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Strong convergence theorems by Halpern-Mann iterations for multi-valued relatively nonexpansive mappings in Banach spaces with applications

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Abstract

In this article, an iterative sequence for relatively nonexpansive multi-valued mapping by modifying Halpern and Mann's iterations is introduced, and then some strong convergence theorems are proved. At the end of the article some applications are given also.

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1 Introduction

Throughout this article, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. Let D be a nonempty closed subset of a real Banach space E. A single-valued mapping $T : D \to D$ is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in D$. Let $N(D)$ and $CB(D)$ denote the family of nonempty subsets and nonempty closed bounded subsets of D, respectively. The Hausdorff metric on $CB(D)$ is defined by

$$H(A_1, A_2) = \max\left\{\sup_{x\in A_1} d(x, A_2), \sup_{y\in A_2} d(y, A_1)\right\},$$

for $A_1,A_2 \in CB(D)$, where $d(x, A_1) = \inf\{\|x - y\|, y \in A_1\}$. The multi-valued mapping $T : D \to CB(D)$ is called nonexpansive if $H(T(x),T(y)) \le \|x - y\|$ for all $x, y \in D$. An element $p \in D$ is called a fixed point of $T : D \to N(D)$ if $p \in T(p)$. The set of fixed points of T is represented by $F(T)$.

Let E be a real Banach space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \left\{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\right\}, x \in E.$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing.

A Banach space E is said to be strictly convex if $\frac{\|x+y\|}{2} < 1$ for all $x, y \in U = \{z \in E : \|z\| = 1\}$ with $x \ne y$. E is said to be uniformly convex if, for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\frac{\|x+y\|}{2} < 1 - \delta$ for all $x, y \in U$ with $\|x - y\| \ge \epsilon$. E is said to be smooth if the limit

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$$

exists for all $x, y \in U$. E is said to be uniformly smooth if the above limit exists uniformly in $x, y \in U$.

Remark 1.1. The following basic properties for Banach space E and for the normalized duality mapping J can be found in Cioranescu [1].

(i) If E is an arbitrary Banach space, then J is monotone and bounded;

(ii) If E is a strictly convex Banach space, then J is strictly monotone;

(iii) If E is a a smooth Banach space, then J is single-valued, and hemi-continuous, i.e., J is continuous from the strong topology of E to the weak star topology of E^*;

(iv) If E is a uniformly smooth Banach space, then J is uniformly continuous on each bounded subset of E;

(v) If E is a reflexive and strictly convex Banach space with a strictly convex dual E^* and J^*: $E^* \to E$ is the normalized duality mapping in E^*, then $J^{-1} = J^*$, $J J^* = I_{E^*}$, and $J^* J = I_E$;

(vi) If E is a smooth, strictly convex, and reflexive Banach space, then the normalized duality mapping J is single-valued, one-to-one and onto;

(vii) A Banach space E is uniformly smooth if and only if E^* is uniformly convex. If E is uniformly smooth, then it is smooth and reflexive.

Next we assume that E is a smooth, strictly convex, and reflexive Banach space and C is a nonempty closed convex subset of E. In the sequel, we always use $\varphi : E \times E \to \mathbb{R}^+$ to denote the Lyapunov functional defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \forall x, y \in E. \tag{1.2}$$

It is obvious from the definition of φ that

$$(\|x\| - \|y\|)^2 \le \phi(x,y) \le (\|x\| - \|y\|)^2, \forall x, y \in E. \tag{1.3}$$

$$\phi\left(x, J^{-1}\left(\lambda Jy + (1-\lambda)Jz\right) \le \lambda\phi(x,y) + (1-\lambda)\phi(x,z)\right), \tag{1.4}$$

for all $\lambda \in [0,1]$ and $x,y,z \in E$.

Following Alber [2], the generalized projection $\Pi_C : E \to C$ is defined by

$$\prod\nolimits_C(x) = \arg\inf_{y\in C}\phi(y,x), \forall x \in E.$$

Let D be a nonempty subset of a smooth Banach space. A mapping $T : D \to E$ is relatively expansive [3-5], if the following properties are satisfied:

(R1) $F(T) \neq \emptyset$;
(R2) $\varphi(p,Tx) \le \varphi(p,x)$ for all $p \in F(T)$ and $x \in D$;
(R3) I - T is demi-closed at zero, that is, whenever a sequence $\{x_n\}$ in D converges weakly to p and $\{x_n - Tx_n\}$ converges strongly to 0, it follows that $p \in F(T)$.

If T satisfies (R1) and (R2), then T is called quasi-φ-nonexpansive [6].

Recently, Nilsrakoo and Saejung [7] introduced the following iterative sequence for finding a fixed point of relatively nonexpansive mapping $T : D \to E$. Given $x_1 \in D$,

$$x_{n+1} = \prod_D J^{-1} \left(\alpha_n Ju + (1-\alpha_n)\left(\beta_n Jx_n + (1-\beta_n)JTx_n\right)\right)$$

where D is nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E, Π_D is the generalized projection of E onto D and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in [0,1].

They proved strong convergence theorems in uniformly convex and uniformly smooth Banach space E.

Iterative methods for approximating fixed points of multi-valued mappings in Banach spaces have been studied by some authors, see for instance [8-15].

Let D be a nonempty closed convex subset of a smooth Banach space E. We define a relatively nonexpansive multi-valued mapping as follows.

Definition 1.2. A multi-valued mapping $T : D \to N(D)$ is called relatively nonexpansive, if the following conditions are satisfied:

(S1) $F(T) \neq \emptyset$
(S2) $\varphi(p,z) \leq \varphi(p, x)$, $\forall x \in D$, $z \in T(x)$, $p \in F(T)$;
(S3) $I - T$ is demi-closed at zero, that is, whenever a sequence $\{x_n\}$ in D which weakly to p and $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$, it follows that $p \in F(T)$.

If T satisfies (S1) and (S2), then multi-valued mapping T is called quasi-φ-nonexpansive.

In this article, inspired by Nilsrakoo and Saejung [7], we introduce the following iterative sequence for finding a fixed point of relatively nonexpansive multi-valued mapping $T : D \to N(D)$. Given $u \in E, x_i \in D$,

$$x_{n+1} = \prod_D J^{-1} \left(\alpha_n Ju + (1-\alpha_n)\left(\beta_n Jx_n + (1-\beta_n)Jw_n\right)\right.$$

where $w_n \in Tx_n$ for all $n \in \mathbb{N}$, D is a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E, Π_D is the generalized projection of E onto D and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in [0,1]. We proved the strong convergence theorems in uniformly convex and uniformly smooth Banach space E.

2 Preliminaries

In the sequel, we denote the strong convergence and weak convergence of the sequence $\{x_n\}$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

First, we recall some conclusions.

Lemma 2.1 [16,17]. Let E be a smooth, strictly convex, and reflexive Banach space and C be a nonempty closed convex subset of E. Then the following conclusions hold:

(a) $\varphi(x, \Pi_C y) + \varphi(\Pi_C y, y) \leq \varphi(x, y)$ for all $x \in C$ and $y \in E$;
(b) If $x \in E$ and $z \in C$, then

$$z = \prod_C x \Leftrightarrow \left\langle z - y, Jx - Jz \right\rangle \geq 0, \forall y \in C;$$

(c) For $x, y \in E$, $\varphi(x, y) = 0$ if and only $x = y$.

Remark 2.2. If E is a real Hilbert space H, then $\varphi(x, y) = \|x - y\|^2$ and Π_C is the metric projection P_C of H onto C.

Lemma 2.3 [18]. Let E be a uniformly convex Banach space, $r > 0$ be a positive number and $B_r(0)$ be a closed ball of E. Then, for any given sequence $\{x_i\}_{i=1}^{\infty} \subset B_r(0)$ and for any given sequence $\{\lambda_i\}_{i=1}^{\infty}$ of positive numbers with $\sum_{i=1}^{\infty} \lambda_i = 1$, then there exists a continuous, strictly increasing, and convex function $g : [0, 2r) \to [0, \infty)$ with $g(0) = 0$ such that for any positive integers i, j with $i < j$,

$$\left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\|^2 \leq \sum_{n=1}^{\infty} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g\left(\|x_i - x_j\|\right) \tag{2.1}$$

In what follows, we need the following lemmas for proof of our main results.

Lemma 2.4 [17]. Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty}\varphi(x_n, y_n) = 0$. Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

Let E be a reflexive, strictly convex, and smooth Banach space. The duality mapping J^* from E^* onto $E^{**} = E$ coincides with the inverse of the duality mapping J from E onto E^*, that is, $J^* = J^{-1}$. We make use the following mapping $V : E \times E^* \to \mathbb{R}$ studied in Alber [19]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 \tag{2.2}$$

for all $x \in E$ and $x^* \in E^*$. Obviously, $V(x, x^*) = \varphi(x, J^{-1}(x^*))$ for all $x \in E$ and $x^* \in E^*$. We know the following lemma.

Lemma 2.5 [20]. Let E be a reflexive, strictly convex, and smooth Banach space, and let V as in (2.2). Then

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^*\rangle \leq V(x, x^* + y^*), \tag{2.3}$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.6 [21]. Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \gamma_n\delta_n,$$

where $\{\gamma_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence such that

(a) $\lim_{n\to\infty}\gamma_n = 0, \sum_{n=1}^{\infty}\gamma_n = \infty$;

(b) $\limsup_{n\to\infty} \leq 0$.

Then $\lim_{n\to\infty}\alpha_n = 0$.

Lemma 2.7 [22]. Let $\{\alpha_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $\alpha_{n_i} < \alpha_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied for all (sufficiently large) numbers $k \in \mathbb{N}$:

$$\alpha_{m_k} \leq \alpha_{m_k+1} \;\; \textit{and} \;\; \alpha_k \leq \alpha_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : \alpha_j < \alpha_{j+1}\}$.

3 Main results

Lemma 3.1 Let E be a strictly convex and smooth Banach space, and D a nonempty closed subset of E. Suppose $T : D \to N(D)$ is a quasi-φ-nonexpansive multi-valued mapping. Then $F(T)$ is closed and convex.

Proof. First, we show $F(T)$ is closed. Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \to x^*$. Since T is quasi-φ-nonexpansive, we have

$$\phi(x_n, z) \leq \phi(x_n, x^*)$$

for all $z \in T(x^*)$ and for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}\phi(x^*, z) &= \lim_{n\to\infty} \phi(x_n, z) \\ &\leq \lim_{n\to\infty} \phi(x_n, x^*) \\ &= \phi(x^*, x^*) \\ &= 0.\end{aligned}$$

By Lemma 2.1(c), we obtain $x^* = z$. Hence, $T(x^*) = \{x^*\}$. So, we have $x^* \in F(T)$. Next, we show $F(T)$ is convex. Let $x, y \in F(T)$ and $t \in (0,1)$, put $p = tx + (1 - t)y$. We show $p \in F(T)$. Let $w \in F(p)$, we have

$$\begin{aligned}\phi(p, w) &= \|p\|^2 - 2\langle p, Jw\rangle + \|w\|^2 \\ &= \|p\|^2 - 2\langle tx + (1-t)y, Jw\rangle + \|w\|^2 \\ &= \|p\|^2 - 2t\langle x, Jw\rangle - 2(1-t)\langle y, Jw\rangle + \|w\|^2 \\ &= \|p\|^2 + t\phi(x, w) + (1-t)\phi(y, p) - t\|x\|^2 - t(1-t)\|p\|^2 \\ &= \|p\|^2 - 2\langle tx + (1-t)y, Jp\rangle + \|p\|^2 \\ &= \|p\|^2 - 2\langle p, Jp\rangle + \|p\|^2 \\ &= 0.\end{aligned}$$

By Lemma 2.1(c), we obtain $p = w$. Hence, $T(p) = \{p\}$. So, we have $p \in F(T)$. Therefore, $F(T)$ is convex.

Lemma 3.2. Let D be a nonempty closed convex subset of a reflexive, strictly convex, and smooth Banach space E and $T : D \to N(D)$ be a relatively nonexpansive multi-valued mapping. If $\{x_n\}$ is a bounded sequence such that $\lim_{n\to\infty} d(x_n, Tx_n)$ and $x^* = \Pi_{F(T)}x$, then

$$\lim_{n\to\infty} \sup\langle x_n - x^*, Jx - Jx^*\rangle \leq 0.$$

Proof. From (S3) of the mapping T, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup y \in F(T)$ and

$$\lim_{n\to\infty} \sup\langle x_n - x^*, Jx - Jx^*\rangle = \lim_{i\to\infty}\langle x_{n_i} - x^*, Jx - Jx^*\rangle.$$

By Lemma 2.1(b), we immediately obtain that

$$\lim_{n\to\infty} \sup\langle x_n - x^*, Jx - Jx^*\rangle = \langle y - x^*, Jx - Jx^*\rangle \leq 0.$$

Lemma 3.3. Let D be a nonempty closed convex subset of a reflexive, strictly convex, and smooth Banach space E and $T : D \to N(D)$ be a relatively nonexpansive multivalued mapping. Let $\{x_n\}$ be a sequence in D defined as follows: $u \in E$, $x_1 \in D$ and

$$x_{n+1} = \prod_D J^{-1}\left(\alpha_n Ju + (1-\alpha_n)\left(\beta_n Jx_n + (1-\beta_n)Jw_n\right)\right), \tag{3.1}$$

where $w_n \in Tx_n$ for all $n \in \mathbb{N}$, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in [0,1]. Then $\{x_n\}$ is bounded.

Proof. Let $p \in F(T)$ and $y_n = J^{-1}\left(\beta_n Jx_n + (1-\beta_n)Jw_n\right)$ for all $n \in \mathbb{N}$. Then

$$x_{n+1} \equiv \prod\nolimits_D J^{-1}\left(\alpha_n Ju + (1-\alpha_n)Jy_n\right)$$

for all $n \in \mathbb{N}$. By using (1.4), we have

$$\begin{aligned}
\phi(p, y_n) &= \phi\left(p, J^{-1}\left(\beta_n Jx_n + (1-\beta_n)Jw_n\right)\right) \\
&\le \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, w_n) \\
&\le \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, x_n) \\
&= \phi(p, x_n)
\end{aligned}$$

and

$$\begin{aligned}
\phi(p, x_{n+1}) &= \phi\left(p, \prod\nolimits_D J^{-1}\left(\alpha_n Ju + (1-\alpha_n)Jy_n\right)\right) \\
&\le \phi\left(p, J^{-1}\left(\alpha_n Ju + (1-\alpha_n)Jy_n\right)\right) \\
&\le \alpha_n\phi(p, u) + (1-\alpha_n)\phi(p, y_n) \\
&\le \alpha_n\phi(p, u) + (1-\alpha_n)\phi(p, x_n) \\
&\le \max\left\{\phi(p, u), \phi(p, x_n)\right\}) \\
&\le \cdots \\
&\le \max\left\{\phi(p, u), \phi(p, x_1)\right\}).
\end{aligned}$$

This implies that $\{x_n\}$ is bounded.

Theorem 3.4 Let D be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and $T : D \to N(D)$ be a relatively nonexpansive multivalued mapping. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in (0,1) satisfying

(C1) $\lim_{n\to\infty}, \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$.

Then $\{x_n\}$ defined by (3.1) converges strongly to $\Pi_{F(T)}u$, where $\Pi_{F(T)}$ is the generalized projection from E onto $F(T)$.

Proof. By Lemma 3.1, $F(T)$ is closed and convex. So, we can define the generalized projection $\Pi_{F(T)}$ onto $F(T)$. Putting $u^* = \Pi_{F(T)}u$, by Lemma 3.3 we know that $\{x_n\}$ is bounded and hence, $\{w_n\}$ is bounded. Let $g : [0,2r] \to [0,\infty)$ be a function satisfying the properties of Lemma 2.3, where $r = \sup\{\|u\|, \|x_n\|, \|w_n\| : n \in \mathbb{N}\}$. Put

$$y_n \equiv J^{-1}\left(\beta_n Ju + (1-\beta_n)Jw_n\right)$$

Then

$$\begin{aligned}\phi(u^*, y_n) &= \phi\left(u^*, J^{-1}\left(\beta_n Jx_n + (1-\beta_n)Jw_n\right)\right)\\ &= \left\|u^*\right\|^2 - 2\left\langle u^*, \beta_n Jx_n + (1-\beta_n)Jw_n\right\rangle + \left\|\beta_n Jx_n + (1-\beta_n)Jw_n\right\|^2\\ &\le \left\|u^*\right\|^2 - 2\beta_n\left\langle u^*, Jx_n\right\rangle - 2(1-\beta_n)\left\langle u^*, Jw_n\right\rangle + \beta_n\|x_n\|^2 + (1-\beta_n)\|w_n\|^2\\ &\quad - \beta_n(1-\beta_n)g\left(\|Jx_n - Jw_n\|\right)\\ &= \phi\left(u^*, x_n\right) - \beta_n\left(1-\beta_n\right)g\left(\|Jx_n - Jw_n\|\right)\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\phi\left(u^*, x_{n+1}\right) &= \phi\left(u^*, \prod\nolimits_D J^{-1}\left(\alpha_n Ju + (1-\alpha_n)Jy_n\right)\right)\\ &\le \phi\left(u^*, J^{-1}\left(\alpha_n Ju + (1-\alpha_n)Jy_n\right)\right)\\ &\le \alpha_n\phi(u^*, u) + (1-\alpha_n)\phi(u^*, y_n)\\ &\le \alpha_n\phi(u^*, u) + (1-\alpha_n)\left(\phi(u^*, x_n) - \beta_n(1-\beta_n)g\left(\|Jx_n - Jw_n\|\right)\right).\end{aligned} \tag{3.3}$$

for all $n \in \mathbb{N}$. Put

$$M = \sup\left\{\left|\phi(u^*, u) - \phi(u^*, x_n)\right| + \beta_n(1-\beta_n)g\left(||Jx_n - Jw_n\right) : n \in \mathbb{N}\right\}$$

It follows from (3.3) that

$$\beta_n(1-\beta_n)g\left(\|Jx_n - Jw_n\|\right) \le \phi(u^*, x_n) - \phi\left(u^*, x_{n+1}\right) + \alpha_n M. \tag{3.4}$$

Let $z_n \equiv J^{-1}\left(\alpha_n Ju + (1-\alpha_n)Jy_n\right)$. Then $x_{n+1} = \prod_{C^z n}$ for all $n \in \mathbb{N}$. It follows from (2.3) and (3.2) that

$$\begin{aligned}&\phi\left(u^*, x_{n+1}\right)\\ &\quad\le \phi\left(u^*, J^{-1}\left(\alpha_n Ju + (1-\alpha_n)Jy_n\right)\right) = V\left(u^*, \alpha_n Ju + (1-\alpha_n)Jy_n\right)\\ &\quad\le V\left(u^*, \alpha_n Ju + (1-\alpha_n)Jy_n - \alpha_n(Ju - Ju^*)\right) - 2\left\langle J^{-1}\left(\alpha_n Ju + (1-\alpha_n)Jy_n\right) - u^*, -\alpha_n\left(Ju - Ju^*\right)\right\rangle\\ &\quad= V\left(u^*, \alpha_n Ju^* + (1-\alpha_n)Jy_n\right) + 2\alpha_n\left\langle z_n - u^*, Ju - Ju^*\right\rangle\\ &\quad= \phi\left(u^*, J^{-1}\left(\alpha_n Ju^* + (1-\alpha_n)Jy_n\right)\right) + 2\alpha_n\left\langle z_n - u^*, Ju - Ju^*\right\rangle\\ &\quad= \left\|u^*\right\|^2 - 2\left\langle u^*, \alpha_n Ju^* + (1-\alpha_n)Jy_n\right\rangle + \left\|\alpha_n Ju^* + (1-\alpha_n)Jy_n\right\|^2 + 2\alpha_n\left\langle z_n - u^*, Ju - Ju^*\right\rangle\\ &\quad\le \left\|u^*\right\|^2 - 2\alpha_n\left\langle u^*, Ju^*\right\rangle - 2(1-\alpha_n)\left\langle u^*, Jy_n\right\rangle + \alpha_n\left\|u^*\right\|^2 + (1-\alpha_n)\left\|y_n\right\|^2 + 2\alpha_n\left\langle z_n - u^*, Ju - Ju^*\right\rangle\\ &\quad= \alpha_n\phi(u, u^*) + (1-\alpha_n)\phi(u^*, y_n) + 2\alpha_n\left\langle z_n - u^*, Ju - Ju^*\right\rangle\\ &\quad\le (1-\alpha_n)\phi(u^*, x_n) + 2\alpha_n\left\langle z_n - u^*, Ju - Ju^*\right\rangle.\end{aligned} \tag{3.5}$$

The rest of proof will be divided into two parts:

Case (1). Suppose that there exists $n_0 \in \mathbb{N}$ such that $\left\{\phi(u^*, x_n)\right\}_{n=n_0}^{\infty}$ is nonincreasing. In this situation, $\{\varphi(u^*, x_n)\}$ is then convergent. Then $\lim_{n\to\infty} (\varphi(u^*, x_n) - \varphi(u^*, x_{n+1})) = 0$. This together with (C1), (C3), and (3.4), we obtain

$$\lim_{n\to\infty} g\left(\|Jx_n - Jw_n\|\right) = 0.$$

Therefore,

$$\lim_{n\to\infty} \|Jx_n - Jw_n\| = 0.$$

Since J^{-1} is uniformly norm-to-norm continuous on every bounded subset of E, we have

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{3.6}$$

Since $d(x_n, Tx_n) \leq \|x_n - w_n\|$, we obtain

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0 \tag{3.7}$$

Then,

$$\begin{aligned}\phi(w_n, y_n) &= \phi\left(w_n, J^{-1}\left(\beta_n Jx_n + (1-\beta_n)Jw_n\right)\right)\\ &\leq \beta_n\phi(w_n, x_n) + (1-\beta_n)\phi(w_n, w_n)\\ &= \beta_n\phi(w_n, x_n) \to 0.\end{aligned} \tag{3.8}$$

and

$$\phi(y_n, z_n) \leq \alpha_n\phi(y_n, u) + (1-\alpha_n)\phi(y_n, y_n) = \alpha_n\phi(y_n, u) \to 0. \tag{3.9}$$

From (3.8), (3.9) and Lemma 2.3, we have

$$\lim_{n\to\infty} \left\|w_n - y_n\right\| = 0$$

and

$$\lim_{n\to\infty} \left\|y_n - z_n\right\| = 0$$

This together with (3.6) gives

$$\lim_{n\to\infty} \|x_n - z_n\| = 0 \tag{3.10}$$

From (3.7), (3.10) and invoking Lemma 3.2, we have

$$\lim_{n\to\infty} \left\langle z_n - u^*, Ju - Ju^*\right\rangle = \lim_{n\to\infty} \left\langle x_n - u^*, Ju - Ju^*\right\rangle \leq 0$$

Hence the conclusion follows by Lemma 2.5.

Case (2). Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\phi(u^*, x_{n_i}) < \phi(u^*, x_{n_i+1})$$

for all $i \in \mathbb{N}$. Then, by Lemma 2.7, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$, $m_k \to \infty$ such that

$$\phi\left(u^*, x_{m_k}\right) \leq \phi\left(u^*, x_{m_k+1}\right) \quad and \quad \phi\left(u^*, x_k\right) \leq \phi\left(u^*, x_{m_k+1}\right)$$

for all $k \in \mathbb{N}$. This together with (3.4) gives

$$\beta_{m_k}(1-\beta_{m_k})g\left(\left\|Jx_{m_k} - Jw_{m_k}\right\|\right) \leq \phi\left(u^*, x_{m_k}\right) - \phi\left(u^*, x_{m_{k+1}}\right) + \alpha_{m_k}M \leq \alpha_{m_k}M$$

for all $k \in N$. Then, by conditions (C1) and (C3)

$$\lim_{k\to\infty} g\left(\left\|Jx_{m_k} - Jw_{m_k}\right\|\right) = 0$$

By the same argument as Case (1), we get

$$\lim_{k\to\infty} \sup\left\langle z_{m_k} - u^*, Ju - Ju^*\right\rangle \leq 0. \tag{3.11}$$

From (3.5), we have

$$\phi\left(u^*, x_{m_k+1}\right) \leq \left(1-\alpha_{m_k}\right)\phi\left(u^*, x_{m_k}\right) + 2\alpha_{m_k}\left\langle z_{m_k} - u^*, Ju - Ju^*\right\rangle \tag{3.12}$$

Since $\phi\left(u^*, x_{m_k}\right) \leq \phi\left(u^*, x_{m_k+1}\right)$, we have

$$\alpha_{m_k}\phi\left(u^*, x_{m_k}\right) \leq \phi\left(u^*, x_{m_k}\right)-\phi\left(u^*, x_{m_k+1}\right)+2\alpha_{m_k}\left\langle z_{m_k}-u^*, Ju-Ju^*\right\rangle \leq 2\alpha_{m_k}\left\langle z_{m_k}-u^*, Ju-Ju^*\right\rangle$$

In particular, since $\alpha_{m_k} > 0$, we get

$$\phi\left(u^*, x_{m_k}\right) \leq 2\left\langle z_{m_k}-u^*, Ju-Ju^*\right\rangle$$

It follows from (3.11) that $\lim_{k\to\infty}\phi\left(u^*, x_{m_k}\right) = 0$. This together with (3.12) gives

$$\lim_{k\to\infty}\phi\left(u^*, x_{m_k+1}\right) = 0$$

But $\phi\left(u^*, x_k\right) \leq \phi\left(u^*, x_{m_k+1}\right)$ for all $k \in \mathbb{N}$. We conclude that $x_k \to u^*$.

This implies that $\lim_{n\to\infty} x_n = u^*$ and the proof is finished.

Letting $\beta_n = \beta$ gives the following result.

Corollary 3.5. Let D be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and $T : D \to N(D)$ be a relatively nonexpansive multivalued mapping. Let $\{x_n\}$ be a sequence in D defined as follows: $u \in E$,$x_1 \in D$ and

$$x_{n+1} = \prod\nolimits_D J^{-1}\left(\alpha_n Ju + (1-\alpha_n)\left(\beta Jx_n + (1-\beta)Jw_n\right)\right),$$

where $w_n \in Tx_n$ for all $n \in \mathbb{N}$, $\{\alpha_n\}$ is a sequence in [0,1] satisfying condition (C1) and (C2), and $\beta \in (0,1)$. Then $\{x_n\}$ converges strongly to $\Pi_{F(T)}u$.

4 Application to zero point problem of maximal monotone mappings

Let E be a smooth, strictly convex, and reflexive Banach space. An operator $A : E \to 2^{E^*}$ is said to be *monotone*, if $\langle x - y, x^* - y^*\rangle \geq 0$ whenever $x, y \in E$, $x^* \in Ax$, $y^* \in Ay$. We denote the zero point set $\{x \in E : 0 \in Ax\}$ of A by $A^{-1}0$. A monotone operator A is said to be *maximal*, if its graph $G(A) := \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. If A is maximal monotone, then $A^{-1}0$ is closed and convex. Let A be a maximal monotone operator, then for each $r > 0$ and $x \in E$, there exists a unique $x_r \in D(A)$ such that $J(x) \in J(x_r) + rA(x_r)$ (see, for example, [19]). We define the *resolvent* of A by $J_r x = x_r$. In other words $J_r = (J + rA)^{-1} J$, $\forall r > 0$. We know that J_r is a single-valued relatively nonexpansive mapping and $A^{-1}0 = F(J_r)$,$\forall r > 0$, where $F(J_r)$ is the set of fixed points of J_r.

We have the following

Theorem 4.1 Let E, $\{\alpha_n\}$, and $\{\beta_n\}$ be the same as in Theorem 3.4. Let $A : E \to 2^{E^*}$ be a maximal monotone operator and $J_r = (J + rA)^{-1}J$ for all $r > 0$ such that $A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by

$$x_{n+1} = J^{-1}\left[\alpha_n Jx_1 + (1-\alpha_n)\left(\beta_n Jx_n + (1-\beta_n)JJ_{r_n}x_n\right)\right],$$

then $\{x_n\}$ converges strongly to $\Pi_{A^{-1}0}x_1$.

Proof. In Theorem 3.4 taking $D = E$, $T = J_r$, $r > 0$, then $T : E \to E$ is a single-valued relatively nonexpansive mapping and $A^{-1}0 = F(T) = F(J_r)$,$\forall r > 0$ is a nonempty closed convex subset of E. Therefore all the conditions in Theorem 3.4 are satisfied. The conclusion of Theorem 4.1 can be obtained from Theorem 3.4 immediately.

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Authors' contributions

All the authors contributed equally to the writing of the present article. And they also read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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