Mongkolkeha and Kumam *Journal of Inequalities and Applications* 2012, **2012**:76
http://www.journalofinequalitiesandapplications.com/content/2012/1/76

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

On *H*-property and uniform Opial property of generalized cesàro sequence spaces

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Abstract

In this article, we define the generalized cesàro sequence spaces $ces_{(p)}(q)$ and consider it equipped with the Luxemburg norm. We show that the spaces $ces_{(p)}(q)$ has the *H*-property and Uniform Opial property. The results of this article, we improve and extend some results of Petrot and Suantai.

Keywords: generalized Cesàro sequence spaces, *H*-property, uniform Opial property

1. Introduction

Let $(X, ||\cdot||)$ be a real Banach space and let $B(X)$ (resp., $S(X)$) be a closed unit ball (resp., the unit sphere) of X. A point $x \in S(X)$ is an H-point of $B(X)$ if for any sequence (x_n) in X such that $||x_n|| \to 1$ as $n \to \infty$, the week convergence of (x_n) to x implies that $||x_n - x|| \to 0$ as $n \to \infty$. If every point in $S(X)$ is an H-point of $B(X)$, then X is said to have the property (H). A Banach space X is said to have the Opial property (see [1]), if every weakly null sequence (x_n) in X satisfies

$$\lim_{n\to\infty} \inf ||x_n|| \le \lim_{n\to\infty} \inf ||x_n - x||,$$

for every $x \in X \backslash\{0\}$. Opial proved in [1] that the sequence space $l_p(1 < p < \infty)$ have this property but $L_p[0, \pi](p \neq 2, 1 < p < \infty)$ do not have it. A Banach space X is said to have the uniform Opial property (see [2]), if for each $\varepsilon > 0$ there exists $\tau > 0$ such that for any weakly null sequence (x_n) in $S(X)$ and $x \in X$ with $||x|| > \varepsilon$ there holds

$$1 + \tau \le \lim_{n\to\infty} \inf ||x_n + x||.$$

For example, the space in [3-5] have the uniform Opial property.

Let l^0 be the space of all real sequences. For $1 \le p < \infty$, the Cesàro sequence space (ces_p, for short) is defined by

$$ces_p = \left\{ x \in l^0 : \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=0}^{n} |x(i)| \right)^p < \infty \right\}$$

equipped with the norm

$$||x|| = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p \right)^{\frac{1}{p}} \tag{1.1}$$

This space was first introduced by Shiue [6]. It is useful in the theory of matrix operators and others (see [7,8]). Suantai [9,10] defined the generalized Cesàro sequence space $\text{ces}_{(p)}$ when $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k \geq 1$ for all $k \in \mathbb{N}$ by

$$\text{ces}_{(p)} = \{x \in l^0 : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

where

$$\varrho(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{k} |x(i)| \right)^{p_n}$$

equipped with the Luxemburg norm

$$||x|| = \inf \left\{ \varepsilon > 0 : \varrho\left(\frac{x}{\varepsilon}\right) \leq 1 \right\}.$$

In the case when $p_k = p$, $1 \leq p < \infty$ for all $k \in \mathbb{N}$, the generalized Cesàro sequence space $\text{ces}_{(p)}$ is the Cesàro sequence space ces_p and the Luxemburg norm is expressed by the formula (1.1). Khan [11] defined the generalized Cesàro sequence space for $1 \leq p < \infty$ with $q = q_k$ is a bounded sequence of positive real numbers by

$$\text{ces}_p(q) = \left\{ x \in l^0 : \left(\sum_{k=1}^{\infty} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i)| \right)^p \right)^{1/p} < \infty \right\},$$

where $Q_k = \sum_{k=1}^{n} q_k, n \in \mathbb{N}$. If $q_k = 1$ for all $k \in \mathbb{N}$, then $\text{ces}_p(q)$ reduces to ces_p.

In this article, we define the generalized Cesàro sequence space for a bounded sequence $p = (p_k)$ and $q = q_k$ of positive real numbers with $p_k \geq 1$ and $q_k \geq 1$ for all $k \in \mathbb{N}$ by

$$\text{ces}_{(p)}(q) = \{x \in l^0 : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

where

$$\varrho(x) = \sum_{k=1}^{\infty} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i)| \right)^{p_k}$$

with $Q_k = \sum_{k=1}^{n} q_k$ and consider $\text{ces}_{(p)}(q)$ equipped with the Luxemburg norm

$$||x|| = \inf \left\{ \varepsilon > 0 : \varrho\left(\frac{x}{\varepsilon}\right) \leq 1 \right\}.$$

Thus, we see that $p_k = p$, $1 \leq p < \infty$ for all $k \in \mathbb{N}$, then $\text{ces}_{(p)}(q)$ reduces to $\text{ces}_p(q)$ and if $q_k = 1$ for all $k \in \mathbb{N}$, then $\text{ces}_{(p)}(q)$ reduces to $\text{ces}_{(p)}$. Throughout this article, for $x \in l^0$, $i \in \mathbb{N}$, we denote

$$e_i = (\overbrace{0, 0, \ldots, 0}^{i-1}, 1, 0, 0, 0, \ldots),$$
$$x|_i = (x(1), x(2), x(3), \ldots, x(i), 0, 0, 0, \ldots),$$
$$x|_{\mathbb{N}-i} = (0, 0, 0, \ldots, x(i+1), x(i+2), \ldots),$$

and $M = \sup_k p_k$ with $p_k > 1$ for all $k \in \mathbb{N}$. First, we start with a brief recollection of basic concepts and facts in modular space. For a real vector space X, a function $\rho: X \to [0, \infty]$ is called a *modular* if it satisfies the following conditions;

(i) $\rho(x) = 0$ if and only if $x = 0$;
(ii) $\rho(\alpha x) = \rho(x)$ for all scalar α with $|\alpha| = 1$;
(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$, for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.
The modular ρ is called convex if
(iv) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$, for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

For modular ρ on X, the space

$$X_\rho = \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0^+\}$$

is called the *modular space*.

A sequence (x_n) in X_ρ is called *modular convergent* to $x \in X_\rho$ if there exists a $\lambda > 0$ such that $\rho(\lambda(x_n - x)) \to 0$ as $n \to \infty$.

A modular ρ is said to satisfy the Δ_2-*condition* ($\rho \in \Delta_2$) if for any $\varepsilon > 0$ there exist a constants $K \geq 2$ and $a > 0$ such that

$$\rho(2u) \leq K\rho(u) + \varepsilon$$

for all $u \in X_\rho$ with $\rho(u) \leq a$.

If ρ satisfies the Δ_2-*condition* for any $a > 0$ with $K \geq 2$ dependent on a, we say that ρ the *strong* Δ_2-*condition* $(\rho \in \Delta_2^s)$.

Lemma 1.1. [[12], Lemma 2.1] *If* $\rho \in \Delta_2^s$, *then for any* $L > 0$ *and* $\varepsilon > 0$, *there exists* $\delta = \delta(L, \varepsilon) > 0$ *such that*

$$|\rho(u + v) - \rho(u)| < \varepsilon,$$

whenever $u, v \in X_\rho$ *with* $\rho(u) \leq L$, *and* $\rho(v) \leq \delta$.

Lemma 1.2. [[12], Lemma 2.3] *Convergences in norm and in modular are equivalent in* X_ρ *if* $\rho \in \Delta_2$.

Lemma 1.3. [[12], Lemma 2.4] *If* $\rho \in \Delta_2^s$, *then for any* $\varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon) > 0$ *such that* $\| x \| \geq 1 + \delta$, *whenever* $\rho(x) \geq 1 + \varepsilon$.

2. Main results

In this section, we prove the property H and uniform Opial property in generalized Cesàro sequence space $\mathrm{ces}_{(p)}(q)$. First, we give some results which are very important for our con-sideration.

Proposition 2.1. *The functional* ϱ *is a convex modular on* $\mathrm{ces}_{(p)}(q)$.

Proof. Let $x, y \in \mathrm{ces}_{(p)}(q)$. It is obvious that $\varrho(x) = 0$ if and only if $x = 0$ and $\varrho(\alpha x) = \varrho(x)$ for scalar α with $|\alpha| = 1$. Let $\alpha \geq 0$, $\beta \geq 0$ with $\alpha + \beta = 1$. By the convexity of the function $t \mapsto |t|^{p_k}$, for all $k \in \mathbb{N}$, we have

$$
\begin{aligned}
\varrho(\alpha x + \beta y) &= \sum_{k=1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|\alpha q_i x(i) + \beta q_i y(i)|\right)^{p_k} \\
&\le \sum_{k=1}^{\infty}\left(\alpha\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x(i)| + \beta\frac{1}{Q_k}\sum_{i=1}^{k}|q_i y(i)|\right)^{p_k} \\
&\le \alpha\sum_{k=1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x(i)|\right)^{p_k} + \beta\sum_{k=1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i y(i)|\right)^{p_k} \\
&= \alpha\varrho(x) + \beta\varrho(y).
\end{aligned}
$$

□

Proposition 2.2. *For* $x \in \operatorname{ces}_{(p)}(q)$, *the modular* ϱ *on* $\operatorname{ces}_{(p)}(q)$ *satisfies the following properties:*

(i) *if* $0 < a < 1$, *then* $a^M\varrho(\frac{x}{a}) \le \varrho(x)$ *and* $\varrho(ax) \le a\varrho(x)$;
(ii) *if* $a > 1$, *then* $\varrho(x) \le a^M\varrho(\frac{x}{a})$;
(iii) *if* $a \ge 1$, *then* $\varrho(x) \le a\varrho(x) \le \varrho(ax)$.

Proof. (*i*) Let $0 < a < 1$. Then we have

$$
\begin{aligned}
\varrho(x) &= \sum_{k=1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x(i)|\right)^{p_k} \\
&= \sum_{k=1}^{\infty}\left(\frac{a}{Q_k}\sum_{i=1}^{k}\left|\frac{q_i x(i)}{a}\right|\right)^{p_k} \\
&= \sum_{k=1}^{\infty}a^{p_k}\left(\frac{1}{Q_k}\sum_{i=1}^{k}\left|\frac{q_i x(i)}{a}\right|\right)^{p_k} \\
&\ge \sum_{k=1}^{\infty}a^{M}\left(\frac{1}{Q_k}\sum_{i=1}^{k}\left|\frac{q_i x(i)}{a}\right|\right)^{p_k} \\
&= a^M\sum_{k=1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=1}^{k}\left|\frac{q_i x(i)}{a}\right|\right)^{p_k} \\
&= a^M\varrho\left(\frac{x}{a}\right).
\end{aligned}
$$

By convexity of modular ϱ, we have $\varrho(ax) \le a\varrho(x)$, so (*i*) is obtained.
(*ii*) Let $a > 1$. Then

$$
\begin{aligned}
\varrho(x) &= \sum_{k=1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x(i)|\right)^{p_k} \\
&= \sum_{k=1}^{\infty}a^{p_k}\left(\frac{1}{Q_k}\sum_{i=1}^{k}\left|\frac{q_i x(i)}{a}\right|\right)^{p_k} \\
&\le a^M\sum_{k=1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=1}^{k}\left|\frac{q_i x(i)}{a}\right|\right)^{p_k} \\
&= a^M\varrho\left(\frac{x}{a}\right).
\end{aligned}
$$

Hence (*ii*) is satisfies. (*iii*) follows from the convexity of ϱ. □

Proposition 2.3. *For any* $x \in \operatorname{ces}_{(p)}(q)$, *we have*

(i) *if* $||x|| < 1$, *then* $\varrho(x) \le ||x||$;
(ii) *if* $||x|| > 1$, *then* $\varrho(x) \ge ||x||$;
(iii) $||x|| = 1$ *if and only if* $\varrho(x) = 1$;
(iv) $||x|| < 1$ *if and only if* $\varrho(x) < 1$;
(v) $||x|| > 1$ *if and only if* $\varrho(x) > 1$.

Proof. (*i*) Let $\varepsilon > 0$ be such that $0 < \varepsilon < 1 - ||x||$, so $||x|| + \varepsilon < 1$. By the definition of $||\cdot||$, then there exits $\lambda > 0$ such that $||x|| + \varepsilon > \lambda$ and $\varrho(\frac{x}{\lambda}) \le 1$. By (*i*) and (*iii*) of Proposition 2.2, we have

$$\begin{aligned}\varrho(x) &\le \varrho\left(\frac{(||x|| + \varepsilon)}{\lambda}x\right)\\ &= \varrho\left((||x|| + \varepsilon)\frac{x}{\lambda}\right)\\ &\le (||x|| + \varepsilon)\varrho\left(\frac{x}{\lambda}\right)\\ &\le ||x|| + \varepsilon,\end{aligned}$$

which implies that $\varrho(x) \le ||x||$. Hence (*i*) is satisfies.

(*ii*) Let $\varepsilon > 0$ such that $0 < \varepsilon < \frac{||x||-1}{||x||}$, then $0 < (1 - \varepsilon)||x|| \le ||x||$. By definition of $||.||$ and Proposition 2.2(*i*), we have $1 < \varrho(\frac{x}{(1-\varepsilon)||x||}) < \frac{x}{(1-\varepsilon)||x||}\varrho(x)$, so $(1 - \varepsilon)||x|| < \varrho(x)$ for all $\varepsilon \in (0, \frac{||x||-1}{||x||})$ which implies that $||x|| \le \varrho(x)$.

(*iii*) Assume that $||x|| = 1$. Let $\varepsilon > 0$ then there exits $\lambda > 0$ such that $1 + \varepsilon > \lambda > ||x||$ and $\varrho(\frac{x}{\lambda}) \le 1$. By Proposition 2.2(*ii*), we have $\varrho(x) \le \lambda^M \varrho(\frac{x}{\lambda}) \le \lambda^M < (1+\varepsilon)^M$, so $(\varrho(x))^{\frac{1}{M}} < 1 + \varepsilon$ for all $\varepsilon > 0$ which implies that $\varrho(x) \le 1$. If $\varrho(x) < 1$, let $a \in (0, 1)$ such that $\varrho(x) < a^M < 1$. From Proposition 2.2(*i*), we have $\varrho(\frac{x}{a}) \le \frac{1}{a^M}\varrho(x) < 1$. Hence $||x|| \le a < 1$, which is contradiction. Thus, we have $\varrho(x) = 1$.

Conversely, assume that $\varrho(x) = 1$. By definition of $||\cdot||$, we conclude that $||x|| \le 1$. If $||x|| < 1$, then we have by (*i*) that $\varrho(x) \le ||x|| < 1$, which is contradiction, so we obtain that $||x|| = 1$. (*iv*) follows from (*i*) and (*iii*), (*v*) follows from (*iii*) and (*iv*). □

Proposition 2.4. *For any* $x \in \operatorname{ces}_{(p)}(q)$, *we have*

(i) *if* $0 < a < 1$ *and* $||x|| > a$, *then* $\varrho(x) > a^M$;
(ii) *if* $a \ge 1$ *and* $||x|| < a$, *then* $\varrho(x) < a^M$.

Proof. (*i*) Let $0 < a < 1$ and $||x|| > a$. Then $||\frac{x}{a}|| > 1$, by Proposition 2.3(*v*), we have $\varrho(\frac{x}{a}) > 1$. Hence by Proposition 2.2(*i*), we have $\varrho(x) \ge a^M \varrho(\frac{x}{a}) > a^M$, so we obtain (*i*).

(*ii*) Suppose $a \ge 1$ and $||x|| < a$. Then $||\frac{x}{a}|| < 1$, by Proposition 2.3(*iv*), we have $\varrho(\frac{x}{a}) < 1$.

If $a = 1$, it is obvious that $\varrho(x) < 1 = a^M$. If $a > 1$, then by Proposition 2.2(*ii*), we obtain that $\varrho(x) \le a^M \varrho(\frac{x}{a}) < a^M$. □

Proposition 2.5. *Let* (x_n) *be a sequence in* $\mathrm{ces}_{(p)}(q)$.

(i) *If* $||x_n|| \to 1$ *as* $n \to \infty$, *then* $\varrho(x_n) \to 1$ *as* $n \to \infty$.
(ii) *If* $\varrho(x_n) \to 0$ *as* $n \to \infty$, *then* $||x_n|| \to 0$ *as* $n \to \infty$.

Proof. (i) Assume that $||x_n|| \to 1$ as $n \to \infty$. Let $\varepsilon \in (0, 1)$. Then there exists $N \in \mathbb{N}$ such that $1 - \varepsilon < ||x_n|| < 1 + \varepsilon$ for all $n \geq N$. By Proposition 2.4, we have $(1 - \varepsilon)^M < \varrho(x_n) < (1 + \varepsilon)^M$ for all $n \geq N$, which implies that $\varrho(x_n) \to 1$ as $n \to \infty$.

(ii) Suppose that $||x_n|| \nrightarrow 0$ as $n \to \infty$. Then there exists $\varepsilon \in (0, 1)$ and a subsequence (x_{n_k}) of (x_n) such that $||x_{n_k}|| > \varepsilon$ for all $k \in \mathbb{N}$. By Proposition 2.4(i) we obtain $\varrho(x_{n_k}) > (\varepsilon)^M$ for all $k \in \mathbb{N}$. This implies that $\varrho(x_n) \nrightarrow 0$ as $n \to \infty$. □

Lemma 2.6. *Let* $x \in \mathrm{ces}_{(p)}(q)$ *and* $(x_n) \subseteq \mathrm{ces}_{(p)}(q)$. *If* $\varrho(x_n) \to \varrho(x)$ *as* $n \to \infty$ *and* $x_n(i) \to x(i)$ *as* $n \to \infty$ *for all* $i \in \mathbb{N}$, *then* $x_n \to x$ *as* $n \to \infty$.

Proof. Let $\varepsilon > 0$ be given. Since $\varrho(x) = \sum_{k=1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x_n(i)|\right)^{p_k} < \infty$, there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0+1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x_n(i)|\right)^{p_k} < \frac{\varepsilon}{3 \cdot 2^{M+1}}. \tag{2.1}$$

Since $\varrho(x_n) - \sum_{k=1}^{k_0}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x_n(i)|\right)^{p_k} \to \varrho(x) - \sum_{k=1}^{k_0}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x(i)|\right)^{p_k}$ and $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that

$$\varrho(x_n) - \sum_{k=1}^{k_0}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x_n(i)|\right)^{p_k} < \varrho(x) - \sum_{k=1}^{k_0}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x(i)|\right)^{p_k} + \frac{\varepsilon}{3 \cdot 2^{M}} \tag{2.2}$$

for all $n \geq n_0$ and

$$\sum_{k=1}^{k_0}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x_n(i) - q_i x(i)|\right)^{p_k} < \frac{\varepsilon}{3}, \tag{2.3}$$

for all $n \geq n_0$. It follow from (2.1), (2.2), and (2.3), for all $n \geq n_0$ we have

$$\begin{aligned}
\varrho(x_n - x) &= \sum_{k=1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x_n(i) - q_i x(i)|\right)^{p_k} \\
&= \sum_{k=1}^{k_0}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x_n(i) - q_i x(i)|\right)^{p_k} + \sum_{k=k_0+1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x_n(i) - q_i x(i)|\right)^{p_k} \\
&< \frac{\varepsilon}{3} + 2^M\left(\sum_{k=k_0+1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x_n(i))|\right)^{p_k} + \sum_{k=k_0+1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x(i))|\right)^{p_k}\right) \\
&= \frac{\varepsilon}{3} + 2^M\left(\varrho(x_n) - \sum_{k=1}^{k_0}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x_n(i)|\right)^{p_k} + \sum_{k=k_0+1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x(i))|\right)^{p_k}\right) \\
&< \frac{\varepsilon}{3} + 2^M\left(\varrho(x) - \sum_{k=1}^{k_0}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x(i)|\right)^{p_k} + \frac{\varepsilon}{3 \cdot 2^M} + \sum_{k=k_0+1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x(i))|\right)^{p_k}\right) \\
&= \frac{\varepsilon}{3} + 2^M\left(\sum_{k=k_0+1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x(i)|\right)^{p_k} + \frac{\varepsilon}{3 \cdot 2^M} + \sum_{k=k_0+1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x(i))|\right)^{p_k}\right) \\
&= \frac{\varepsilon}{3} + 2^M\left(2\sum_{k=k_0+1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x(i)|\right)^{p_k} + \frac{\varepsilon}{3 \cdot 2^M}\right) \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}$$

This show that $\varrho(x_n - x) \to 0$ as $n \to \infty$. Hence, by Proposition 2.5(ii), we have $||x_n - x|| \to 0$ as $\to \infty$. □

Theorem 2.7. *The space* $\mathrm{ces}_{(p)}(q)$ *has the property* (*H*).

Proof. Let $x \in S(\mathrm{ces}_{(p)}(q))$ and $(x_n) \subseteq \mathrm{ces}_{(p)}(q)$ such that $||x_n|| \to 1$ and $x_n \xrightarrow{w} x$ as $n \to \infty$. By Proposition 2.3(iii), we have $\varrho(x) = 1$, so it follow form Proposition 2.5(i), we get $\varrho(x_n) \to \varrho(x)$ as $n \to \infty$. Since the mapping π_i: $\mathrm{ces}_{(p)}(q) \to \mathbb{R}$ defined by $\pi_i(y) = y(i)$, is a continuous linear functional on $\mathrm{ces}_{(p)}(q)$, it follow that $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$. Thus by Lemma 2.6, we obtain $x_n \to x$ as $n \to \infty$, and hence the space $\mathrm{ces}_{(p)}(q)$ has the property (*H*). □

Corollary 2.8. *For any* $1 < p < \infty$, *the space* $\mathrm{ces}_p(q)$ *has the property* (*H*).

Corollary 2.9. [9, Theorem 2.6] *The space* $\mathrm{ces}_{(p)}$ *has the property* (*H*).

Corollary 2.10. *For any* $1 < p < \infty$, *the space* ces_p *has the property* (*H*).

Theorem 2.11. *The space* $\mathrm{ces}_{(p)}(q)$ *has uniform Opial property.*

Proof. Take any $\varepsilon > 0$ and $x \in \mathrm{ces}_{(p)}(q)$ with $||x|| \geq \varepsilon$. Let (x_n) be weakly null sequence in $S(\mathrm{ces}_{(p)}(q))$. By $\sup_k p_k < \infty$, i.e., $\varrho \in \Delta_2^s$, hence by Lemma 1.2 there exists $\delta \in (0, 1)$ independent of x such that $\varrho(x) > \delta$. Also, by $\varrho \in \Delta_2^s$ and Lemma 1.1 asserts that there exists $\delta_1 \in (0, \delta)$ such that

$$|\varrho(y+z) - \varrho(y)| < \frac{\delta}{4} \tag{2.4}$$

whenever, $\varrho(y) \leq 1$ and $\varrho(z) \leq \delta_1$. Choose $k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0+1}^{\infty} \left(\frac{1}{Q_k} \sum_{i=k_0+1}^{k} |q_i x(i)| \right)^{p_k} < \sum_{k=k_0+1}^{\infty} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i)| \right)^{p_k} < \frac{\delta_1}{4}. \tag{2.5}$$

So, we have

$$\begin{aligned} \delta &< \sum_{k=1}^{k_0} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i)| \right)^{p_k} + \sum_{k=k_0+1}^{\infty} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i)| \right)^{p_k} \\ &\leq \sum_{k=1}^{k_0} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i)| \right)^{p_k} + \frac{\delta_1}{4}, \end{aligned} \tag{2.6}$$

which implies that

$$\begin{aligned} \sum_{k=1}^{k_0} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x(i)| \right)^{p_k} &> \delta - \frac{\delta_1}{4} \\ &> \delta - \frac{\delta}{4} \\ &= \frac{3\delta}{4}. \end{aligned} \tag{2.7}$$

Since $x_n \xrightarrow{w} 0$, then there exists $n_0 \in \mathbb{N}$ such that

$$\frac{3\delta}{4} \leq \sum_{k=1}^{k_0} \left(\frac{1}{Q_k} \sum_{i=1}^{k} |q_i x_n(i) + q_i x(i)| \right)^{p_k} \tag{2.8}$$

for all $n > n_0$, since weak convergence implies coordinatewise convergence. Again, by $x_n \xrightarrow{w} 0$, then there exists $n_1 \in \mathbb{N}$ such that

$$||x_{n|_{k_o}}|| < 1 - \left(1 - \frac{\delta}{4}\right)^{\frac{1}{M}} \tag{2.9}$$

for all $n > n_1$ where $p_k \leq M$ for all $k \in \mathbb{N}$. Hence, by the triangle inequality of the norm, we get

$$||x_{n|_{\mathbb{N}-k_o}}|| > \left(1 - \frac{\delta}{4}\right)^{\frac{1}{M}}. \tag{2.10}$$

It follows by the definition of $||\cdot||$, we have

$$\begin{aligned} 1 &\leq \varrho\left(\frac{x_{n|_{\mathbb{N}-k_o}}}{\left(1-\frac{\delta}{4}\right)^{\frac{1}{M}}}\right) \\ &= \sum_{k=k_0+1}^{\infty}\left(\frac{\frac{1}{Q_k}\sum_{i=k_0+1}^{k}|q_i x_n(i)|}{\left(1-\frac{\delta}{4}\right)^{\frac{1}{M}}}\right)^{p_k} \\ &\leq \left(\frac{1}{\left(1-\frac{\delta}{4}\right)^{\frac{1}{M}}}\right)^{M}\sum_{k=k_0+1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=k_0+1}^{k}|q_i x_n(i)|\right)^{p_k} \end{aligned} \tag{2.11}$$

implies that

$$\sum_{k=k_0+1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=k_0+1}^{\infty}|q_i x_n(i)|\right)^{p_k} \geq 1 - \frac{\delta}{4} \tag{2.12}$$

for all $n > n_1$. By inequality (2.4), (2.5), (2.8), and (2.12), yields for any $n > n_1$ that

$$\begin{aligned} \varrho(x_n + x) &= \sum_{k=1}^{k_0}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x_n(i) + q_i x(i)|\right)^{p_k} + \sum_{k=k_0+1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x_n(i) + q_i x(i)|\right)^{p_k} \\ &> \sum_{k=1}^{k_0}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i x_n(i) + q_i x(i)|\right)^{p_k} + \sum_{k=k_0+1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=k_0+1}^{k}|q_i x_n(i) + q_i x(i)|\right)^{p_k} \\ &\geq \frac{3\delta}{4} + \sum_{k=k_0+1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=k_0+1}^{k}|q_i x_n(i)|\right)^{p_k} - \frac{\delta}{4} \\ &\geq \frac{3\delta}{4} + \left(1 - \frac{\delta}{4}\right) - \frac{\delta}{4} \\ &\geq 1 + \frac{\delta}{4}. \end{aligned}$$

Since $\varrho \in \Delta_2^s$ and by Lemma 1.3 there exists τ depending on δ only such that $\| x_n + x \| \geq 1+\tau$, which implies that $\liminf_{n\to\infty} \|x_n + x\| \geq 1 + \tau$, hence the proof is complete. □

Corollary 2.12. *For any* $1 < p < \infty$, *the space* $\mathrm{ces}_p(q)$ *has the uniform Opial property.*

Corollary 2.13. [5, Theorem 2.6] *The space* $\mathrm{ces}_{(p)}$ *has the uniform Opial property.*

Corollary 2.14. [4, Theorem 2] *For any* $1 < p < \infty$, *the space* ces_p *has the uniform Opial property.*

Acknowledgements
Mr. Chirasak Mongkolkeha was supported from the Thailand Research Fund through the Royal Golden Jubilee Program under Grant PHD/0029/2553 for the Ph.D program at KMUTT, Thailand. Moreover, the authors would like to thank the Higher Education Research Promotion and National Research University Project of Thailand, Office of the Higher Education Commission (NRU-CSEC No.54000267) for financial support.

Authors' contributions
The authors have equitably contributed in obtaining the new results presented in this article. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 29 October 2011 Accepted: 30 March 2012 Published: 30 March 2012

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doi:10.1186/1029-242X-2012-76
Cite this article as: Mongkolkeha and Kumam: On *H*-property and uniform Opial property of generalized cesàro sequence spaces. *Journal of Inequalities and Applications* 2012 **2012**:76.