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A new version of the Gleason-Kahane-Żelazko theorem in complete random normed algebras

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Abstract

In this article we first present the notion of multiplicative L^0-linear function. Moreover, we establish a new version of the Gleason-Kahane-Żelazko theorem in unital complete random normed algebras.

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1 Introduction

Gleason [1] and, independently, Kahane and Żelazko [2] proved the so-called Gleason-Kahane-Żelazko theorem which is a famous theorem in classical Banach algebras. There are various extensions and generalizations of this theorem [3]. The Gleason-Kahane-Żelazko theorem in an unital complete random normed algebra as a random generalization of the classical Gleason-Kahane-Żelazko theorem is given in [4].

Based on the study of [5], we will establish a new version of the Gleason-Kahane-Żelazko theorem in an unital complete random normed algebra. In this article we first present the notion of multiplicative L^0-functions. Then, we give the new version of the Gleason-Kahane-Żelazko theorem in an unital complete random normed algebra as another random generalization of the classical Gleason-Kahane-Żelazko theorem.

The remainder of this article is organized as follows: in Section 2 we give some necessary definitions and lemmas and in Section 3 we give the main results and proofs.

2 Preliminary

Throughout this article, N denotes the set of positive integers, K the scalar field R of real numbers or C of complex numbers, $\bar{R}$ (or [-∞, +∞]) the set of extended real numbers, $(\Omega, \mathcal{F}, P)$ a probability space, $\bar{\mathcal{L}}^0(\mathcal{F}, R)$ the set of extended real-valued $\mathscr{F}$-random variables on Ω, $\bar{L}^0(\mathcal{F}, R)$ the set of equivalence classes of extended real-valued $\mathscr{F}$-random variables on Ω, $\mathcal{L}^0(\mathcal{F}, K)$ the algebra of K-valued $\mathscr{F}$-random variables on Ω under the ordinary pointwise addition, multiplication and scalar multiplication operations, $L^0(\mathcal{F}, K)$ the algebra of equivalence classes of K-valued $\mathscr{F}$-random variables on Ω, i.e., the quotient algebra of $\mathcal{L}^0(\mathcal{F}, K)$, and 0 and 1 the null and unit elements, respectively.

It is well known from [6] that $\bar{L}^0(\mathcal{F}, R)$ is a complete lattice under the ordering ≤: $\xi \le \eta$ iff $\xi^0(\omega) \le \eta^0(\omega)$ for P-almost all ω in Ω (briefly, a.s.), where ξ^0 and η^0 are arbitrarily chosen representatives of ξ and η, respectively. Furthermore, every subset A of $\bar{L}^0(\mathcal{F}, R)$ has a supremum, denoted by $\vee A$, and an infimum, denoted by $\wedge A$, and there exist two sequences $\{a_n, n \in N\}$ and $\{b_n, n \in N\}$ in A such that $\vee_{n\ge1} a_n = \vee A$ and $\wedge_{n\ge1} b_n = \wedge A$. If, in addition, A is directed (accordingly, dually directed), then the above $\{a_n, n \in N\}$ (accordingly, $\{b_n, n \in N\}$) can be chosen as nondecreasing (accordingly, nonincreasing). Finally $L^0(\mathcal{F}, R)$, as a sublattice of $\bar{L}^0(\mathcal{F}, R)$, is complete in the sense that every subset with an upper bound has a supremum (equivalently, every subset with a lower bound has an infimum).

Specially, let $\bar{L}^0_+(\mathcal{F}) = \{\xi \in \bar{L}^0(\mathcal{F}, R) | \xi \ge 0\}$ and $L^0_+(\mathcal{F}) = \{\xi \in L^0(\mathcal{F}, R) | \xi \ge 0\}$.

The following notions of generalized inverse, absolute value, complex conjugate and sign of an element in $L^0(\mathcal{F}, K)$ bring much convenience to this article.

Definition 2.1. [7] Let ξ be an element in $L^0(\mathcal{F}, K)$. For an arbitrarily chosen representative ξ^0 of ξ, define two $\mathscr{F}$-random variables $(\xi^0)^{-1}$ and $|\xi^0|$, respectively, by

$$(\xi^0)^{-1}(\omega) = \begin{cases} \dfrac{1}{\xi^0(\omega)} & \text{if } \xi^0(\omega) \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\left|\xi^0\right|(\omega) = \left|\xi^0(\omega)\right|, \quad \forall \omega \in \Omega.$$

Then the equivalence class of $(\xi^0)^{-1}$, denoted by ξ^{-1}, is called the generalized inverse of ξ; the equivalence class of $|\xi^0|$, denoted by $|\xi|$, is called the absolute value of ξ. When $\xi \in L^0(\mathcal{F}, C)$, set $\xi = u + iv$, where $u, v \in L^0(\mathcal{F}, R), \bar{\xi} := u - iv$ is called the complex conjugate of ξ and $\text{sgn}(\xi) := |\xi|^{-1} \cdot \xi$ is called the sign of ξ. It is obvious that $|\xi| = \left|\bar{\xi}\right|, \xi \cdot \text{sgn}\ (\bar{\xi}) = |\xi|, \left|\text{sgn}\ (\xi)\right| = \tilde{I}_A,\ \xi^{-1} \cdot \xi = \xi \cdot \xi^{-1} = \tilde{I}_A$, where $A = \{\omega \in \Omega : \xi^0(\omega) \neq 0\}$ and $\tilde{I}_A$ denotes the equivalence class of the characteristic function I_A of A. Throughout this article, the symbol $\tilde{I}_A$ is always understood as above unless stated otherwise.

Besides the equivalence classes of $\mathscr{F}$-random variables, we also use the equivalence classes of $\mathscr{F}$-measurable sets. Let $A \in \mathcal{F}$, then the equivalence class of A, denoted by $\tilde{A}$, is defined by $\tilde{A} = \{B \in \mathcal{F} : P(A \Delta B) = 0\}$, where $A\Delta B = (A \backslash B) \cup (B \backslash A)$ is the symmetric difference of A and B, and $P(\tilde{A})$ is defined to be $P(A)$. For two $\mathscr{F}$-measurable sets G and D, $G \subset D$ a.s. means $P(G \backslash D) = 0$, in which case we also say $\tilde{G} \subset \tilde{D}$; $\tilde{G} \cap \tilde{D}$ denotes the the equivalence class determined by $G \cap D$. Other similar notations are easily understood in an analogous manner.

As usual, we also make the following convention: for any $\xi,\ \eta \in L^0(\mathcal{F}, R), \xi > \eta$ means $\xi \ge \eta$ and $\xi \ne \eta$; $[\xi > \eta]$ stands for the equivalence class of the $\mathscr{F}$-measurable set $\{\omega \in \Omega : \xi^0(\omega) > \eta^0(\omega)\}$ (briefly, $[\xi^0 > \eta^0]$), where ξ^0 and η^0 are arbitrarily selected representatives of ξ and η, respectively, and $I_{[\xi > \eta]}$ stands for $\tilde{I}_{[\xi^0 > \eta^0]}$. If $A \in \mathcal{F}$, then $\xi > \eta$ on $\tilde{A}$ means $\xi^0(\omega) > \eta^0(\omega)$ a.s. on A, similarly $\xi \ne \eta$ on $\tilde{A}$ means that $\xi^0(\omega) \ne \eta^0(\omega)$ a.s. on A, also denoted by $\tilde{A} \subset [\xi \neq \eta]$.

Definition 2.2. [7] An ordered pair $(S, ||\cdot||)$ is called a random normed module (briefly, an RN module) over K with base $(\Omega, \mathcal{F}, P)$ if S is a left module over the algebra $L^0(\mathcal{F}, K)$ and $||\cdot||$ is a mapping from S to $L^0_+(\mathcal{F})$ such that the following conditions are satisfied:

(RNM-1) $||\xi x|| = |\xi|||x||, \forall \xi \in L^0(\mathcal{F}, K), x \in S$;
(RNM-2) $||x + y|| \leq ||x|| + ||y||, \forall x, y \in S$;
(RNM-3) $||x|| = 0$ implies $x = 0$(the zero element in S).

Where $||x||$ is called the L^0-norm of the vector x in S.

In this article, given an RN module $(S, ||\cdot||)$ over K with base $(\Omega, \mathcal{F}, P)$ it is always assumed that $(S, ||\cdot||)$ is endowed with its (ϵ, λ)-topology: for any $\epsilon > 0$, $0 < \lambda < 1$, let $N(\epsilon, \lambda) = \{x \in S \mid P\{\omega \in \Omega : ||x||(\omega) < \epsilon\} > 1 - \lambda\}$, then the family $\mathcal{U}_0 = \{N(\varepsilon, \lambda) | \varepsilon > 0, 0 < \lambda < 1\}$ forms a local base at the null element 0 of some metrizable linear topology for S, called the (ϵ, λ)-topology for S. It is well known that a sequence $\{x_n, n \geq 1\}$ in S converges in the (ϵ, λ)-topology to some x in S if $\{||x_n - x||, n \geq 1\}$ converges in probability P to 0, and that S is a topological module over the topological algebra $L^0(\mathcal{F}, K)$, namely the module multiplication $\cdot : L^0(\mathcal{F}, K) \times S \to S$ is jointly continuous (see [7] for details). Besides, let $L^0(\mathcal{F}, K)$ be the RN module of equivalence classes of X-valued $\mathscr{F}$-random variables on $(\Omega, \mathcal{F}, P)$, where X is an ordinary normed space, then it is easy to see that the (ϵ, λ)-topology on $L^0(\mathcal{F}, K)$ is exactly the topology of convergence in probability and $L^0(\mathcal{F}, K)$ is complete iff X is complete, in particular $L^0(\mathcal{F}, K)$ is complete.

Definition 2.3. [5] An ordered pair $(S, ||\cdot||)$ is called a random normed algebra (briefly, an RN algebra) over K with base $(\Omega, \mathcal{F}, P)$ if $(S, ||\cdot||)$ is an RN module over K with base $(\Omega, \mathcal{F}, P)$ and also a ring such that the following two conditions are satisfied:

(1) $(\xi \cdot x)y = x(\xi \cdot y) = \xi \cdot (xy)$, for all $\xi \in L^0(\mathcal{F}, K)$ and all $x, y \in S$;
(2) the L^0-norm $||\cdot||$ is submultiplicative, that is, $||xy|| \leq ||x||||y||$, for all $x, y \in S$.

Furthermore, the RN algebra is said to be unital if it has the identity element e and $||e|| = 1$. As usual, the RN algebra $(S, ||\cdot||)$ is said to be complete if the RN module $(S, ||\cdot||)$ is complete.

Example 2.1. [5] Let $(X, ||\cdot||)$ be a normed algebra over C and $L^0(\mathcal{F}, X)$ be the RN module of equivalence classes of X-valued $\mathscr{F}$-random variables on $(\Omega, \mathcal{F}, P)$. Define a multiplication $\cdot : L^0(\mathcal{F}, X) \times L^0(\mathcal{F}, X) \to L^0(\mathcal{F}, X)$ by $x \cdot y$ = the equivalence class determined by the $\mathscr{F}$-random variable x^0y^0, which is defined by $(x^0y^0)(\omega) = (x^0(\omega)) \cdot (y^0(\omega))$, $\forall \omega \in \Omega$, where x^0 and y^0 are arbitrarily chosen representatives of x and y in $L^0(\mathcal{F}, X)$, respectively. Then $(L^0(\mathcal{F}, X), \|\cdot\|)$ is an RN algebra, in particular $L^0(\mathcal{F}, C)$ is a unital RN algebra with identity 1.

Example 2.2. [5] It is easy to see that $L^\infty_{\mathcal{F}}(\varepsilon, C)$ is a unital RN algebra with identity 1 (see [8,9] for the construction of $L^\infty_{\mathcal{F}}(\varepsilon, C)$.

Definition 2.4. [5] Let $(S, ||\cdot||)$ be an RN algebra with identity e over C with base $(\Omega, \mathcal{F}, P)$, and A be any given element in $\mathscr{F}$ such that $P(A) > 0$. An element $x \in S$ is invertible on A if there exists $y \in S$ such that $\tilde{I}_A \cdot xy = \tilde{I}_A \cdot yx = \tilde{I}_A \cdot e$. Clearly, $\tilde{I}_A \cdot y$ is unique and called the inverse on A of x, denoted by x_A^{-1}. Let $G(S, A)$ denote the set of elements of S which are invertible on A. Then $\tilde{I}_A \cdot G(S, A)$ is also a group, and $(xy)_A^{-1} = y_A^{-1} x_A^{-1}$ for any x and y in $\tilde{I}_A \cdot G(S, A)$. For any $x \in S$, the sets

$$\sigma(x, S, A) = \left\{ \xi \in L^0(\mathcal{F}, C) : \tilde{I}_A \cdot (\xi \cdot e - x) \notin \tilde{I}_A \cdot G(S, A) \right\},$$
$$\sigma(x, S) = \bigcap_{A \in \mathcal{F}} \sigma(x, S, A)$$

are called the random spectrum on A of x in S and the random spectrum of x in S, respectively, and further their complements $\rho(x, S, A) = L^0(\mathcal{F}, C) \backslash \sigma(x, S, A)$ and $\rho(x, S) = L^0(\mathcal{F}, C) \backslash \sigma(x, S)$ are called the random resolvent set on A of x and the random resolvent set of x, respectively.

Definition 2.5. [5] Let $(S, ||\cdot||)$ be an RN algebra with identity e over C with base $(\Omega, \mathcal{F}, P)$. For any $x \in S$, $r(x) = \vee\{|\xi| : \xi \in \sigma(x, S)\}$ is called the random spectral radius of x.

Besides, $\wedge \left\{ \|x^n\|^{\frac{1}{n}} | n \in N \right\}$ is denoted by $r_p(x)$, for any x in an RN algebra over K with base $(\Omega, \mathcal{F}, P)$.

Lemma 2.1. [5] Let $(S, ||\cdot||)$ be a unital complete RN algebra with identity e over C with base $(\Omega, \mathcal{F}, P)$. Then for any $x \in S$, $\sigma(x, S)$ is nonempty and $r(x) = r_p(x)$.

3 Main results and proofs

Definition 3.1. Let S be a random normed algebra, $A \in \mathcal{F}$ and f be an L^0-linear function on S, i.e., a mapping from S to $L^0(\mathcal{F}, C)$ such that $f(\xi \cdot x + \eta \cdot y) = \xi f(x) + \eta f(y)$ for all $\xi, \eta \in L^0(\mathcal{F}, C)$ and $x, y \in S$. Then f is called multiplicative if $f(xy) = f(x)f(y)$ for all $x, y \in S$ and is called nonzero if there exists $x \in S$ such that $[f(x) \neq 0] = \tilde{\Omega}$.

Lemma 3.1. Let S be a random normed algebra with identity e, and let f be an L^0-function on S satisfying $f(e) = 1$ and $f(x^2) = f(x)^2$ for all $x \in S$. Then f is multiplicative.

Proof. By assumption we obtain

$$\begin{aligned} f(x^2) + f(xy + yx) + f(y^2) &= f(x^2 + xy + yx + y^2) \\ &= f((x + y)^2) \\ &= f(x + y)^2 \\ &= f(x)^2 + 2f(x)f(y) + f(y)^2, \end{aligned}$$

and hence

$$f(xy + yx) = 2f(x)f(y)$$

for all $x, y \in S$. So it remains to verify that $f(xy) = f(yx)$. For $a, b \in S$, the identity

$$(ab - ba)^2 + (ab + ba)^2 = 2[a(bab) + (bab)a]$$

implies

$$\begin{aligned} f(ab-ba)^2+4f(a)^2f(b)^2 &= f((ab-ba)^2)+f(ab+ba)^2 \\ &= f((ab-ba)^2+(ab+ba)^2) \\ &= f((ab-ba)^2+(ab+ba)^2) \\ &= 2f(a(bab)+(bab)a) \\ &= 4f(a)f(bab). \end{aligned}$$

Taking $a = x - f(x) \cdot e$, so that $f(a) = 0$, and $b = y$ we get $f(ay) = f(ya)$ and hence $f(xy) = f(yx)$. This completes the proof of Lemma 3.1.

The following theorem is a new version of the Gleason-Kahane-Żelazko theorem.

Theorem 3.1 Let S be an unital complete random normed algebra with identity e, and let f be an L^0-linear function on S. Then the following conditions are equivalent.

(1) f is nonzero and multiplicative.

(2) $f(e) = 1$ and $f(x) \neq 0$ on $\tilde{A}$ for any $A \in \mathcal{F}$ with $P(A) > 0$ and $x \in G(S, A)$.

(3) $f(x) \in \sigma(x, S)$ for every $x \in S$.

Proof If f is multiplicative, then $f(e) = f(e^2) = f(e)f(e)$. Since f is nonzero, we have $f(e) = 1$ and hence $\tilde{I}_A = \tilde{I}_A f(e) = f(xx_A^{-1}) = f(x)f(x_A^{-1})$ for any $A \in \mathcal{F}$ with $P(A) > 0$ and $x \in G(S, A)$. Thus (1)⇒(2). (2)⇒(3) is clear since if $\xi \in \rho(x, S)$, then there exists $A \in \mathcal{F}$ with $P(A) > 0$ such that $\tilde{I}_A(\xi - f(x)) = f[\tilde{I}_A \cdot (\xi \cdot e - x)] \neq 0$ on $\tilde{A}$ and hence $f(x) \in \sigma(x, S)$. Assume (3), then $f(e) = 1$ since $f(e) \in \sigma(e, S)$. Now, let $n \geq 2$ and consider the random polynomial

$$p(\lambda) = f((\lambda \cdot e - x)^n)$$

of degree n. Therefore we can find $\lambda_i \in L^0(\mathcal{F}, C)(i = 1, 2 \ldots n)$ such that

$$0 = p(\lambda_i) = f((\lambda_i \cdot e - x)^n) \in \sigma((\lambda_i \cdot e - x)^n, S)$$

for each λ_i. This implies that $\lambda_i \in \sigma(x, S)$ and hence $|\lambda_i| < r_p(x)$ by Lemma 2.1. Note that

$$\prod_{i=1}^{n}(\lambda - \lambda_i) = p(\lambda) = \lambda^n - nf(x)\lambda^{n-1} + C_n^2 f(x^2)\lambda^{n-2} + \cdots + (-1)^n f(x^n).$$

Comparing coefficients we can see that

$$\sum_{i=1}^{n}\lambda_i = nf(x), \qquad \sum_{1\leq i<j\leq n}\lambda_i\lambda_j = C_n^2 f(x^2).$$

On the other hand, by the second equation,

$$\left(\sum_{i=1}^{n}\lambda_i\right)^2 = \sum_{i=1}^{n}\lambda_i^2 + 2\sum_{1\leq i<j\leq n}\lambda_i\lambda_j = \sum_{i=1}^{n}\lambda_i^2 + n(n-1)f(x^2).$$

Combining these equalities yields

$$n^2\left|f(x)^2 - f(x^2)\right| = \left|-nf(x^2) + \sum_{i=1}^{n}\lambda_i^2\right| \leq n\left|f(x)^2\right| + nr_p(x)^2.$$

Hence

$$\left|f(x)^2 - f(x^2)\right| \leq \frac{1}{n}[\left|f(x^2)\right| + r_p(x)^2].$$

Letting $n \to \infty$, we then obtain $f(x^2) = f(x)^2$ for all $x \in S$. It follows from Lemma 3.1 that f is multiplicative. Clearly, f is nonzero. Thus (3)⇒(1). This completes the proof of Theorem 3.1.

Remark 3.1. When the base space $(\Omega, \mathcal{F}, P)$ of the RN module is a trivial probability space, i.e., $\mathcal{F} = \{\Omega, \emptyset\}$, the new version of the Gleason-Kahane-Żelazko theorem automatically degenerates to the classical case.

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Competing interests
The author declares that they have no competing interests.

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