ quasi-paranormal;

paranormal $\Rightarrow$ quasi-paranormal $\Rightarrow$ algebraically quasi-paranormal.

In [4], it was observed that there are examples which are quasi-paranormal but not paranormal, as well as quasi-paranormal but not quasi-class A. We give a more simple example which is quasi-paranormal but not quasi-class A. To construct this example we recall the following lemma in [4].

Lemma 2.3. An operator $T \in B(\mathcal{H})$ is quasi-paranormal if and only if

$$T^*(T^{2^*}T^2 - 2\lambda T^*T + \lambda^2)T \geq 0 \text{ for all } \lambda > 0.$$

Example 2.4. $T = \begin{pmatrix} I & 0 \\ I & 0 \end{pmatrix} \in B(\ell_2 \oplus \ell_2)$. Then it is quasi-paranormal but not quasi-class

A.

Proof. Since $T^* = \begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix}$, $|T^2| = \sqrt{(T^*)^2T^2} = \sqrt{\begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix}^2 \begin{pmatrix} I & 0 \\ I & 0 \end{pmatrix}^2} = \begin{pmatrix} \sqrt{2}I & 0 \\ 0 & 0 \end{pmatrix}$

Therefore $T^*\,|T^2|\,T = \begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{2}I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ I & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2}I & 0 \\ 0 & 0 \end{pmatrix}$

On the other hand, since $|T^2| = T^*T = \begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ I & 0 \end{pmatrix} = \begin{pmatrix} 2I & 0 \\ 0 & 0 \end{pmatrix}$,

$T^*\,|T^2|\,T = \begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ I & 0 \end{pmatrix} = \begin{pmatrix} 2I & 0 \\ 0 & 0 \end{pmatrix}$. Hence T is not quasi-class A.

However, since

$$T^{2*}T^2 - 2\lambda T^*T + \lambda^2 = \begin{pmatrix} (2 - 4\lambda + \lambda^2)I & 0 \\ 0 & \lambda^2 I \end{pmatrix},$$

we have

$$T^*(T^{2*}T^2 - 2\lambda T^*T + \lambda^2)T = \begin{pmatrix} 2(1-\lambda)^2 I & 0 \\ 0 & 0 \end{pmatrix} \geq 0$$

for all $\lambda > 0$. Therefore T is quasi-paranormal. □

The following example provides an operator which is algebraically quasi-paranormal but not quasi-paranormal.

Example 2.5 Let $T = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \in B(\ell_2 \oplus \ell_2)$. Then it is algebraically quasi-paranormal but not quasi-paranormal.

Proof. Since $T^* = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$, we have

$$T^{2*}T^2 - 2\lambda T^*T + \lambda^2 = \begin{pmatrix} (\lambda^2 - 4\lambda + 5)I & (-2\lambda + 2)I \\ (-2\lambda + 2)I & (\lambda^2 - 2\lambda + 1)I \end{pmatrix}.$$

Therefore

$$T^*(T^{2*}T^2 - 2\lambda T^*T + \lambda^2)T = \begin{pmatrix} (2\lambda^2 - 10\lambda + 10)I & (\lambda^2 - 4\lambda + 3)I \\ (\lambda^2 - 4\lambda + 3)I & (\lambda^2 - 2\lambda + 1)I \end{pmatrix}.$$

Since $(2\lambda^2 - 10\lambda + 10)I$ is not a positive operator for $\lambda = 2$, $T^*(T^{2*}T^2 - 2\lambda T^*T + \lambda^2)T \not\geq 0$ for $\lambda > 0$. Therefore T is not quasi-paranormal. On the other hand, consider the complex polynomial $h(z) = (z - 1)^2$. Then $h(T) = 0$, and hence T is algebraically quasi-paranormal.

□

The following facts follow from the above definition and some well known facts about quasi-paranormal operators [4]:

(i) If $T \in B(\mathcal{H})$ is algebraically quasi-paranormal, then so is $T-\lambda$ for each $\lambda \in \mathbb{C}$.
(ii) If $T \in B(\mathcal{H})$ is algebraically quasi-paranormal and $\mathcal{M}$ is a closed T-invariant subspace

of $\mathcal{H}$, then $T|\mathcal{M}$ is algebraically quasi-paranormal.

(iii) If T is algebraically quasi-paranormal, then T has SVEP.
(iv) Suppose T does not have dense range. Then we have:

T is quasi-paranormal $\Leftrightarrow T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$ on $\mathcal{H} = \overline{T(\mathcal{H})} \oplus N(T^*)$,

where $A = T|\overline{T(\mathcal{H})}$ is paranormal.

An operator $T \in B(\mathcal{H})$ is called *isoloid* if iso $\sigma(T) \subseteq \sigma_p(T)$ and an operator $T \in B(\mathcal{H})$ is called *polaroid* if iso $\sigma(T) \subseteq \pi(T)$.

In general, the following implications hold:

$$T \text{ polaroid} \Rightarrow T \text{ isoloid}.$$

However, each converse is not true. Consider the following example: let $T \in B(\ell_2)$ be defined by

$$T(x_1, x_2, x_3, \ldots) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \ldots).$$

Then T is a compact quasinilpotent operator with $\alpha(T) = 1$, and so T is isoloid. However, since $q(T) = \infty$, T is not polaroid.

An important subspace in local spectral theory is the *quasi-nilpotent part* of T defined by

$$H_0(T) := \left\{ x \in \mathcal{H} : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0 \right\}.$$

If $T \in B(\mathcal{H})$, then the *analytic core* $K(T)$ is the set of all $x \in \mathcal{H}$ such that there exists a constant $c > 0$ and a sequence of elements $x_n \in \mathcal{H}$ such that $x_0 = x$, $Tx_n = x_{n-1}$, and $\|x_n\| \leq c^n \|x\|$ for all $n \in \mathbb{N}$, see [6] for information on $K(T)$.

Let $\mathcal{P}(\mathcal{H})$ denotes the class of all operators for which there exists $p := p(\lambda) \in \mathbb{N}$ for which

$$H_0(T-\lambda) = N(T-\lambda)^p \text{ for all } \lambda \in \mathbb{C},$$

and $\mathcal{P}_1(\mathcal{H})$ denotes the class of all operators for which there exists $p := p(\lambda) \in \mathbb{N}$ for which

$$H_0(T-\lambda) = N(T-\lambda)^p \text{ for all } \lambda \in E(T).$$

Evidently, $\mathcal{P}(\mathcal{H}) \subseteq \mathcal{P}_1(\mathcal{H})$. Now we give a characterization of $\mathcal{P}_1(\mathcal{H})$.

Theorem 2.6. $T \in \mathcal{P}_1(\mathcal{H})$ if and only if $\pi(T) = E(T)$.

Proof. Suppose $T \in \mathcal{P}_1(\mathcal{H})$ and let $\lambda \in E(T)$. Then there exists $p \in \mathbb{N}$ such that $H_0(T-\lambda) = N(T-\lambda)^p$. Since λ is an isolated point of $\sigma(T)$, it follows from [6, Theorem 3.74] that

$$\mathcal{H} = H_0(T-\lambda) \oplus K(T-\lambda) = N(T-\lambda)^p \oplus K(T-\lambda).$$

Therefore, we have

$$(T-\lambda)^p(\mathcal{H}) = (T-\lambda)^p(K(T-\lambda)) = K(T-\lambda),$$

and hence $\mathcal{H} = N(T-\lambda)^p \oplus (T-\lambda)^p(\mathcal{H})$, which implies, by [6, Theorem 3.6], that $p(T-\lambda) = q(T-\lambda) \leq p$. But $\alpha(T-\lambda) > 0$, hence $\lambda \lfloor \pi(T)$. Therefore $E(T) \subseteq \pi(T)$. Since the opposite inclusion holds for every operator T, we then conclude that $\pi(T) = E(T)$. Conversely, suppose $\pi(T) = E(T)$. Let $\lambda \lfloor E(T)$. Then $p := p(T-\lambda) = q(T-\lambda) < \infty$. By [6, Theorem 3.74], $H_0(T-\lambda) = N(T-\lambda)^p$. Therefore $T \in \mathcal{P}_1(\mathcal{H})$. □

From Theorem 2.6, we can give a simple example which belongs to $\mathcal{P}_1(\mathcal{H})$ but not $\mathcal{P}(\mathcal{H})$. Let U be the unilateral shift on ℓ_2 and let $T = U^*$. Then T does not have SVEP at 0, and so $H_0(T)$ is not closed. Therefore $T \notin \mathcal{P}(\mathcal{H})$. However, since $\sigma(T) = \bar{\mathbb{D}}, \pi(T) = E(T) = \emptyset$, where $\mathbb{D}$ is an open unit disk in $\mathbb{C}$. Hence $T \in \mathcal{P}_1(\mathcal{H})$ by Theorem 2.6.

Before we state our main theorem (Theorem 2.9) in this section, we need some preliminary results.

Lemma 2.7. Let $T \in B(\mathcal{H})$ be a quasinilpotent algebraically quasi-paranormal operator. Then T is nilpotent.

Proof. We first assume that T is quasi-paranormal. We consider two cases:

Case I: Suppose T has dense range. Then clearly, it is paranormal. Therefore T is nilpotent by [7, Lemma 2.2].

Case II: Suppose T does not have dense range. Then we can represent T as the upper triangular matrix

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \text{ on } \mathcal{H} = \overline{T(\mathcal{H})} \oplus N(T^*),$$

where $A := T|\overline{T(\mathcal{H})}$ is an paranormal operator. Since T is quasinilpotent, $\sigma(T) = \{0\}$. But $\sigma(T) = \sigma(A) \cup \{0\}$, hence $\sigma(A) = \{0\}$. Since A is paranormal, $A = 0$ and therefore T is nilpotent. Thus if T is a quasinilpotent quasi-paranormal operator, then it is nilpotent. Now, we suppose T is algebraically quasi-paranormal. Then there exists a non-constant polynomial p such that $p(T)$ is quasi-paranormal. If $p(T)$ has dense range, then $p(T)$ is paranormal. So T is algebraically paranormal, and hence T is nilpotent by [7, Lemma 2.2]. If $p(T)$ does not have dense range, we can represent $p(T)$ as the upper triangular matrix

$$p(T) = \begin{pmatrix} C & D \\ 0 & 0 \end{pmatrix} \text{ on } \mathcal{H} = \overline{p(T)(\mathcal{H})} \oplus N\left(p(T)^*\right),$$

where $C := p(T)|\overline{p(T)(\mathcal{H})}$ is paranormal. Since T is quasinilpotent, $\sigma(p(T)) = p(\sigma(T)) = \{p(0)\}$. But $\sigma(p(T)) = \sigma(C) \cup \{0\}$ by [8, Corollary 8], hence $\sigma(C) \cup \{0\} = \{p(0)\}$. So $p(0) = 0$, and hence $p(T)$ is quasinilpotent. Since $p(T)$ is quasi-paranormal, by the previous argument $p(T)$ is nilpotent. On the other hand, since $p(0) = 0$, $p(z) = cz^m(z - \lambda_1)(z - \lambda_2) \dots (z - \lambda_n)$ for some natural number m. Therefore $p(T) = cT^m(T - \lambda 1)(T - \lambda_2) \dots (T - \lambda_n)$. Since $p(T)$ is nilpotent and $T - \lambda_i$ is invertible for every $\lambda_i \neq 0$, T is nilpotent. This completes the proof. □

Theorem 2.8. Let $T \in B(\mathcal{H})$ be algebraically quasi-paranormal. Then $T \in \mathcal{P}_1(\mathcal{H})$.

Proof. Suppose T is algebraically quasi-paranormal. Then $h(T)$ is a quasi-paranormal operator for some nonconstant complex polynomial h. Let $\lambda \in E(T)$. Then λ is an isolated point of $\sigma(T)$ and $\alpha(T - \lambda) > 0$. Using the spectral projection $P := \frac{1}{2\pi i}\int_{\partial D}(\mu - T)^{-1}d\mu$, where D is a closed disk of center λ which contains no other points of $\sigma(T)$, we can represent T as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \text{ where } \sigma(T_1) = \{\lambda\} \text{ and } \sigma(T_2) = \sigma(T) \setminus \{\lambda\}.$$

Since T_1 is algebraically quasi-paranormal, so is $T_1 - \lambda$. But $\sigma(T_1 - \lambda) = \{0\}$, it follows from Lemma 2.7 that $T_1 - \lambda$ is nilpotent. Therefore $T_1 - \lambda$ has finite ascent and descent. On the other hand, since $T_2 - \lambda$ is invertible, clearly it has finite ascent and descent. Therefore λ is a pole of the resolvent of T, and hence $\lambda \in \pi(T)$. Hence $E(T) \subseteq \pi(T)$. Since $\pi(T) \subseteq E(T)$ holds for any operator T, we have $\pi(T) = E(T)$. It follows from Theorem 2.6 that $T \in \mathcal{P}_1(\mathcal{H})$. □

We now show that generalized Weyl's theorem holds for algebraically quasi-paranormal operators. In the following theorem, recall that $H(\sigma(T))$ is the space of functions analytic in an open neighborhood of $\sigma(T)$.

Theorem 2.9. Suppose that T or T^* is an algebraically quasi-paranormal operator. Then $f(T) \in g\mathcal{W}$ for each $f \in H(\sigma(T))$.

Proof. Suppose T is algebraically quasi-paranormal. We first show that $T \in g\mathcal{W}$. Suppose that $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Then $T - \lambda$ is B-Weyl but not invertible. It follows from [9, Lemma 4.1] that we can represent $T - \lambda$ as the direct sum

$$T - \lambda = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \text{ where} T_1 \text{ is Weyl and} T_2 \text{ is nilpotent.}$$

Since T is algebraically quasi-paranormal, it has SVEP. So T_1 and T_2 have both finite ascent. But T_1 is Weyl, hence T_1 has finite descent. Therefore T-λ has finite ascent and descent, and so $\lambda \in E(T)$. Conversely, suppose that $\lambda \in E(T)$. Since T is algebraically quasi-paranormal, it follows from Theorem 2.8 that $T \in \mathcal{P}_1(\mathcal{H})$. Since $\pi(T) = E(T)$ by Theorem 2.6, $\lambda \in E(T)$. Therefore $T - \lambda$ has finite ascent and descent, and so we can represent $T - \lambda$ as the direct sum

$$T - \lambda = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \text{ where} T_1 \text{ is invertible and} T_2 \text{ is nilpotent.}$$

Therefore $T - \lambda$ is B-Weyl, and so $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Thus $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$, and hence $T \in g\mathcal{W}$.

Next, we claim that $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$ for each $f \in H(\sigma(T))$. Since $T \in g\mathcal{W}$, $T \in g\mathcal{B}$. It follows from [5, Theorem 2.1] that $\sigma_{BW}(T) = \sigma_D(T)$. Since T is algebraically quasi-paranormal, $f(T)$ has SVEP for each $f \in H(\sigma(T))$. Hence $f(T) \in g\mathcal{B}$ by [5, Theorem 2.9], and so $\sigma_{BW}(f(T)) = \sigma_D(f(T))$. Therefore we have

$$\sigma_{BW}\left(f(T)\right) = \sigma_D\left(f(T)\right) = f\left(\sigma_D(T)\right) = f\left(\sigma_{BW}(T)\right).$$

Since T is algebraically quasi-paranormal, it follows from the proof of Theorem 2.8 that it is isoloid. Hence for any $f \in H(\sigma(T))$ we have

$$\sigma\left(f(T)\right) \setminus E\left(f(T)\right) = f\left(\sigma(T) \setminus E(T)\right).$$

Since $T \in g\mathcal{W}$, we have

$$\sigma\left(f(T)\right) \setminus E\left(f(T)\right) = f\left(\sigma(T) \setminus E(T)\right) = f\left(\sigma_{BW}(T)\right) = \sigma_{BW}\left(f(T)\right),$$

which implies that $f(T) \in g\mathcal{W}$.

Now suppose that T^* is algebraically quasi-paranormal. We first show that $T \in g\mathcal{W}$. Let $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Observe that $\sigma(T^*) = \overline{\sigma(T)}$ and $\sigma_{BW}(T^*) = \overline{\sigma_{BW}(T)}$. So $\overline{\lambda} \in \sigma\left(T^*\right) \setminus \sigma_{BW}\left(T^*\right)$, and so $\overline{\lambda} \in E\left(T^*\right)$ because $T^* \in g\mathcal{W}$. Since T^* is algebraically quasi-paranormal, it follows from Theorem 2.8 that $\overline{\lambda} \in \pi(T^*)$. Hence $T - \lambda$ has finite ascent and descent, and so $\lambda \in E(T)$. Conversely, suppose $\lambda \in E(T)$. Then λ is an isolated point of $\sigma(T)$ and $\alpha(T - \lambda) > 0$. Since $\sigma(T^*) = \overline{\sigma(T)}$, $\bar{\lambda}$ is an isolated point of $\sigma(T^*)$. Since T^* is isoloid, $\overline{\lambda} \in E(T^*)$. But $E(T^*) = \pi(T^*)$ by Theorem 2.8, hence we have $T - \lambda$ has finite ascent and descent. Therefore we can represent $T - \lambda$ as the direct sum

$$T - \lambda = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \text{ where } T_1 \text{ is invertible and } T_2 \text{ is nilpotent.}$$

Therefore $T - \lambda$ is B-Weyl, and so $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Thus $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$, and hence $T \in g\mathcal{W}$. If T^* is algebraically quasi-paranormal then T is isoloid. It follows from the first part of the proof that $f(T) \in g\mathcal{W}$. This completes the proof. □

From the proof of Theorem 2.9 and [10, Theorem 3.4], we obtain the following useful consequence.

Corollary 2.10. Suppose T or T^* is algebraically quasi-paranormal. Then

$$\sigma_{BW}\left(f(T)\right) = f\left(\sigma_{BW}(T)\right) \quad \text{for every } f \in H\left(\sigma(T)\right).$$

An operator $X \in B(\mathcal{H})$ is called a *quasiaffinity* if it has trivial kernel and dense range. $S \in B(\mathcal{H})$ is said to be a *quasiaffine transform of* $T \in B(\mathcal{H})$(notation: $S \prec T$) if there is a quasiaffinity $X \in B(\mathcal{H})$ such that $XS = TX$. If both $S \prec T$ and $T \prec S$, then we say that S and T are *quasisimilar*.

Corollary 2.11. Suppose T is algebraically quasi-paranormal and $S \prec T$. Then $f(S) \in ga\mathcal{B}$ for each $f \in H(\sigma(S))$.

Proof. Suppose T is algebraically quasi-paranormal. Then T has SVEP. Since $S \prec T$, $f(S)$ has SVEP by [7, Lemma 3.1]. It follows from [11, Theorem 3.3.6] that $f(S)$ has SVEP. Therefore $f(S) \in ga\mathcal{B}$ by [12, Corollary 2.5]. □

3. Generalized Weyl's theorem for perturbations of algebraically quasi-paranormal operators

An operator T is said to be *algebraic* if there exists a nontrivial polynomial h such that $h(T) = 0$. From the spectral mapping theorem it easily follows that the spectrum of an algebraic operator is a finite set. It is known that generalized Weyl's theorem is not generally transmitted to perturbation of operators satisfying generalized Weyl's theorem. In [13], they proved that if T is paranormal and F is an algebraic operator commuting with T, then Weyl's theorem holds for $T + F$. We now extend this result to generalized Weyl's theorem for algebraically quasi-paranormal operators. We begin with the following lemma.

Lemma 3.1. Let $T \in B(\mathcal{H})$. Then the following statements are equivalent:

(1) $T \in g\mathcal{W}$;
(2) T has SVEP at every $\lambda \in \mathbb{C}\backslash\sigma_{BW}(T)$ and $\pi(T) = E(T)$.

Proof. Observe that $T \in g\mathcal{B}$ if and only if $\sigma_{BW}(T) = \sigma_D(T)$. *So* $T \in g\mathcal{B}$if and only if T has SVEP at every $\lambda \in \mathbb{C}\backslash\sigma_{BW}(T)$. Therefore we obtain the desired conclusion. □

From this lemma, we obtain the following corollary

Corollary 3.2. Let $T \in B(\mathcal{H})$. Suppose T has SVEP. Then

$$T \in g\mathcal{W} \text{ if and only if } T \in \mathcal{P}_1(\mathcal{H}).$$

Proof. Since T has SVEP, $T \in g\mathcal{B}$*by* Lemma 3.1. So $\sigma(T) \setminus \sigma_{BW}(T) = \pi(T)$. Therefore $T \in g\mathcal{W}$ if and only if $T \in \mathcal{P}_1(\mathcal{H})$ by Theorem 2.6. □

Lemma 3.3. Suppose $T \in B(\mathcal{H})$ and N is nilpotent such that $TN = NT$. Then $T \in \mathcal{P}_1(\mathcal{H})$ if and only if $T + N \in \mathcal{P}_1(\mathcal{H})$.

Proof. Suppose $N^p = 0$ for some $p \in \mathbb{N}$. Observe that without any assumption on T we have

$$N(T) \subseteq N(T+N)^p \text{and } N(T+N) \subseteq N(T^p). \tag{3.3.1}$$

Suppose now that $T \in \mathcal{P}_1(\mathcal{H})$, or equivalently $\pi(T) = E(T)$. We show first $E(T) = E(T+N)$. Let $\lambda \in E(T)$. Without loss of generality, we may assume that $\lambda = 0$. From $\sigma(T+N) = \sigma(T)$, we see that 0 is an isolated point of $\sigma(T+N)$. Since $0 \in E(T)$, $\alpha(T) > 0$ and hence by the first inclusion in (3.3.1) we have $\alpha(T+N)^p > 0$. Therefore $\alpha(T+N) > 0$, and hence $0 \in E(T+N)$. Thus the inclusion $E(T) \subseteq E(T+N)$ is proved. To show the opposite inclusion, assume that $0 \in E(T+N)$. Then 0 is an isolated point of $\sigma(T)$ because $\sigma(T+N) = \sigma(T)$. Since $\alpha(T+N) > 0$, the second inclusion in (3.3.1) entails that $\alpha(T^p) > 0$. Therefore $\alpha(T) > 0$, and hence $0 \in E(T)$. So the equality $E(T) = E(T+N)$ is proved. Suppose $T \in \mathcal{P}_1(\mathcal{H})$. Then $\pi(T) = E(T)$ by Theorem 2.6, and so $\pi(T+N) = \pi(T) = E(T) = E(T+N)$. Therefore $T + N \in \mathcal{P}_1(\mathcal{H})$. Conversely, if $T + N \in \mathcal{P}_1(\mathcal{H})$ by symmetry we have $\pi(T) = \pi(T+N) = E(T+N) = E((T+N)-N) = E(T)$, so the proof is complete. □

The following theorem is a generalization of [13, Theorem 2.5]. The proof of the following theorem is strongly inspired to that of it.

Theorem 3.4. Suppose T is algebraically quasi-paranormal. If F is algebraic with $TF = FT$, then $T + F \in g\mathcal{W}$.

Proof. Since F is algebraic, $\sigma(F)$ is finite. Let $\sigma(F) = \{\mu_1,\mu_2,...,\mu_n\}$. Denote by P_i the spectral projection associated with F and the spectral set $\{\mu_i\}$. Let $Y_i := R(P_i)$ and $Z_i := N(P_i)$. Then $H = Y_i \oplus Z_i$ and the closed subspaces Y_i and Z_i are invariant under T and F. Moreover, $\sigma(F|Y_i) = \{\mu_i\}$. Define $F_i := F|Y_i$ and $Ti := T|Y_i$. Then clearly, the restrictions T_i and F_i commute for every $i = 1, 2,...,n$ and

$$\sigma(T+F) = \sigma((T+F)|Y_i) \cup \sigma((T+F)|Z_i).$$

Let h be a nontrivial complex polynomial such that $h(F) = 0$. Then $h(F_i) = h(F|Y_i) = h(F)|Y_i = 0$, and from $\{0\} = \sigma(h(F_i)) = h(\sigma(F_i)) = h(\{\mu_i\})$, we obtain that $h(\mu_i) = 0$. Write $h(\mu) = (\mu - \mu_i)^m g(\mu)$ with $g(\mu_i) = 0$. Then $0 = h(F_i) = (F - \mu_i)^m g(F_i)$, where $g(F_i)$ is invertible. Hence $N_i := F_i - \mu_i$ are nilpotent for all $i = 1, 2,...,n$. Observe that

$$T_i + F_i = (T_i + \mu_i) + (F_i - \mu_i) = T_i + N_i + \mu_i. \tag{3.4.1}$$

Since $T_i + \mu_i$ is algebraically quasi-paranormal for all $i = 1, 2,...,n$, $T_i + \mu_i$ has SVEP. Moreover, since N_i is nilpotent with $T_iN_i = N_iT_i$, it follows from [6, Corollary 2.12] that $T_i + N_i + \mu_i$ has SVEP, and hence $T_i + F_i$ has SVEP. From [6, Theorem 2.9] we obtain that

$$T+F = \bigoplus_{i=1}^{n} (T_i + F_i) \text{ has SVEP.}$$

Now, we show that $T + F \in \mathcal{P}_1(\mathcal{H})$. Since $T_i + \mu_i$ is algebraically quasi-paranormal, $T_i + \mu_i \in \mathcal{P}_1(Y_i)$ by Theorem 2.8. By Lemma 3.3 and (3.4.1), $T_i + F_i \in \mathcal{P}_1(Y_i)$ for every $i = 1, 2,...,n$. Now assume that $\lambda_0 \in E(T + F)$. Fix $i \in \mathbb{N}$ such that $1 \le i \le n$. Since the equality $T_i + N_i - \lambda_0 + \mu_i = T_i + F_i - \lambda_0$ holds, we consider two cases:

Case I: Suppose that $T_i - \lambda_0 + \mu_i$ is invertible. Since N_i is quasi-nilpotent commuting with $T_i - \lambda_0 + \mu_i$, it is clear that $T_i + F_i - \lambda_0$ is also invertible. Hence $H_0(T_i + F_i - \lambda_0) = N(T_i + F_i - \lambda_0) = \{0\}$.

Case II: Suppose that $T_i - \lambda_0 + \mu_i$ is not invertible. Then $\lambda_0 - \mu_i \in \sigma(T_i)$. We claim that $\lambda_0 \in E(T_i + F_i)$. Note that $\lambda_0 \in \sigma(T_i + \mu_i) = \sigma(T_i + F_i)$. Since $\sigma(T_i + F_i) \in \sigma(T + F)$ and $\lambda_0 \in \text{iso}\,\sigma(T + F)$, $\lambda_0 \in \text{iso}\,\sigma(T_i + N_i + \mu_i)$. Therefore $\lambda_0 - \mu_i \in \text{iso}\,\sigma(T_i + N_i) = \text{iso}\,\sigma(T_i)$. Since $T_i - \lambda_0 + \mu_i$ is algebraically quasi-paranormal, $\lambda_0 - \mu_i \in \pi(T_i)$. Since $\pi(T_i) = E(T_i)$ by Theorem 2.6 and $T_i \in g\mathcal{W}$ by Theorem 2.9, $\lambda_0 - \mu_i \in E(T_i) = \sigma(T_i) \setminus \sigma_{BW}(T_i)$. But N_i is nilpotent with $T_iN_i = N_iT_i$, hence $\sigma_D(T_i) = \sigma_D(T_i + N_i)$ and $T_i + N_i \in g\mathcal{B}$. Therefore we have $\sigma_{BW}(T_i + N_i) = \sigma_D(T_i + N_i)$. Hence

$$E(T_i) = \sigma(T_i) \setminus \sigma_{BW}(T_i) = \sigma(T_i + N_i) \setminus \sigma_{BW}(T_i + N_i).$$

Hence $T_i + F_i - \lambda_0$ is *B-Weyl*. Assume to the contrary that $T_i + F_i - \lambda_0$ is injective. Then $\beta(T_i + F_i - \lambda_0) = \alpha(T_i + F_i - \lambda_0) = 0$. Therefore $T_i + F_i - \lambda_0$ is invertible, and so $\lambda_0 \notin \sigma(T_i + F_i)$. This is a contradiction. Hence $\lambda_0 \in E(T_i + F_i)$. Since $T_i + F_i \in \mathcal{P}_1(Y_i)$ by Theorem 2.6, there exists a positive integer m_i such that $H_0(T_i + F_i - \lambda_0) = N(T_i + F_i - \lambda_0)^{mi}$.

From Cases I and II we have

$$\begin{aligned} H_0(T+F-\lambda_0) &= \bigoplus_{i=1}^{n} H_0(T_i+F_i-\lambda_0) \\ &= \bigoplus_{i=1}^{n} N(T_i+F_i-\lambda_0)^{m_i} \\ &= N(T+F-\lambda_0)^{m}, \end{aligned}$$

where $m := \max\{m_1, m_2, \ldots, m_n\}$. Since the last equality holds for every $\lambda_0 \in E(T+F)$, $T + F \in \mathcal{P}_1(\mathcal{H})$. Therefore $T + F \in g\mathcal{W}$by Corollary 3.2. □

It is well known that if for an operator $F \in B(\mathcal{H})$ there exists a natural number n for which F^n is finite-dimensional, then F is algebraic.

Corollary 3.5. Suppose $T \in B(\mathcal{H})$ is algebraically quasi-paranormal and F is an operator commuting with T such that F^n is a finite-dimensional operator for some $n \in \mathbb{N}$. Then $T + F \in g\mathcal{W}$.

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