Feng and Yao *Journal of Inequalities and Applications* 2012, **2012**:78
http://www.journalofinequalitiesandapplications.com/content/2012/1/78

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On (λ, μ)-anti-fuzzy subgroups

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Abstract
We introduced the notions of (λ, μ)-anti-fuzzy subgroups, studied some properties of them and discussed the product of them.

Keywords: product, (λ, μ)-fuzzy, subgroup, ideal

1 Introduction and preliminaries

Fuzzy sets was first introduced by Zadeh [1] and then the fuzzy sets have been used in the reconsideration of classical mathematics. Yuan et al. [2] introduced the concept of fuzzy subgroup with thresholds. A fuzzy subgroup with thresholds λ and μ is also called a (λ, μ)-fuzzy subgroup. Yao continued to research (λ, μ)-fuzzy normal subgroups, (λ, μ)-fuzzy quotient subgroups and (λ, μ)-fuzzy subrings in [3-5].

Shen researched anti-fuzzy subgroups in [6] and Dong [7] studied the product of anti-fuzzy subgroups.

By a fuzzy subset of a nonempty set X we mean a mapping from X to the unit interval 0[1]. If A is a fuzzy subset of X, then we denote $A_{(\alpha)} = \{x \in X | A(x) < \alpha\}$ for all $\alpha \in$ 0[1].

Throughout this article, we will always assume that $0 \le \lambda < \mu \le 1$.

Let G, G_1, and G_2 always denote groups in the following. 1, 1_1, and 1_2 are identities of G, G_1, and G_2, respectively.

2 (λ, μ)-anti-fuzzy subgroups

Definition 1. *A fuzzy set A of a group G is called a (λ, μ)-anti-fuzzy subgroup of G if $\forall a, b, c \in G$.*

$$A(ab) \wedge \mu \le (A(a) \vee A(b)) \vee \lambda$$

and

$$A(c^{-1}) \wedge \mu \le A(c) \vee \lambda$$

where c^{-1} is the inverse element of c.

Proposition 1. *If A is a (λ, μ)-anti-fuzzy subgroup of a group G, then $A(1) \wedge \mu \le A(x) \vee \lambda$ for all $x \in G$, where 1 is the identity of G.*

Proof. $\forall x \in G$ and let x^{-1} be the inverse element of x. Then $A(1) \wedge \mu = A(xx^{-1}) \wedge \mu = (A(xx^{-1}) \wedge \mu) \wedge \mu \le ((A(x) \vee A(x^{-1})) \vee \lambda) \wedge \mu = (A(x) \wedge \mu) \vee (A(x^{-1}) \wedge \mu) \vee (\lambda \wedge \mu) \le A(x) \vee (A(x) \vee \lambda) \vee \lambda = A(x) \vee \lambda$.

Theorem 1. *Let A be a fuzzy subset of a group G. Then A is a (λ, μ)-anti-fuzzy subgroup of G if and only if $A(x^{-1}y) \wedge \mu \le (A(x) \vee A(y)) \vee \lambda$, $\forall x, y \in G$.*

Proof. Let A is a (λ, μ)-anti-fuzzy subgroup of G, then

$$\begin{aligned} A(x^{-1}y) \wedge \mu &= A(x^{-1}y) \wedge \mu \wedge \mu \\ &\leq ((A(x^{-1}) \vee A(y)) \vee \lambda) \wedge \mu \\ &= (A(x^{-1}) \wedge \mu \vee A(y)) \vee (\lambda \wedge \mu) \\ &\leq ((A(x) \vee \lambda) \vee A(y)) \vee \lambda \\ &= (A(x) \vee A(y)) \vee \lambda. \end{aligned}$$

Conversely, suppose $A(x^{-1}y) \wedge \mu \leq (A(x) \vee A(y)) \vee \lambda$, $\forall x, y \in G$, then
$A(1) \wedge \mu = A(x^{-1}x) \wedge \mu \leq A(x) \vee A(x) \vee \lambda = A(x) \vee \lambda$.
So
$A(x^{-1}) \wedge \mu = A(x^{-1}1) \wedge \mu = A(x^{-1}1) \wedge \mu \wedge \mu \leq (A(x) \vee A(1) \vee \lambda) \wedge \mu = (A(1) \wedge \mu) \vee ((A(x) \vee \lambda) \wedge \mu) \leq (A(x) \vee \lambda) \vee ((A(x) \vee \lambda) \wedge \mu) = A(x) \vee \lambda$.
$A(xy) \wedge \mu = A((x^{-1})^{-1} y) \wedge \mu = A((x^{-1})^{-1}y) \wedge \mu \wedge \mu \leq (A(x^{-1}) \vee A(y) \vee \lambda) \wedge \mu = (A(x^{-1}) \wedge \mu) \vee ((A(y) \vee \lambda) \wedge \mu) \leq (A(x) \vee \lambda) \vee (A(y) \vee \lambda) = (A(x) \vee A(y)) \vee \lambda$.
So A is a (λ, μ)-anti-fuzzy subgroup of G.

Theorem 2. *Let A be a fuzzy subset of a group G. Then the following are equivalent:*

(1) A is a (λ, μ)-anti-fuzzy subgroup of G;
(2) $A_{(\alpha)}$ is a subgroup of G, for any $\alpha \in (\lambda, \mu]$, where $A_{(\alpha)} \neq \emptyset$.

Proof. "(1) ⇒ (2)"
Let A be a (λ, μ)-anti-fuzzy subgroup of G. For any $\alpha \in (\lambda, \mu]$, such that $A_\alpha \neq \emptyset$, we need to show that $x^{-1} y \in A_{(\alpha)}$, for all $x,y \in A_{(\alpha)}$.
Since $A(x) < \alpha$ and $A(y) < \alpha$, Then $A(x^{-1} y) \wedge \mu \leq A(x) \vee A(y) \vee \lambda < \alpha \vee \alpha \vee \lambda = \alpha \vee \lambda = \alpha$. Note that $\alpha \leq \mu$, we obtain $A(x^{-1} y) < \alpha$. So $x^{-1} y \in A_{(\alpha)}$.
"(2) ⇒ (1)"
Conversely, let $A_{(\alpha)}$ is a subgroup of G. We need to prove that $A(x^{-1} y) \wedge \mu \leq A(x) \vee A(y) \vee \lambda$, $\forall x \in G$. If there exist x_0, $y_0 \in G$ such that $A\left(x_0^{-1}y_0\right) \wedge \mu = \alpha > A(x_0) \vee A\left(y_0\right) \vee \lambda$, then $A(x_0) < \alpha$, $A(y_0) < \alpha$ and $\alpha \in (\lambda, \mu]$. Thus $x_0 \in A_\alpha$ and $y_0 \in A_\alpha$. But $A\left(x_0^{-1} y_0\right) \geq \alpha$, that is $x_0^{-1} y_0 \notin A_{(\alpha)}$. This is a contradiction with that $A_{(\alpha)}$ is a subgroup of G. Hence $A(x^{-1} y) \wedge \mu \leq A(x) \vee A(y) \vee \lambda$ holds for any $x, y \in G$.
Therefore, A is a (λ, μ)-anti-fuzzy subgroup of G.
We set $\inf \emptyset = 1$, where $\emptyset$ is the empty set.

Theorem 3. *Let f: $G_1 \to G_2$ be a homomorphism and let A be a (λ, μ)-anti-fuzzy subgroup of G_1. Then $f(A)$ is a (λ, μ)-anti-fuzzy subgroup of G_2, where*

$$f(A)\left(y\right) = \inf_{x \in G_1} \left\{A(x) \mid f(x) = y\right\}, \quad \forall_y \in G_2.$$

Proof. If $f^{-1}(y_1) = \emptyset$ or $f^{-1}(y_2) = \emptyset$ for any y_1, $y_2 \in G_2$, then $\left(f(A)\left(y_1^{-1}y_2\right)\right) \wedge \mu \leq 1 = \left(f(A)\left(y_1\right) \vee f(A)\left(y_2\right)\right) \vee \lambda$.
Suppose that $f^{-1}(y_1) \neq \emptyset$, $f^{-1}(y_2) = \emptyset$ for any $y_1, y_2 \in G_2$. Then

For any $y_1, y_2 \in G_2$, we have

$$\begin{aligned} f(A)\left(y_1^{-1}y_2\right) \wedge \mu &= \inf_{t \in G_1}\left\{A(t) \mid f(t) = y_1^{-1}y_2\right\} \wedge \mu \\ &= \inf_{t \in G_1}\left\{(A(t)) \wedge \mu \mid f(t) = y_1^{-1}y_2\right\} \\ &\leq \inf_{x_1, x_2 \in G_1}\left\{\left(A\left(x_1^{-1}x_2\right)\right) \wedge \mu \mid f(x_1) = y_2, f(x_2) = y_2\right\} \\ &\leq \inf_{x_1, x_2 \in G_1}\left\{(A(x_1) \vee A(x_2)) \vee \lambda \mid f(x_1) = y_1, f(x_2) = y_2\right\} \\ &= \left(\inf_{x_1 \in s_1}\left\{A(x_1) \mid f(x_1) = y_1\right\} \vee \inf_{x_2 \in S_1}\left\{A(x_2) \mid f(x_2) = y_2\right\}\right) \vee \lambda \\ &= \left(f(A)\left(y_1\right) \vee f(A)\left(y_2\right)\right) \vee \lambda. \end{aligned}$$

So, $f(A)$ is a (λ, μ)-anti-fuzzy subgroup of G_2.

Theorem 4. *Let $f: G_1 \to G_2$ be a homomorphism and let B be a (λ, μ)-anti-fuzzy subgroup of G_2. Then $f^{-1}(B)$ is a (λ, μ)-anti-fuzzy subgroup of G_1, where*

$$f^{-1}(B)(x) = B\left(f(x)\right), \quad \forall_x \in G_1.$$

Proof. For any $x_1, x_2 \in G_1$,

$$\begin{aligned} f^{-1}(B)\left(x_1^{-1}x_2\right) \wedge \mu &= B\left(f\left(x_1^{-1}x_2\right)\right) \wedge \mu \\ &= B\left(\left(f(x_1)\right)^{-1} f(x_2)\right) \wedge \mu \\ &\leq \left(B\left(f(x_1)\right) \vee B\left(f(x_2)\right)\right) \vee \lambda \\ &= \left(f^{-1}(B)(x_1) \vee f^{-1}(B)(x_2)\right) \vee \lambda. \end{aligned}$$

So, $f^{-1}(B)$ is a (λ, μ)-anti-fuzzy subgroup of G_1.

Let G_1 be a group with the identity 1_1 and G_2 be a group with the identity 1_2, then $G_1 \times G_2$ is a group with the identity $(1_1, 1_2)$ if we define $(x_1, y_1)(x_2, y_2) = (x_1x_2, y_1y_2)$ for all $(x_1, y_1), (x_2, y_2) \in G_1 \times G_2$. Moreover, the inverse element of any $(x, a) \in G_1 \times G_2$ is $(y, b) \in G_1 \times G_2$ if and only if y is the inverse element of x in G_1 and b is the inverse element of a in G_2.

Theorem 5. *Let A, B be two (λ, μ)-anti-fuzzy subgroups of groups G_1 and G_2, respectively. The product of A and B, denoted by $A \times B$, is a (λ, μ)-anti-fuzzy subgroup of the group $G_1 \times G_2$, where*

$$(A \times B)\left(x, y\right) = A(x) \vee B\left(y\right), \quad \forall \left(x, y\right) \in G_1 \times G_2.$$

Proof. Let (x^{-1}, a^{-1}) be the inverse element of (x, a) in $G_1 \times G_2$. Then x^{-1} is the inverse element of x in G_1 and a^{-1} is the inverse element of a in G_2. Hence $A(x^{-1}) \wedge \mu \leq A(x) \vee \lambda$ and $B(a^{-1}) \wedge \mu \leq B(a) \vee \lambda$. For all $(y, b) \in G_1 \times G_2$. We have

$$\begin{aligned} \left((A \times B)(x, a)^{-1}\left(y, b\right)\right) \wedge \mu &= \left((A \times B)\left(x^{-1}, a^{-1}\right)\left(y, b\right)\right) \wedge \mu \\ &= \left(A\left(x^{-1}y\right) \vee B\left(a^{-1}b\right)\right) \wedge \mu \\ &= \left(A\left(x^{-1}y\right) \wedge \mu\right) \vee \left(B\left(a^{-1}b\right) \wedge \mu\right) \\ &\leq \left(A(x) \vee A\left(y\right) \vee \lambda\right) \vee \left(B(a) \vee B(b) \vee \lambda\right) \\ &= \left(A(x) \vee B(a)\right) \vee \left(A\left(y\right) \vee B(b)\right) \vee \lambda \\ &= \left((A \times B)(x, a)\right) \vee \left((A \times B)\left(y, b\right)\right) \vee \lambda. \end{aligned}$$

Hence $A \times B$ is a (λ, μ)-anti-fuzzy subgroup of $G_1 \times G_2$.

Theorem 6. *Let A and B be two fuzzy subsets of groups G_1 and G_2, respectively. If A × B is a (λ, μ)-anti-fuzzy subgroup of $G_1 \times G_2$, then at least one of the following statements must hold.*

$$A(1_1) \wedge \mu \le B(a) \vee \lambda, \quad \forall_a \in G_2$$

and

$$B(1_2) \wedge \mu \le A(x) \vee \lambda, \quad \forall x \in G_1.$$

Proof. Let $A \times B$ be a (λ, μ)-anti-fuzzy subgroup of the group $G_1 \times G_2$.

By contraposition, suppose that none of the statements hold. Then we can find $x \in G_1$ and $a \in G_2$ such that $A(x) \vee \lambda < B(1_2) \wedge \mu$ and $B(a) \vee \lambda < A(1_1) \wedge \mu$. Now

$(A{\times}B)\ (x,\ a) \vee \lambda = (A(x) \vee B(a)) \vee \lambda = (A(x) \vee \lambda) \vee (B(a) \vee \lambda) < (A(1_1) \wedge \mu) \vee (B(1_2) \wedge \mu) = (A{\times}B)\ (1_1, 1_2) \wedge \mu$.

Thus $A \times B$ is a (λ, μ)-anti-fuzzy subgroup of the group $G_1 \times G_2$ satisfying $(A \times B)(x, a) \vee \lambda < (A \times B)\ (1_1, 1_2) \wedge \mu$. This is a contradict with that $(1_1, 1_2)$ iss the identity of $G_1 \times G_2$.

Theorem 7. *Let A and B be fuzzy subsets of groups G_1 and G_2, respectively, such that $B(1_2) \wedge \mu \le A(x) \vee \lambda$ for all $x \in G_1$. If $A \times B$ is a (λ, μ)-anti-fuzzy subgroup of $G_1 \times G_2$, then A is a (λ, μ)-anti-fuzzy subgroup of G_1.*

Proof. From $B(1_2) \wedge \mu \le A(x) \vee \lambda$ we obtain that $\mu \le A(x) \vee \lambda$ or $B(1_2) \le A(x) \vee \lambda$, for all $x \in G_1$.

Let $x, y \in G_1$, then $(x, 1_2), (y, 1_2) \in G_1 \times G_2$.
Two cases are possible:
(1) If $\mu \le A(x) \vee \lambda$ for all $x \in G_1$. Then
$A(xy) \wedge \mu \le \mu \le A(x) \vee \lambda \le (A(x) \vee A(y)) \vee \lambda$
and $A(1_1) \wedge \mu \le \mu \le A(x) \vee \lambda$.
(2) If $B(1_2) \le A(x) \vee \lambda$ for all $x \in G_1$. Then

$$\begin{aligned} A(xy) \wedge \mu &\le \left(A(xy) \vee B(1_2 1_2)\right) \wedge \mu \\ &= \left((A \times B)\left(xy, 1_2 1_2\right)\right) \wedge \mu \\ &= \left((A \times B)\left((x, 1_2)(y, 1_2)\right)\right) \wedge \mu \\ &\le \left((A \times B)(x, 1_2) \vee (A \times B)(y, 1_2)\right) \vee \lambda \\ &= A(x) \vee B(1_2) \vee A(y) \vee B(1_2) \vee \lambda \\ &= \left(A(x) \vee A(y)\right) \vee \lambda. \end{aligned}$$

and

$$\begin{aligned} A(1_1) \wedge \mu &\le (A(1_1) \vee B(1_2)) \wedge \mu \\ &= ((A \times B)(1_1, 1_2)) \wedge \mu \\ &\le (A \times B)(x, 1_2) \vee \lambda \\ &= A(x) \vee B(1_2) \vee \lambda \\ &= A(x) \vee \lambda. \end{aligned}$$

Hence A is a (λ, μ)-anti-fuzzy subgroup of G_1.

Analogously, we have

Theorem 8. *Let A and B be fuzzy subsets of groups* G_1 *and* G_2*, respectively, such that* $A(1_1) \wedge \mu \leq B(a) \vee \lambda$ for all $a \in G_2$. *If* $A \times B$ *is a* (λ, μ)*-anti-fuzzy subgroup of* $G_1 \times G_2$*, then B is a* (λ, μ)*-anti-fuzzy subgroup of* G_2.

From the previous theorems, we have the following corollary

Corollary 1. *Let A and B be fuzzy subsets of groups* G_1 *and* G_2*, respectively. If* $A \times B$ *is a* (λ, μ)*-anti-fuzzy subgroup of* $G_1 \times G_2$*, then either A is a* (λ, μ)*-anti-fuzzy subgroup of* G_1 *or B is a* (λ, μ)*-anti-fuzzy subgroup of* G_2.

Acknowledgements
YF wished to thank Prof. Michela for her help with the language.

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Authors' contributions
YF had posed ideals and typed this article with a computer. BY had given some good advice. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 19 December 2011 Accepted: 3 April 2012 Published: 3 April 2012

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doi:10.1186/1029-242X-2012-78
Cite this article as: Feng and Yao: On (λ, μ)-anti-fuzzy subgroups. *Journal of Inequalities and Applications* 2012 2012:78.