Rahim Chowdhury and Cho *Journal of Inequalities and Applications* 2012, **2012**:79
http://www.journalofinequalitiesandapplications.com/content/2012/1/79

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Existence theorems of generalized quasi-variational-like inequalities for η-h-pseudo-monotone type I operators on non-compact sets

Mohammad Showkat Rahim Chowdhury[1] and Yeol Je Cho[2*]

* Correspondence: yjcho@gnu.ac.kr
[2]Department of Mathematics Education and RINS, Gyeongsang National University, Chinju 660-701, Korea
Full list of author information is available at the end of the article

Abstract

In this article, we prove the existence results of solutions for a new class of generalized quasi-variational-like inequalities (GQVLI) for η-h-pseudo-monotone type I operators defined on non-compact sets in locally convex Hausdorff topological vector spaces. In obtaining our results on GQVLI for η-h-pseudo-monotone type I operators, we use Chowdhury and Tan's generalized version of Ky Fan's minimax inequality as the main tool.

Keywords: generalized quasi-variational-like inequalities, η-h-pseudo-monotone type I operators, locally convex Hausdorff topological vector spaces

1. Introduction

If X is a nonempty set, then we denote by 2^X the family of all non-empty subsets of X and by $\mathcal{F}(x)$ the family of all non-empty finite subsets of X. Let E be a topological vector space over Φ, F be a vector space over Φ and X be a non-empty subset of E. Let $\langle \cdot, \cdot \rangle$ $F \times E \to \Phi$ be a bilinear functional. Throughout this article, Φ denotes either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$.

For each $x_0 \in E$, each nonempty subset A of E and each $\epsilon > 0$, let $W(x_0; \epsilon) := \{y \in F : |\langle y, x_0 \rangle| < \epsilon\}$ and $U(A; \epsilon) := \{y \in F : \sup_{x \in A} |\langle y, x \rangle| < \epsilon\}$. Let $\sigma\langle F, E\rangle$ be the (weak) topology on F generated by the family $\{W(x; \epsilon) : x \in E, \epsilon > 0\}$ as a subbase for the neighborhood system at 0 and $\delta\langle F, E\rangle$ be the (strong) topology on F generated by the family $\{U(A; \epsilon) : A$ is a non-empty bounded subset of E and $\epsilon > 0\}$ as a base for the neighborhood system at 0. We note then that F, when equipped with the (weak) topology $\sigma\langle F, E\rangle$ or the (strong) topology $\delta\langle F, E\rangle$, becomes a locally convex topological vector space which is not necessarily Hausdorff. But, if the bilinear functional $\langle \cdot, \cdot \rangle : F \times E \to \Phi$ separates points in F, i.e., for each $y \in F$ with $y \neq 0$, there exists $x \in E$ such that $\langle y, x \rangle \neq 0$, then F also becomes Hausdorff. Furthermore, for any net $\{y_\alpha\}_{\alpha \in \Gamma}$ in F and $y \in F$,

(*a*) $y_\alpha \to y$ in $\sigma\langle F, E\rangle$ if and only if $\langle y_\alpha, x\rangle \to \langle y, x\rangle$ for each $x \in E$;

(*b*) $y_\alpha \to y$ in $\delta\langle F, E\rangle$ if and only if $\langle y_\alpha, x\rangle \to \langle y, x\rangle$ uniformly for each $x \in A$, where A is a nonempty bounded subset of E.

Suppose that, for the sets X, E, and F mentioned above, $S : X \to 2^X$, $T : X \to 2^F$ are two set-valued mappings, $f : X \to F$, $\eta : X \times X \to E$ are two single-valued mappings and $h : X \times X \to \mathbb{R}$ is a real-valued function. As introduced by Shih and Tan [1], the generalized quasi-variational inequality in infinite dimensional spaces is defined as follows: Find $\hat{y} \in S(\hat{y})$ and $\hat{w} \in T(\hat{y})$ such that

$$\operatorname{Re}\langle \hat{w}, \hat{y} - x\rangle \leq 0$$

for all $x \in S(\hat{y})$.

Now, we introduce the following definition:

Definition 1.1. Let X, E, and F be the sets and the mappings S, T, η, and h be as defined above. Then the generalized quasi-variational-like inequality problem is defined as follows: Find $\hat{y} \in S(\hat{y})$ and $\hat{w} \in T(\hat{y})$ such that

$$\operatorname{Re}\langle \hat{w}, \eta(\hat{y}, x)\rangle + h(\hat{y}, x) \leq 0$$

for all $x \in S(\hat{y})$.

For more results related to the generalized quasi-variational-like inequality problem, refer to [2-5], and therein.

The following definition is a slight modifications of pseudo-monotone operators defined in [6, Definition 1] and of pseudo-monotone type I operators defined in [7] (see also [8]):

Definition 1.2. Let X be a non-empty subset of a topological vector space E over Φ, F be a vector space over Φ which is equipped with $\sigma\langle F, E\rangle$-topology, where $\langle \cdot, \cdot\rangle : F \times E \to \Phi$ is a bilinear functional. Let $h : X \times X \to \mathbb{R}$, $\eta : X \times X \to E$, and $T : X \to 2^F$ be three mappings. Then T is said to be:

(1) an (η, h)-pseudo-monotone type I operator if, for each $y \in X$ and every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y (respectively, weakly to y) with

$$\limsup_{\alpha} \left[\inf_{u \in T(y_\alpha)} \operatorname{Re}\langle u, \eta(y_\alpha, y)\rangle + h(y_\alpha, y) \right] \leq 0,$$

we have

$$\begin{aligned} &\limsup_{\alpha} \left[\inf_{u \in T(y_\alpha)} \operatorname{Re}\langle u, \eta(y_\alpha, y)\rangle + h(y_\alpha, x) \right] \\ &\geq \inf_{w \in T(y)} \operatorname{Re}\langle w, \eta(y, x)\rangle + h(y, x) \end{aligned}$$

for all $x \in X$;

(2) an h-pseudo-monotone type I operator if T is an (η, h)-pseudo-monotone type I operator with $\eta(x, y) = x - y$ and, for some $h' : X \to \mathbb{R}$, $h(x, y) = h'(x) - h'(y)$ for all $x, y \in X$.

Note that, if $F = E^*$, the topological dual space of E, then the notions of h-pseudo-monotone type I operators coincide with those in [6].

Pseudo-monotone type I operators were first introduced by Chowdhury and Tan [6] with a slight variation in the name of this operator. Later, these operators were renamed as pseudo-monotone type I operators by Chowdhury [7]. The pseudo-monotone type I operators are set-valued generalization of the classical (single-valued) pseudo-monotone operators with slight variations. The classical definition of a single-valued pseudo-monotone operator was introduced by Brézis et al. [9].

In this article, we obtain some general theorems on solutions for a new class of generalized quasi-variational-like inequalities for pseudo-monotone type I operators defined on non-compact sets in topological vector spaces. For the main results, we mainly use the following generalized version of Ky Fan's minimax inequality [10] due to Chowdhury and Tan [6].

Theorem 1.1. *Let E be a topological vector space, X be a nonempty convex subset of E and $f : X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be such that*

(a) *for each $A \in \mathcal{F}(X)$ and each fixed $x \in co(A)$, $y \mapsto f(x, y)$ is lower semi-continuous on $co(A)$;*

(b) *for each $A \in \mathcal{F}(X)$ and $y \in co(A)$, $\min_{x \in A} f(x, y) \le 0$;*

(c) *for each $A \in \mathcal{F}(X)$ and $x, y \in co(A)$, every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y with $f(tx + (1 - t)y, y_\alpha) \le 0$ for all $\alpha \in \Gamma$ and $t \in [0, 1]$, we have $f(x, y) \le 0$;*

(d) *there exist a nonempty closed and compact subset K of X and $x_0 \in K$ such that $f(x_0, y) > 0$ for all $y \in X \setminus K$.*

Then there exists $\hat{y} \in K$ such that $f(x, \hat{y}) \le 0$ for all $x \in X$.

Definition 1.3. A function $\varphi : X \times X \to \mathbb{R} \cup \{\pm\infty\}$ is said to be 0-diagonally concave (in short, 0-DCV) in the second argument [14] if, for any finite set $\{x_1, \ldots, x_n\} \subset X$ and $\lambda_i \ge 0$ with $\sum_{i=1}^{n} \lambda_i = 1$, we have $\sum_{i=1}^{n} \lambda_i \phi(y, x_i) \le 0$, where $y = \sum_{i=1}^{n} \lambda_i x_i$.

Now, we state the following definition given in [8]:

Definition 1.4. Let X, E, F be be the sets defined before and $T : X \to 2^F$, $\eta : X \times X \to E$, $g : X \to E$ be mappings.

(1) The mappings T and η are said to have 0-diagonally concave relation (in short, 0-DCVR) if the function $\varphi : X \times X \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\phi(x, y) = \inf_{w \in T(x)} \mathrm{Re}\langle w, \eta(x, y)\rangle$$

is 0-DCV in y;

(2) The mappings T and g are said to have 0-diagonally concave relation if T and $\eta(x, y) = g(x) - g(y)$ have the 0-DCVR.

2. Preliminaries

Now, we start with some earlier studies which will be needed for our main results. We first state the following result which is Lemma 1 of Shih and Tan [1]:

Lemma 2.1. *Let X be a nonempty subset of a Hausdorff topological vector space E and $S : X \to 2^E$ be an upper semi-continuous map such that $S(x)$ is a bounded subset of E for each $x \in X$. Then, for each continuous linear functional p on E, the mapping $f_p : X \to \mathbb{R}$ defined by $f_p(y) = \sup_{x \in S(y)} \mathrm{Re}\langle p, x\rangle$ is upper semi-continuous, i.e., for each $\lambda \in \mathbb{R}$, the set $\{y \in X : f_p(y) = \sup_{x \in S(y)} \mathrm{Re}\langle p, x\rangle < \lambda\}$ is open in X.*

The following result is Takahashi [[11], Lemma 3] (see also [[12], Lemma 3]):

Lemma 2.2. *Let X and Y be topological spaces, $f : X \to \mathbb{R}$ be non-negative and continuous and $g : Y \to \mathbb{R}$ be lower semi-continuous. Then the mapping $F : X \times Y \to \mathbb{R}$ defined by $F(x, y) = f(x)g(y)$ for all $(x, y) \in X \times Y$ is lower semi-continuous.*

The following result which follows from slight modification of Chowdhury and Tan [6, Lemma 3]:

Lemma 2.3. *Let E be a Hausdorff topological vector space over Φ, $A \in \mathcal{F}(E)$ and $X =$ co(A). Let F be a vector space over Φ which is equipped with $\sigma\langle F, E\rangle$ -topology such* *that, for each $w \in F$, $x \mapsto \langle w, x\rangle$ is continuous. Let $\eta : X \times X \to E$ be continuous in the first argument. Let $T : X \mapsto 2F \setminus \varnothing$ be upper semi-continuous from X to the $\sigma\langle F, E\rangle$-topology on F such that each $T(x)$ is $\sigma\langle F, E\rangle$ -compact. Let $f : X \times X \to \mathbb{R}$ be defined by $f(x, y) = \inf_{w \in T(y)} \mathrm{Re}\langle w, \eta(y, x)\rangle$ for all $x, y \in X$. Then, for each fixed $x \in X$, $y \mapsto f(x, y)$ is lower semi-continuous on X.*

We need the following Kneser's minimax theorem in [13] (see also Aubin [14]):

Theorem 2.1. *Let X be a non-empty convex subset of a vector space and Y be a none-mpty compact convex subset of a Hausdorff topological vector space. Suppose that f is a real-valued function on $X \times Y$ such that for each fixed $x \in X$, the map $y \mapsto f(x, y)$, i.e., $f(x, \cdot)$ is lower semi-continuous and convex on Y and, for each fixed $y \in Y$, the mapping $x \mapsto f(x, y)$, i.e., $f(\cdot, y)$ is concave on X. Then*

$$\min_{y\in Y} \sup_{x\in X} f(x, y) = \sup_{x\in X} \min_{y\in Y} f(x, y).$$

3. Existence theorems for generalized quasi-variational-like inequalities for η-h-pseudo-monotone type I operators

In this section, we prove some existence theorems for the solutions to the generalized quasi-variational-like inequalities for pseudo-monotone type I operators T with non-compact domain in locally convex Hausdorff topological vector spaces. Our results extend and or generalize the corresponding results in [1].

First, we establish the following result:

Theorem 3.1. *Let E be a locally convex Hausdorff topological vector space over Φ, X be a non-empty para-compact convex and bounded subset of E and F be a vector space over Φ with $\sigma\langle F, E\rangle$ -topology, where $\langle \cdot, \cdot\rangle : F \times E \to \Phi$ is a bilinear functional such that, for each $w \in F$, the function $x \mapsto \mathrm{Re}\langle w, x\rangle$ is continuous. Let $S : X \to 2^X$, $T : X \to 2^F$, $\eta : X \times X \to F$, and $h : E \times E \to \mathbb{R}$ be the mappings such that*

(*a*) *S is upper semi-continuous such that each $S(x)$ is compact and convex;*

(*b*) *$h(X \times X)$ is bounded;*

(*c*) *T is an (η, h)-pseudo-monotone type I operator and upper semi-continuous from $co(A)$ to the $\sigma\langle F, E\rangle$-topology on F for each $A \in \mathcal{F}(X)$such that each $T(x)$ is $\sigma\langle F, E\rangle$ -compact and convex and $T(X)$ is $\delta\langle F, E\rangle$ -bounded;*

(*d*) *T and η have the 0 - DCV R;*

(*e*) *for each fixed $y \in X$, $x \mapsto \eta(x, y)$, i.e., $\eta(\cdot, y)$ is continuous and $x \mapsto h(x, y)$, i.e., $h(\cdot, y)$ is lower semi-continuous on $co(A)$ for each $A \in \mathcal{F}(X)$and for each $x \in X$, $h(x, \cdot)$ and $\eta(x, \cdot)$ are concave, and $h(x, x) = 0$, $\eta(x, x) = 0$;*

(*f*) *the set $\Sigma = \{y \in X : \sup_{x\in S(y)} [\inf_{w\in T(y)} \mathrm{Re}\langle w, \eta(y, x)\rangle + h(y, x)] > 0\}$ is open in X.*

Suppose further that there exist a non-empty compact convex subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\inf_{w\in T(y)} \mathrm{Re}\langle w, \eta(y, x_0)\rangle + h(y, x_0) > 0$ for all $y \in X \setminus K$. Then there exist a point $\hat{y} \in X$ such that

(1) *$\hat{y} \in S(\hat{y})$;*

(2) *there exists a point $\hat{w} \in T(\hat{y})$ with $\mathrm{Re}\langle \hat{w}, \eta(\hat{y}, x)\rangle + h(\hat{y}, x) \le 0$ for all $x \in S(\hat{y})$.*

Proof. Let us first show that there exist a point $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x\in S(\hat{y})} \left[\inf_{w\in T(\hat{y})} \mathrm{Re}\langle w, \eta(\hat{y}, x)\rangle + h(\hat{y}, x) \right] \le 0.$$

Now, we prove this by contradiction. So, we assume that, for each $y \in X$, either $y \notin S(y)$ or there exists $x \in S(y)$ such that $\inf_{w\in T(y)} \mathrm{Re}\langle w, \eta(y, x) \rangle + h(y, x) > 0$, that is, for each $y \in X$, either $y \notin S(y)$ or $y \in \Sigma$. If $y \notin S(y)$, then, by a slight modification of a separation theorem for convex sets in locally convex Hausdorff topological vector spaces, there exists a continuous linear functional p on E such that

$$\mathrm{Re}\langle p, y\rangle - \sup_{x\in S(y)} \mathrm{Re}\langle p, x\rangle > 0.$$

For each $y \in X$, set

$$\gamma(y) := \sup_{x\in S(y)} \left[\inf_{w\in T(y)} \mathrm{Re}\ \langle w, \eta(y, x) + h(y, x)\right],$$
$$V_0 := \sum = \{y \in X : \gamma(y) > 0\}$$

and, for each continuous linear functional p on E,

$$V_p := \left\{y \in X : \mathrm{Re}\langle p, y\rangle - \sup_{x\in S(y)} \mathrm{Re}\langle p, x\rangle > 0\right\}.$$

Then we have

$$X = V_0 \cup \bigcup_{p\in LF(E)} V_p,$$

where $LF(E)$ denotes the set of all continuous linear functionals on E. Since V_0 is open by hypothesis and each V_p is open in X by Lemma 2.1 ([[12], Lemma 1]), $\{V_0, V_p : p \in LF(E)\}$ is an open covering for X. Since X is para-compact, there exists a continuous partition of unity $\{\beta_0, \beta_p : p \in LF(E)\}$ for X subordinated to the open cover $\{V_0, V_p : p \in LF(E)\}$. Note that, for each $y \in X$ and $A \in \mathcal{F}(X)$, $x \mapsto h(x, y)$, i.e., $h(\cdot, y)$ is continuous on $co(A)$ (see [[15], Corollary 10.1.1]). Define a function $\varphi : X \times X \to \mathbb{R}$ by

$$\phi(x, y) = \beta_0(y) \left[\min_{w\in T(y)} \mathrm{Re}\ \langle w, \eta(y, x)\rangle + h(y, x)\right] + \sum_{p\in LF(E)} \beta_p(y) \mathrm{Re}\ \langle p, \eta(y, x)\rangle$$

for all $x, y \in X$. Then we have the following:

(I) Since E is Hausdorff, for each $A \in \mathcal{F}(X)$ and each fixed $x \in co(A)$, the mapping

$$y \mapsto \inf_{w\in T(y)} \mathrm{Re}\langle w, \eta(y, x)\rangle + h(y, x)$$

is lower semi-continuous on $co(A)$ by Lemma 2.3 and the fact that h is continuous on $co(A)$ and so the mapping

$$y \mapsto \beta_0(y) \left[\inf_{w\in T(y)} \mathrm{Re}\langle w, \eta(y, x)\rangle + h(y, x)\right]$$

is lower semi-continuous on $co(A)$ by Lemma 2.2. Also, for each fixed $x \in X$,

$$y \mapsto \sum_{p\in LF(E)} \beta_p(y)\ \mathrm{Re}\langle p, \eta(y, x)\rangle$$

is continuous on X. Hence, for each $A \in \mathcal{F}(X)$ and each fixed $x \in co(A)$, the mapping $y \mapsto \varphi(x, y)$ is lower semi-continuous on $co(A)$.

(II) For each $A \in \mathcal{F}(X)$ *and* $y \in co(A)$, $\min_{x \in A} \varphi(x, y) \leq 0$. Indeed, if this were false, then, for some $A = \{x_1, x_2, \ldots, x_n\} \in \mathcal{F}(X)$ and some $y \in co(A)$ (say $y = \sum_{i=1}^{n} \lambda_i x_i$, where $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$), we have $\min_{1 \leq i \leq n} \varphi(x_i, y) > 0$. Then, for each $i = 1, 2, \ldots, n$,

$$\beta_0(y)\left[\inf_{w \in T(y)} \mathrm{Re}\langle w, \eta(y, x_i)\rangle + h(y, x_i)\right] + \sum_{p \in LF(E)} \beta_p(y)\, \mathrm{Re}\langle p, \eta(y, x_i)\rangle > 0$$

and so

$$\begin{aligned}
0 &= \phi(y, y) \\
&= \beta_0(y)\left[\inf_{w \in T(y)} \mathrm{Re}\ \langle w, \eta(y, \sum_{i=1}^{n} \lambda_i x_i)\rangle + h\left(y, \sum_{i=1}^{n} \lambda_i x_i\right)\right] \\
&\quad + \sum_{p \in LF(E)} \beta_p(y)\ \mathrm{Re}\ \langle p, \eta(y, \sum_{i=1}^{n} \lambda_i x_i)\rangle \\
&\geq \sum_{i=1}^{n} \lambda_i \left(\beta_0(y)\left[\inf_{w \in T(y)} \mathrm{Re}\ \langle w, \eta(y, x_i)\rangle + h(y, x_i)\right]\right. \\
&\quad \left. + \sum_{p \in LF(E)} \beta_p(y)\ \mathrm{Re}\ \langle p, \eta(y, x_i)\rangle\right) \\
&> 0,
\end{aligned}$$

which is a contradiction.

(III) Suppose that $A \in \mathcal{F}(X)$, $x, y \in co(A)$ and $\{y_\alpha\}_{\alpha \in \Gamma}$ is a net in X converging to y with $\varphi(tx + (1 - t)y, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$ and all $t \in [0, 1]$.

Case 1: $\beta_0(y) = 0$. Note that $\beta_0(y_\alpha) \geq 0$ for each $\alpha \in \Gamma$ and $\beta_0(y_\alpha) \to 0$. Since $T(X)$ is strongly bounded and $\{y_\alpha\}_{\alpha \in \Gamma}$ is a bounded net, it follows that

$$\limsup_{\alpha}\left[\beta_0(y_\alpha) \min_{w \in T(y_\alpha)} \mathrm{Re}\ \langle w, \eta(y_\alpha, x)\rangle + h(y_\alpha, x)\right] = 0. \tag{3.1}$$

Also, we have

$$\beta_0(y)\left[\min_{w \in T(y)} \mathrm{Re}\langle w, \eta(y, x)\rangle + h(y, x)\right] = 0.$$

Thus it follows from (3.1) that

$$\begin{aligned}
&\limsup_{\alpha}\left[\beta_0(y_\alpha) \min_{w \in T(y_\alpha)} \mathrm{Re}\langle w, \eta(y_\alpha, x)\rangle + h(y_\alpha, x)\right] + \sum_{p \in LF(E)} \beta_p(y)\, \mathrm{Re}\langle p, \eta(y, x)\rangle \\
&= \sum_{p \in LF(E)} \beta_p(y)\, \mathrm{Re}\langle p, \eta(y, x)\rangle \\
&= \beta_0(y)\left[\min_{w \in T(y)} \mathrm{Re}\langle w, \eta(y, x)\rangle + h(y, x)\right] + \sum_{p \in LF(E)} \beta_p(y)\, \mathrm{Re}\langle p, \eta(y, x)\rangle.
\end{aligned} \tag{3.2}$$

When $t = 1$, we have $\varphi(x, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$, i.e.,

$$\beta_0(y_\alpha)\left[\min_{w \in T(y_\alpha)} \mathrm{Re}\langle w, \eta(y_\alpha, x)\rangle + h(y_\alpha, x)\right] + \sum_{p \in LF(E)} \beta_p(y_\alpha)\, \mathrm{Re}\langle p, \eta(y_\alpha, x)\rangle \leq 0 \tag{3.3}$$

for all $\alpha \in \Gamma$. Therefore, by (3.3), we have

$$\begin{aligned}
&\limsup_{\alpha} \left[\beta_0(y_\alpha) \min_{w \in T(y_\alpha)} \operatorname{Re} \langle w, \eta(y_\alpha, x) \rangle + h(y_\alpha, x) \right] \\
&\quad + \liminf_{\alpha} \left[\sum_{p \in LF(E)} \beta_p(y_\alpha) \operatorname{Re} \langle p, \eta(y_\alpha, x) \rangle \right] \\
&\le \limsup_{\alpha} \left[\beta_0(y_\alpha) \min_{w \in T(y_\alpha)} \operatorname{Re} \langle w, \eta(y_\alpha, x) \rangle + h(y_\alpha, x) + \sum_{p \in LF(E)} \beta_p(y_\alpha) \operatorname{Re} \langle p, \eta(y_\alpha, x) \rangle \right] \\
&\le 0
\end{aligned}$$

and so

$$\limsup_{\alpha} \left[\beta_0(y_\alpha) \min_{w \in T(y_\alpha)} \operatorname{Re} \langle w, \eta(y_\alpha, x) \rangle + h(y_\alpha, x) \right] + \sum_{p \in LF(E)}^{n} \beta_p(y) \operatorname{Re} \langle p, \eta(y, x) \rangle \le 0. \tag{3.4}$$

Hence, by (3.2) and (3.4), we have $\varphi(x, y) \le 0$.

Case 2. $\beta_0(y) > 0$. Since $\beta_0(y_\alpha) \to \beta_0(y)$, there exists $\lambda \in \Gamma$ such that $\beta_0(y_\alpha) > 0$ for all $\alpha \ge \lambda$. When $t = 0$, we have $\varphi(y, y_\alpha) \le 0$ for all $\alpha \in \Gamma$, i.e.,

$$\beta_0(y_\alpha) \left[\inf_{w \in T(y_\alpha)} \operatorname{Re} \langle w, \eta(y_\alpha, y) \rangle + h(y_\alpha, y) \right] + \sum_{p \in LF(E)} \beta_p(y_\alpha) \operatorname{Re} \langle p, \eta(y_\alpha, y) \rangle \le 0$$

for all $\alpha \in \Gamma$ and so

$$\limsup_{\alpha} \left[\beta_0(y_\alpha) \inf_{w \in T(y_\alpha)} \operatorname{Re} \langle w, \eta(y_\alpha, y) \rangle + h(y_\alpha, y) + \sum_{p \in LF(E)} \beta_p(y_\alpha) \operatorname{Re} \langle p, \eta(y_\alpha, y) \rangle \right] \le 0. \tag{3.5}$$

Hence, by (3.5), we have

$$\begin{aligned}
&\limsup_{\alpha} \left[\beta_0(y_\alpha) \inf_{w \in T(y_\alpha)} \operatorname{Re} \langle w, \eta(y_\alpha, y) \rangle + h(y_\alpha, y) \right] \\
&\quad + \liminf_{\alpha} \left[\sum_{p \in LF(E)} \beta_p(y_\alpha) \operatorname{Re} \langle p, \eta(y_\alpha, y) \rangle \right] \\
&\le \limsup_{\alpha} \left[\beta_0(y_\alpha) \inf_{w \in T(y_\alpha)} \operatorname{Re} \langle w, \eta(y_\alpha, y) \rangle + h(y_\alpha, y) + \sum_{p \in LF(E)} \beta_p(y_\alpha) \operatorname{Re} \langle p, \eta(y_\alpha, y) \rangle \right] \\
&\le 0.
\end{aligned}$$

Since $\liminf_{\alpha} [\sum_{p \in LF(E)} \beta_p(y_\alpha) \operatorname{Re} \langle p, \eta(y_\alpha, y) \rangle] = 0$, we have

$$\limsup_{\alpha} \left[\beta_0(y_\alpha) \min_{w \in T(y_\alpha)} \operatorname{Re} \langle w, \eta(y_\alpha, y) \rangle + h(y_\alpha, y) \right] \le 0. \tag{3.6}$$

Since $\beta_0(y_\alpha) > 0$ for all $\alpha \ge \lambda$, it follows that

$$\begin{aligned}
&\beta_0(y) \limsup_{\alpha} \left[\min_{w \in T(y_\alpha)} \operatorname{Re} \langle w, \eta(y_\alpha, y) \rangle + h(y_\alpha, y) \right] \\
&= \limsup_{\alpha} \left[\beta_0(y_\alpha) \min_{w \in T(y_\alpha)} \operatorname{Re} \langle w, \eta(y_\alpha, y) \rangle + h(y_\alpha, y) \right].
\end{aligned} \tag{3.7}$$

Since $\beta_0(y) > 0$, by (3.6) and (3.7), we have

$$\limsup_{\alpha} \left[\min_{w \in T(y_\alpha)} \operatorname{Re}\langle w, \eta(y_\alpha, y) \rangle + h(y_\alpha, y) \right] \le 0.$$

Since T is an (η, h)-pseudo-monotone type I operator, we have

$$\begin{aligned} &\limsup_{\alpha} \left[\min_{w \in T(y_\alpha)} \operatorname{Re} \langle w, \eta(y_\alpha, x) \rangle + h(y_\alpha, x) \right] \\ &\ge \min_{w \in T(y)} \operatorname{Re} \langle w, \eta(y, x) \rangle + h(y, x) \end{aligned}$$

for all $x \in X$. Since $\beta_0(y) > 0$, we have

$$\begin{aligned} &\beta_0(y) \left[\limsup_{\alpha} \min_{w \in T(y_\alpha)} \operatorname{Re}\langle w, \eta(y_\alpha, x) \rangle + h(y_\alpha, x) \right] \\ &\ge \beta_0(y) \, [\min_{w \in T(y)} \operatorname{Re}\langle w, \eta(y, x) \rangle + h(y, x) \end{aligned}$$

and thus

$$\begin{aligned} &\beta_0(y) \left[\limsup_{\alpha} \min_{w \in T(y_\alpha)} \operatorname{Re}\langle w, \eta(y_\alpha, x) \rangle + h(y_\alpha, x) \right] + \sum_{p \in LF(E)} \beta_p(y) \operatorname{Re}\langle p, \eta(y, x) \rangle \\ &\ge \beta_0(y) \left[\min_{w \in T(y)} \operatorname{Re}\langle w, \eta(y, x) \rangle + h(y, x) \right] + \sum_{p \in LF(E)} \beta_p(y) \operatorname{Re}\langle p, \eta(y, x) \rangle. \end{aligned} \tag{3.8}$$

When $t = 1$, we have $\varphi(x, y_\alpha) \le 0$ for all $\alpha \in \Gamma$, i.e.,

$$\begin{aligned} &\beta_0(y_\alpha) \left[\min_{w \in T(y_\alpha)} \operatorname{Re}\langle w, \eta(y_\alpha, x) \rangle + h(y_\alpha, x) \right] + \sum_{p \in LF(E)} \beta_p(y_\alpha) \operatorname{Re}\langle p, \eta(y_\alpha, x) \rangle \\ &\le 0 \end{aligned}$$

for all $\alpha \in \Gamma$ and so, by (3.8),

$$\begin{aligned} 0 &\ge \limsup_{\alpha} \Bigg[\beta_0(y_\alpha) \min_{w \in T(y_\alpha)} \operatorname{Re}\langle w, \eta(y_\alpha, x) \rangle + h(y_\alpha, x) \\ &\quad + \sum_{p \in LF(E)} \beta_p(y_\alpha) \operatorname{Re}\langle p, \eta(y_\alpha, x) \rangle \Bigg] \\ &\ge \limsup_{\alpha} \left[\beta_0(y_\alpha) \min_{w \in T(y_\alpha)} \operatorname{Re}\langle w, \eta(y_\alpha, x) \rangle + h(y_\alpha, x) \right] \\ &\quad + \liminf_{\alpha} \left[\sum_{p \in LF(E)} \beta_p(y_\alpha) \operatorname{Re}\langle p, \eta(y_\alpha, x) \rangle \right] \\ &= \beta_0(y) \left[\limsup_{\alpha} \min_{w \in T(y_\alpha)} \operatorname{Re}\langle w, \eta(y_\alpha, x) \rangle + h(y_\alpha, x) \right] \\ &\quad + \sum_{p \in LF(E)} \beta_p(y) \operatorname{Re}\langle p, \eta(y, x) \rangle \\ &\ge \beta_0(y) \left[\min_{w \in T(y)} \operatorname{Re}\langle w, \eta(y, x) \rangle + h(y, x) \right] \\ &\quad + \sum_{p \in LF(E)} \beta_p(y) \operatorname{Re}\langle p, \eta(y, x) \rangle. \end{aligned} \tag{3.9}$$

Hence we have $\varphi(x, y) \leq 0$.

(IV) By hypothesis, there exists a non-empty compact (and so closed) subset K of X and a point $x_0 \in X$ such that

$$x_0 \in K \cap S(y), \inf_{w \in T(y)} [\mathrm{Re}\langle w, \eta(y, x_0)\rangle + h(y, x_0)] > 0$$

for all $y \in X \setminus K$. Thus, for all $y \in X \setminus K$, $\beta_0(y)[\inf_{w \in T(y)} \mathrm{Re}\langle w, \eta(y, x_0)\rangle + h(y, x_0)] > 0$ whenever $\beta_0(y) > 0$ and $\mathrm{Re}\langle p, \eta(y, x_0)\rangle > 0$, whenever $\beta_p(y) > 0$ for any $p \in LF(E)$. Consequently, we have

$$\phi(x_0, y) = \beta_0(y)\left[\inf_{w \in T(y)} \mathrm{Re}\langle w, \eta(y, x_0)\rangle + h(y, x_0)\right] + \sum_{p \in LF(E)} \beta_p(y)\,\mathrm{Re}\langle p, \eta(y, x_0)\rangle > 0$$

for all $y \in X \setminus K$ and so φ satisfies all the hypotheses of Theorem 1.1. Hence, by Theorem 1.1, there exists a point $\hat{y} \in K$ such that $\varphi(x, \hat{y}) \leq 0$ for all $x \in X$, i.e.,

$$\beta_0(\hat{y})\left[\inf_{w \in T(\hat{y})} \mathrm{Re}\langle w, \eta(\hat{y}, x)\rangle + h(\hat{y}, x)\right] + \sum_{p \in LF(E)} \beta_p(\hat{y})\,\mathrm{Re}\langle p, \eta(\hat{y}, x)\rangle 0 \qquad (3.10)$$

for all $x \in X$.

Now, the rest of the proof of this part is similar to the proof in Step 1 of [16, Theorem 2.1]. Hence we have shown that there exist a point $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x \in S(\hat{y})}\left[\inf_{w \in T(\hat{y})} \mathrm{Re}\langle w, \eta(\hat{y}, x)\rangle + h(\hat{y}, x)\right] \leq 0.$$

By following the proof of Step 2 in [16, Theorem 2.1] and applying Theorem 2.1 (Keneser's Minimax Theorem) above, we can show that there exist a point $\hat{w} \in T(\hat{y})$ such that $\mathrm{Re}\langle \hat{w}, \eta(\hat{y}, x)\rangle + h(\hat{y}, x) \leq 0$ for all $x \in S(\hat{y})$. This completes the proof.

When X is compact, we obtain the following immediate consequence of Theorem 3.1:

Theorem 3.2. *Let E be a locally convex Hausdorff topological vector space over Φ, X be a nonempty compact convex subset of E and F be a vector space over Φ with $\sigma\langle F, E\rangle$ - topology where $\langle\cdot, \cdot\rangle : F \times E \to \Phi$ is a bilinear functional such that, for each $w \in F$, the function $x \mapsto \mathrm{Re}\langle w, x\rangle$ is continuous. Let $S : X \to 2^X$, $T : X \to 2^F$, $\eta : X \times X \to F$, and $h : E \times E \to \mathbb{R}$ be the mappings such that*

(*a*) *S is upper semi-continuous such that each $S(x)$ is closed and convex;*

(*b*) *$h(X \times X)$ is bounded;*

(*c*) *T is an (η, h)-pseudo-monotone type I operator and is upper semi-continuous from $co(A)$ to the $\sigma\langle F, E\rangle$-topology on F for each $A \in \mathcal{F}(X)$such that each $T(x)$ is $\sigma\langle F, E\rangle$-compact and convex and $T(X)$ is $\delta\langle F, E\rangle$ -bounded;*

(*d*) *T and η have the 0 - DCV R;*

(*e*) *for each fixed $y \in X$, $x \mapsto \eta(x, y)$, i.e., $\eta(\cdot, y)$ is continuous and $x \mapsto h(x, y)$, i.e., $h(\cdot, y)$ is lower semi-continuous on $co(A)$ for each $A \in \mathcal{F}(X)$and, for each $x \in X$, $h(x, \cdot)$ and $\eta(x, \cdot)$ are concave and $h(x, x) = 0$, $\eta(x, x) = 0$;*

(*f*) *the set $\Sigma = \{y \in X : \sup_{x \in S(y)}[\inf_{w \in T(y)} \mathrm{Re}\langle w, \eta(y, x)\rangle + h(y, x)] > 0\}$ is open in X.*

Then there exist a point $\hat{y} \in X$ such that

$\hat{y} \in S(\hat{y})$;

there exists a point $\hat{w} \in T(\hat{y})$ with $\mathrm{Re}\langle \hat{w}, \eta(\hat{y}, x)\rangle + h(\hat{y}, x) \leq 0$ for all $x \in S(\hat{y})$.

Note that, if the mapping $S : X \to 2^X$ is, in addition, lower semi-continuous and, for each $y \in \Sigma$, T is upper semi-continuous at y in X, then the set Σ in Theorem 3.1 is always open in X and so we obtain the following theorem:

Theorem 3.3. *Let E be a locally convex Hausdorff topological vector space over Φ, X be a nonempty para-compact convex and bounded subset of E and F be a vector space over Φ with $\sigma \langle F, E\rangle$-topology, where $\langle\cdot, \cdot\rangle : F \times E \to \Phi$ is a bilinear functional such that, for each $w \in F$, the function $x \mapsto \mathrm{Re}\langle w, x\rangle$ is continuous. Let $S : X \to 2^X$, $T : X \to 2^F$, $\eta : X \times X \to F$ and $h : E \times E \to \mathbb{R}$ be mappings such that*

(*a*) *S is continuous such that each $S(x)$ is compact and convex;*

(*b*) *$h(X \times X)$ is bounded;*

(*c*) *T is an (η, h)-pseudo-monotone type I operator and is upper semi-continuous from $co(A)$ to the $\sigma\langle F, E\rangle$-topology on F for each $A \in \mathcal{F}(X)$such that each $T(x)$ is $\sigma\langle F, E\rangle$-compact and convex and $T(X)$ is $\delta\langle F, E\rangle$-bounded;*

(*d*) *T and η have the 0 - DCV R;*

(*e*) *for each fixed $y \in X$, $x \mapsto \eta(x, y)$, i.e., $\eta(\cdot, y)$ is continuous and $x \mapsto h(x, y)$, i.e., $h(\cdot, y)$ is lower semi-continuous on $co(A)$ for each $A \in \mathcal{F}(X)$and, for each $x \in X$, $h(x, \cdot)$ and $\eta(x, \cdot)$ are concave and $h(x, x) = 0$, $\eta(x, x) = 0$.*

Suppose that, for each $y \in \Sigma = \{y \in X : \sup_{x\in S(y)} [\inf_{w\in T(y)} \mathrm{Re}\langle w, \eta(y, x)\rangle + h(y, x)] >0\}$, T is upper semi-continuous at y from the relative topology on X to the $\delta\langle F, E\rangle$-topology on F. Further, suppose that there exist a non-empty compact convex subset K of X and a point $x_0 \in X$ such that

$$x_0 \in K \cap S(y), \quad \inf_{w\in T(y)} \mathrm{Re}\langle w, \eta(y, x_0)\rangle + h(y, x_0) > 0$$

for all $y \in X \setminus K$. Then there exist a point $\hat{y} \in X$ such that

(1) *$\hat{y} \in S(\hat{y})$;*

(2) *there exists a point $\hat{w} \in T(\hat{y})$ with $\mathrm{Re}\langle \hat{w}, \eta(\hat{y}, x)\rangle + h(\hat{y}, x) \leq 0$ for all $x \in S(\hat{y})$.*

The proof is exactly similar to the proof of Theorems 2.3 and 3.3 in [16] and so is omitted.

When X is compact, we obtain the following theorem:

Theorem 3.4. *Let E be a locally convex Hausdorff topological vector space over Φ, X be a non-empty compact convex subset of E and F be a vector space over Φ with $\sigma\langle F, E\rangle$-topology where $\langle\cdot, \cdot\rangle : F \times E \to \Phi$ is a bilinear functional such that for each $w \in F$, the function $x \mapsto \mathrm{Re}\langle w, x\rangle$ is continuous. Let $S : X \to 2^X$, $T : X \to 2^F$, $\eta : X \times X \to F$, and $h : E \times E \to \mathbb{R}$ be mappings such that*

(*a*) *S is continuous such that each $S(x)$ is closed and convex*;

(*b*) *$h(X \times X)$ is bounded*;

(*c*) *T is an (η, h)-pseudo-monotone type I operator and is upper semi-continuous from $co(A)$ to the $\sigma\langle F, E\rangle$-topology on F for each $A \in \mathcal{F}(X)$such that each $T(x)$ is $\sigma\langle F, E\rangle$-compact and convex and $T(X)$ is $\delta\langle F, E\rangle$-bounded*;

(*d*) *T and η have the 0 - DCV R;*

(*e*) *for each fixed $y \in X$, $x \mapsto \eta(x, y)$, i.e., $\eta(\cdot, y)$ is continuous and $x \mapsto h(x, y)$, i.e., $h(\cdot, y)$ is lower semi-continuous on $co(A)$ for each $A \in \mathcal{F}(X)$and, for each $x \in X$, $h(x, \cdot)$ and $\eta(x, \cdot)$ are concave and $h(x, x) = 0$, $\eta(x, x) = 0$.*

Suppose that, for each $y \in \Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} \mathrm{Re} \langle w, \eta(y, x) \rangle + h(y, x)] > 0\}$, *T is upper semi-continuous at y from the relative topology on X to the* $\delta\langle F, E\rangle$-*topology on F. Then there exist a point* $\hat{y} \in X$ *such that*

(1) $\hat{y} \in S(\hat{y})$;

(2) *there exists a point* $\hat{w} \in T(\hat{y})$ *with* $\mathrm{Re} \langle \hat{w}, \eta(\hat{y}, x) \rangle + h(\hat{y}, x) \le 0$ *for all* $x \in S(\hat{y})$.

Remark 3.1. (1) Theorems 3.1 and 3.3 of this article are further generalizations of the results obtained in [16, Theorems 3.1 and 3.3], respectively, into generalized quasi-variational-like inequalities of (η, h)-pseudo-monotone type I operators on non-compact sets.

(2) Shih and Tan [1] obtained results on generalized quasi-variational inequalities in locally convex topological vector spaces and their results were obtained on compact sets where the set-valued mappings were either lower semi-continuous or upper semi-continuous. Our present article is another extension of the original study in [1] using (η, h)-pseudo-monotone type I operators on non-compact sets.

(3) The results in [16] were obtained on non-compact sets where one of the set-valued mappings is a pseudo-monotone type I operators which were defined first in [6] and later renamed by pseudo-monotone type I operators in [7]. Our present results are extensions of the results in [16] using an extension of the operators defined in [7] (and originally in [6]).

Acknowledgment
This study was supported by the Korea Research Foundation Grant funded by the Korean Government (KRF-2008-313-C00050).

Author details
[1]Department of Mathematics, University of Engineering & Technology (UET), Lahore 54890, Pakistan [2]Department of Mathematics Education and RINS, Gyeongsang National University, Chinju 660-701, Korea

Authors' contributions
All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 16 August 2011 Accepted: 4 April 2012 Published: 4 April 2012

References

1. Shih, M-H, Tan, K-K: Generalized quasi-variational inequalities in locally convex topological vector spaces. J. Math. Anal. Appl. **108**, 333–343 (1985)
2. Cho, YJ, Lan, HY: A new class of generalized nonlinear multi-valued quasi-variational-like-inclusions with *H*-monotone mappings. Math. Inequal. Appl. **10**, 389–401 (2007)
3. Chowdhury, MSR, Cho, YJ: Generalized bi-quasi-variational inequalities for quasi-pseudo-monotone type II operators on non-compact sets. J. Inequal. Appl **2010**, 1–17 (2010). Article ID 237191
4. Fang, YP, Cho, YJ, Huang, NJ, Kang, SM: Generalized nonlinear implicit quasi-variational-like inequalities for set-valued mappings in Banach spaces. Math. Inequal. Appl. **6**, 331–337 (2003)
5. Lan, HY, Cho, YJ, Huang, NJ: Stability of iterative procedures for a class of generalized nonlinear quasi-variational-like inclusions involving maximal η-monotone mappings. In: Cho YJ, Kim JK, Kang SM (eds.) Fixed Point Theory and Applications, vol. 6, pp. 107–116. Nova Science Publishers, New York, Inc. (2006)
6. Chowdhury, MSR, Tan, K-K: Generalization of Ky Fan's minimax inequality with applications to generalized variational inequalities for pseudo-monotone operators and fixed theorems. J. Math. Anal. Appl. **204**, 910–929 (1996)
7. Chowdhury, MSR: The surjectivity of upper-hemicontinuous and pseudo-monotone type II operators in reflexive Banach Spaces. Ganit: J. Bangladesh Math. Soc. **20**, 45–53 (2000)
8. Ding, XP, Tarafdar, E: Generalized variational-like inequalities with pseudo-monotone set-valued mappings. Arch. Math. **74**, 302–313 (2000)
9. Brézis, H, Nirenberg, L, Stampacchia, G: A remark on Ky Fan's minimax principle. Boll. U.M.I. **6**, 293–300 (1972)
10. Fan, K: A minimax inequality and applications. In: Shisha O (ed.) Inequalities, III. pp. 103–113. Academic Press, San Diego (1972)
11. Takahashi, W: Nonlinear variational inequalities and fixed point theorems. J. Math. Soc. Japan. **28**, 166–181 (1976)
12. Shih, M-H, Tan, K-K: Generalized bi-quasi-variational inequalities. J. Math. Anal. Appl. **143**, 66–85 (1989)
13. Kneser, H: Sur un theoreme fundamental de la theorie des jeux. C.R. Acad. Sci. Paris. **234**, 2418–2420 (1952)

14. Aubin, JP: Applied Functional Analysis. Wiley-Interscience, New York (1979)
15. Rockafeller, RT: Convex Analysis. Princeton Univ. Press, Princeton (1970)
16. Chowdhury, MSR, Tan, K-K: Applications of pseudo-monotone operators with some kind of upper semi-continuity in generalized quasi-variational inequalities on non-compacts. Proc. Amer. Math. Soc. **126**, 2957–2968 (1998)

doi:10.1186/1029-242X-2012-79
Cite this article as: Rahim Chowdhury and Cho: **Existence theorems of generalized quasi-variational-like inequalities for** η**-**h**-pseudo-monotone type I operators on non-compact sets.** *Journal of Inequalities and Applications* 2012 **2012**:79.