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Interpolatory curl-free wavelets on bounded domains and characterization of Besov spaces

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Abstract

Based on interpolatory Hermite splines on rectangular domains, the interpolatory curl-free wavelets and its duals are first constructed. Then we use it to characterize a class of vector-valued Besov spaces. Finally, the stability of wavelets that we constructed are studied.

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1 Introduction

Due to its potential use in many physical problems, like the simulation of incompressible fluids or in electromagnetism, curl-free wavelet bases have been advocated in several articles and most of the study focus on the cases of R^2 and R^3 [1-4]. However, it is reasonable to study the corresponding wavelet bases on bounded domains because of some practical use. At the same time, the stability and the characterization of function spaces are also necessary in some applications, such as the adaptive wavelet methods. In recent years, divergence-free and curl-free wavelets on bounded domains begin to be studied [5-8]. In particular, [8] use the truncation method to obtain interpolatory spline wavelets on rectangular domains from [3]. Inspired by this, we mainly study the interpolatory 3D curl-free wavelet bases on the cube and its applications for characterizing the vector-valued Besov spaces.

In Section 2, we first give the construction of interpolatory curl-free wavelets and its duals on the cube. The characterization of a class of vector-valued Besov spaces are given in part 3. Finally, we also study the stability of the corresponding curl-free wavelets.

Now, we begin with some notations and formulae, which will be used later on. Let ξ_1^+ and ξ_2^+ stand for two cubic Hermite splines:

$$\xi_1^+(x) =: (1-3x^2-2x^3)\mathcal{X}_{[-1,0)}(x) + (1-3x^2+2x^3)\mathcal{X}_{[0,1)}(x),$$
$$\xi_2^+(x) =: (x+2x^2+x^3)\mathcal{X}_{[-1,0)}(x) + (x-2x^2+x^3)\mathcal{X}_{[0,1)}(x).$$

Similarly, the quadratic Hermite splines are defined as

$$\xi_1^-(x) =: (-6x-6x^2)\mathcal{X}_{[-1,0]}, \xi_2^-(x) =: (1+4x+3x^2)\mathcal{X}_{[-1,0]}(x)+(1-4x+3x^2)\mathcal{X}_{[0,1)}(x).$$

Let $Z_j^0 =: \{0, 1, ..., 2^j\}, Z_j^1 =: \{1, 2, ..., 2^j\}, Z_j^2 =: \{0, 1, ..., 2^j - 1\}$, and $Z_j^3 =: \{1, 2, ..., 2^j - 1\}$. For each $j \geq j_0$, define the scaling functions on [0,1]:

$$\xi_{m;j,k}^{\Delta,+} =: \xi_m^+(2^j x - k)\mathcal{X}_{[0,1]}, k \in Z_j^0;$$
$$\xi_{1;j,k}^{\Delta,-} =: \xi_1^-(2^j x - k)\mathcal{X}_{[0,1]} = \xi_1^-(2^j x - k), k \in Z_j^1; \xi_{2;j,k}^{\Delta-} =: \xi_2^-(2^j x - k)\mathcal{X}_{[0,1]}, k \in Z_j^0.$$

Let $V_j^{\Delta,+} =: \text{span}\left\{\xi_{1;j,k}^{\Delta,+}, \xi_{2;j,k}^{\Delta,+} : k \in Z_j^0\right\}, V_j^{\Delta,-} =: \text{span}\left\{\xi_{1;j,k_1}^{\Delta,-}, \xi_{2;j,k_2}^{\Delta,-} : k_1 \in Z_j^1, k_2 \in Z_j^0\right\}$, then $\{V_j^{\Delta,+}\}$ and $\{V_j^{\Delta,-}\}$ are two MRAs on $L^2([0,1])$ [8]. The corresponding duals $\widetilde{\xi}_{m;j,k}^{\Delta,\pm}$ are given in the sense of distributions:

$$\tilde{\xi}_1^+ =: \delta_0, \left\langle f, \widetilde{\xi}_{1;j,k}^{\Delta,+}\right\rangle = f(2^{-j}k), k \in Z_j^0; \tilde{\xi}_2^+ =: -\delta'_0, \left\langle f, \widetilde{\xi}_{2;j,k}^{\Delta,+}\right\rangle = 2^{-j} f'(2^{-j}k), k \in Z_j^0;$$
$$\widetilde{\xi}_1^- =: \mathcal{X}_{[-1,0]}, \left\langle f, \widetilde{\xi}_{1;j,k}^{\Delta,-}\right\rangle = 2^j \int_{2^{-j}(k-1)}^{2^{-j}k} f(x)dx, k \in Z_j^1; \widetilde{\xi}_2^- =: \delta_0, \left\langle f, \widetilde{\xi}_{2;j,k}^{\Delta,-}\right\rangle = f(2^{-j}k), k \in Z_j^0.$$

The inperpolating multi-wavelets $\eta_{m;j,k}^{\Delta,\pm}$ on [0,1] as well as the wavelet spaces are defined by

$$\eta_{m;j,k}^{\Delta,\pm}(x) =: \eta_{m;j,k}^{\pm}(x)\mathcal{X}_{[0,1]} = \eta_{m;j,k}^{\pm}(x), k \in Z_j^2; W_j^{\Delta,\pm} =: \text{span}\left\{\eta_{m;j,k}^{\Delta,\pm} : m = 1, 2, k \in Z_j^2\right\}$$

with $\eta_m^+ =: \xi_m^+(2\cdot -1), m = 1, 2; \eta_1^- =: \xi_1^-(2\cdot -1) - \xi_1^-(2\cdot -2); \eta_2^- =: \xi_2^-(2\cdot -1)$. Here and after, $h_{j,k}(\cdot) = h(2^j \cdot -k)$. The corresponding duals are given by

$$\tilde{\eta}_1^+ =: \delta_{\frac{1}{2}} - \frac{1}{2}\delta_0 - \frac{1}{2}\delta_1 + \frac{1}{8}\delta'_0 - \frac{1}{8}\delta'_1, \ \tilde{\eta}_2^+ =: \frac{3}{4}\delta_0 - \frac{3}{4}\delta_1 - \frac{1}{2}\delta'_{\frac{1}{2}} - \frac{1}{8}\delta'_0 - \frac{1}{8}\delta'_1,$$
$$\tilde{\eta}_1^- =: \mathcal{X}_{\left[0,\frac{1}{2}\right]} - \mathcal{X}_{\left[\frac{1}{2},1\right]} - \frac{1}{4}\delta_0 + \frac{1}{4}\delta_1, \ \tilde{\eta}_2^- =: \delta_{\frac{1}{2}} + \frac{1}{4}\delta_0 + \frac{1}{4}\delta_1 - \frac{3}{2}\mathcal{X}_{[0,1]}$$

and $\left\langle f, \widetilde{\eta}_{m;j,k}^{\Delta,\pm}\right\rangle = \left\langle f\left(\frac{\cdot+k}{2^j}\right), \widetilde{\eta}_m^{\pm}\right\rangle$. Moreover, there is the following differential relations

$$\begin{gathered}\frac{d}{dx}\xi_{1;j,k}^{\Delta,+}(x) = 2^j\left(\xi_{1;j,k}^{\Delta,-}(x) - \xi_{1;j,k}^{\Delta,-}(x - 2^{-j})\right), k \in Z_j^3; \frac{d}{dx}\xi_{1;j,0}^{\Delta,+}(x) = -2^j\xi_{1;j,1}^{\Delta,-}(x),\\ \frac{d}{dx}\xi_{1;j,2^j}^{\Delta,+} = 2^j\xi_{1;j,2^j}^{\Delta,-}; \frac{d}{dx}\xi_{2;j,k}^{\Delta,+} = 2^j\xi_{2;j,k}^{\Delta,-}, k \in Z_j^0; \frac{d}{dx}\eta_{m;j,k}^{\Delta,+} = 2^{j+1}\eta_{m;j,k}^{\Delta,-}, k \in Z_j^2.\\ \frac{d}{dx}\widetilde{\xi}_{1;j,k}^{\Delta,-} = -2^j\left(\frac{d}{dx}\widetilde{\xi}_{1;j,k}^{\Delta,+} - \frac{d}{dx}\widetilde{\xi}_{1;j,k-1}^{\Delta,+}\right), \frac{d}{dx}\widetilde{\xi}_{2;j,k}^{\Delta,-} = -2^j\frac{d}{dx}\widetilde{\xi}_{2;j,k}^{\Delta,+}.\end{gathered} \tag{1.1}$$

2 Curl-free wavelets on the cube

For $\overrightarrow{u}(x, y, z) = (u_1, u_2, u_3)^T$, the 3D curl-operator is defined as

$$\text{curl}\overrightarrow{u} = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1)^T.$$

Let $I \subseteq \{1, 2, 3\} =: I_0$, define scaling functions

$$\varphi_m^I(x_1, x_2, x_3) =: \prod_{\nu=1}^{3}\xi_{m_\nu,\nu}^I(x_\nu), \ m = (m_1, m_2, m_3)^T \in \{1, 2\}^3$$

with $\xi_{\mu,\nu}^I = \begin{cases}\xi_\mu^+, \nu \in I\\ \xi_\mu^-, \nu \notin I.\end{cases}$

The corresponding wavelets are

$$\psi^I_{e,m}(x_1,x_2,x_3) =: \prod_{\nu=1}^{3} \vartheta^I_{e_\nu,m_\nu,\nu}(x_\nu), e \in E_3^*, m = (m_1,m_2,m_3)^T \in \{1,2\}^3.$$

Here and after, E_3^* denotes the non-zero apexes of the unite cube and $\vartheta^I_{\ell,\mu,\nu} = \begin{cases} \xi^I_{\mu,\nu}, \ell = 0, \\ \eta^I_{\mu,\nu}, \ell = 1. \end{cases}$

Let $\varphi^{\Delta,I}_{m;j,k} =: \varphi^I_{m;j,k}\mathcal{X}_{[0,1]^3}$, which is the tensor product of corresponding interpolatory scaling functions on the interval. The corresponding duals are given similarly. Furthermore, define

$$\overrightarrow{\varphi}^{\,\Delta}_{m,i;j,k} =: \varphi^{\Delta,I_0\backslash\{i\}}_{m;j,k}\delta_i,\ \overrightarrow{V}^{\Delta}_j =: \text{span}\left\{\overrightarrow{\varphi}^{\,\Delta}_{m,i;j,k} : m \in \{1,2\}^3, k \in \nabla_j, 1 \le i \le 3\right\}$$

and the projection operators:

$$\overrightarrow{\Lambda}^{\Delta}_j =: \Lambda^{\Delta,\{2,3\}}_j\delta_1 + \Lambda^{\Delta,\{1,3\}}_j\delta_2 + \Lambda^{\Delta,\{1,2\}}_j\delta_3,\ \overrightarrow{\Lambda}^{\Delta,*}_j =: \Lambda^{\Delta,\{1\}}_j\delta_1 + \Lambda^{\Delta,\{2\}}_j\delta_2 + \Lambda^{\Delta,\{3\}}_j\delta_3.$$

Lemma 2.1 [8]. For smooth functions $f : [0,1]^3 \to R$, $\frac{\partial}{\partial x_i}\Lambda^{\Delta,I}_j f = \Lambda^{\Delta,I\backslash\{i\}}_j\left(\frac{\partial f}{\partial x_i}\right), i \in I.$.

Proposition 2.1. For $\overrightarrow{f} \in \overrightarrow{C}(\text{curl}; [0,1]^3) =: \{\overrightarrow{v} \in (C([0,1]^3))^3 : \text{curl}\,\overrightarrow{v} \in (C([0,1]^3))^3\}$, there has $\text{curl}\left(\overrightarrow{\Lambda}^{\Delta}_j\overrightarrow{f}\right) = \overrightarrow{\Lambda}^{\Delta,*}_j(\text{curl}\,\overrightarrow{f})$.

Proof. Note that Lemma 2.1, then

$$\begin{aligned}
\overrightarrow{\Lambda}^{\Delta,*}_j(\text{curl}\,\overrightarrow{f}) &= \Lambda^{\Delta,\{1\}}_j\left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right)\delta_1 + \Lambda^{\Delta,\{2\}}_j\left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right)\delta_2 + \Lambda^{\Delta,\{3\}}_j\left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)\delta_3 \\
&= \left(\frac{\partial}{\partial x_2}\Lambda^{\Delta,\{1,2\}}_j f_3 - \frac{\partial}{\partial x_3}\Delta^{\Delta,\{1,3\}}_j f_2\right)\delta_1 + \left(\frac{\partial}{\partial x_3}\Lambda^{\Delta,\{2,3\}}_j f_1 - \frac{\partial}{\partial x_1}\Lambda^{\Delta,\{1,2\}}_j f_3\right)\delta_2 \\
&\quad + \left(\frac{\partial}{\partial x_1}\Lambda^{\Delta,\{1,3\}}_j f_2 - \frac{\partial}{\partial x_2}\Lambda^{\Delta,\{2,3\}}_j f_1\right)\delta_3,
\end{aligned}$$

which is $\text{curl}\left(\overrightarrow{\Lambda}^{\Delta}_j\overrightarrow{f}\right)$ by definition.

Proposition 2.1 is important, because it tells us that $\overrightarrow{\Lambda}^{\Delta}_j$ keeps curl-free property. In general, vector-valued wavelets and wavelet spaces are given, respectively, by

$$\overrightarrow{\psi}^{\,\Delta}_{e,m,i;j,k} =: \psi^{\Delta,I_0\backslash\{i\}}_{e,m;j,k}\delta_i, \overrightarrow{W}^{\Delta}_j =: \text{span}\left\{\overrightarrow{\psi}^{\,\Delta}_{e,m,i;j,k} : e \in E_3^*, m \in \{1,2\}^3, 1 \le i \le 3, k \in \nabla^0_j\right\}.$$

For $e \in E_3^*$ and $m \in \{1,2\}^3$, we define

$$\overrightarrow{\psi}^{\,\Delta,c}_{e,m;j,k} =: 2^{-j}\text{grad}\psi^{\Delta,I_0}_{e,m;j,k} = 2^{-j}\frac{\partial}{\partial x_1}\psi^{\Delta,I_0}_{e,m;j,k}\delta_1 + 2^{-j}\frac{\partial}{\partial x_2}\psi^{\Delta,I_0}_{e,m;j,k}\delta_2 + 2^{-j}\frac{\partial}{\partial x_3}\psi^{\Delta,I_0}_{e,m;j,k}\delta_3, k \in \nabla^1_j.$$

Clearly, $\text{curl}\,(\overrightarrow{\psi}^{\,\Delta,c}_{e,m;j,k}) = \overrightarrow{0}$.

To give a decomposition for $\overrightarrow{W}^{\Delta}_j$, take

$$\overrightarrow{\psi}^{\,\Delta,\text{non}}_{e,m,i;j,k} =: \psi^{\Delta,I_0\backslash\{i\}}_{e,m;j,k}\delta_i, k \in \nabla^2_j$$

for $i \in I_0\backslash\{i_e\}$. Here, we choose i_e such that $e_{i_e} = 1$.

Proposition 2.2. The vector-valued function system $\{\vec{\psi}^{\Delta,c}_{e,m;j,k_1}, \vec{\psi}^{\Delta,\text{non}}_{e,m,i;j,k_2}, e \in E_3^*, m \in \{1,2\}^3, i \neq i_e, k_1 \in \nabla_j^1, k_2 \in \nabla_j^2\}$ is complete in $\vec{W}_j^{\Delta}$.

Proof. It is sufficient to show the statement for $j = 0$. Let $\vec{f} \in \vec{W}_0^{\Delta}$ satisfy

$$\left(\vec{f}, \vec{\psi}^{\Delta,c}_{e,m;0,k}\right) = \left(\vec{f}, \vec{\psi}^{\Delta,\text{non}}_{e,m,i;0,k}\right) = 0$$

for all $e \in E_3^*, m \in \{1,2\}^3, k \in \nabla_j^1 \cup \nabla_j^2\}, 1 \le i \le 3 \textit{and } i \neq i_e$. Here, the inner product is in $L^2([0,1]^3)$. Without loss of generality, one assumes $i_e = 1$. Then, $\left(\vec{f}, \vec{\psi}^{\Delta,\text{non}}_{e,m,i;0,k}\right) = 0$ leads to

$$\left(f_2, \psi^{\Delta,I_0\backslash\{2\}}_{e,m;0,k}\right) = \left(f_3, \psi^{\Delta,I_0\backslash\{3\}}_{e,m;0,k}\right) = 0.$$

By the definition of ψ^I and differential relations (1.1), one knows

$$\left(f_2, \frac{\partial}{\partial x_2}\psi^{\Delta,I_0}_{e,m;0,k}\right) = \left(f_3, \frac{\partial}{\partial x_3}\psi^{\Delta,I_0}_{e,m;0,k}\right) = 0.$$

Moreover, $\left(\vec{f}, \vec{\psi}^{\Delta,c}_{e,m;0,k}\right) = 0$ reduces to $\left(f_1, \frac{\partial}{\partial x_1}\psi^{\Delta,I_0}_{e,m;0,k}\right) = 0$. Now, it follows that $\left(f_1, \psi^{\Delta,I_0\backslash\{1\}}_{e,m;0,k}\right) = 0$ from $i_e = 1$. Finally, $\left(\vec{f}, \vec{\psi}^{\Delta}_{e,m,i;0,k}\right) = 0$ and $\vec{f} = \vec{0}$ follows from the definition of $\vec{W}_0^{\Delta}$.

To give the bi-orthogonal decomposition, we define

$$\vec{\tilde{\psi}}^{\Delta,c}_{e,m;j,k} = \frac{1}{2}\tilde{\psi}^{\Delta,I_0\backslash\{i_e\}}_{e,m;j,k}\delta_{i_e}. \tag{2.1}$$

Assume $I = \{i, i_e, i'\}$, then

$$\text{curl } \vec{\psi}^{\Delta,\text{non}}_{e,m,i;j,k} = \text{curl } \psi^{\Delta,I_0\backslash\{i\}}_{e,m;j,k}\delta_i = 2^{j+1}\varepsilon_1\psi^{\Delta,I_0\backslash\{i,i_e\}}_{e,m;j,k}\delta_{i'} + \varepsilon_2\frac{\partial}{\partial x_{i'}}\psi^{\Delta,I_0\backslash\{i\}}_{e,m;j,k}\delta_{i_e}$$

with $|\varepsilon_1| = |\varepsilon_2| = 1$ and $\varepsilon_1\varepsilon_2 = -1$. Now, define

$$\vec{\tilde{\psi}}^{\Delta,\text{non}}_{e,m,i;j,k} =: \frac{1}{2^{j+1}}\text{curl } \varepsilon_1\tilde{\psi}^{\Delta,I_0\backslash\{i,i_e\}}_{e,m;j,k}\delta_{i'}. \tag{2.2}$$

Here, the derivatives are meant in the sense of distributions. Now, we state the main result:

Proposition 2.3. The set $\{\vec{\psi}^{\Delta,c}_{e,m;j,k_1}, \vec{\psi}^{\Delta,\text{non}}_{e,m,i;j,k_2}, e \in E_3^*, m \in \{1,2\}^3, i \neq i_e, j \ge j_0, k_1 \in \nabla_j^1, k_2 \in \nabla_j^2\}$ is a bi-orthogonal wavelet basis of $L^2([0,1]^3)^3$ with duals defined in (2.1) and (2.2).

Proof. According to Proposition 2.2, one only need show

(*i*) $\left\langle \vec{\psi}^{\Delta,\text{non}}_{e',m',i;j',k'}, \vec{\tilde{\psi}}^{\Delta,c}_{e,m;j,k}\right\rangle = 0;$

(*ii*) $\left\langle \vec{\psi}^{\Delta,c}_{e,m;j,k}, \vec{\tilde{\psi}}^{\Delta,\text{non}}_{e',m',i;j',k'}\right\rangle = 0;$

(*iii*) $\left\langle \vec{\psi}^{\Delta,c}_{e,m;j,k}, \vec{\tilde{\psi}}^{\Delta,c}_{e',m';j',k'}\right\rangle = \delta_{e,e'}\delta_{m,m'}\delta_{j,j'}\delta_{k,k'};$

$$(iv)\ \left\langle \overrightarrow{\psi}^{\Delta,\mathrm{non}}_{e_1,m_1,i_1;j_1,k_1}, \overrightarrow{\widetilde{\psi}}^{\Delta,\mathrm{non}}_{e_2,m_2,i_2;j_2,k_2} \right\rangle = \delta_{e_1,e_2}\delta_{m_1,m_2}\delta_{i_1,i_2}\delta_{j_1,j_2}\delta_{k_1,k_2}.$$

The identity (i) holds obviously for $i \neq i_e$. For $i = i_e$, since $i \neq i_{e'}$, then $i_e \neq i_{e'}$, which means $e \neq e'$. Finally, the result (i) follows from the bi-orthogonality of $\psi^{\Delta,I_0\setminus\{i\}}_{e',m';j',k'}$ and $\widetilde{\psi}^{\Delta,I_0\setminus\{i\}}_{e,m;j,k}$.

Note that $\left\langle \overrightarrow{f}, \mathrm{curl}\ \overrightarrow{g} \right\rangle = \left\langle \mathrm{curl}\ \overrightarrow{f}, \overrightarrow{g} \right\rangle$. Then (ii) follows from curl·grad = 0. Furthermore,

$$\left\langle \overrightarrow{\psi}^{\Delta,c}_{e,m;j,k}, \overrightarrow{\widetilde{\psi}}^{\Delta,c}_{e',m';j',k'} \right\rangle = \frac{1}{2}\times 2^{-j}\left\langle \frac{\partial}{\partial x_{i_{e'}}}\psi^{\Delta,I_0}_{e,m;j,k}, \widetilde{\psi}^{\Delta,I_0\setminus\{i_{e'}\}}_{e',m';j',k'} \right\rangle = -\frac{1}{2}\times 2^{-j}\left\langle \psi^{\Delta,I_0}_{e,m;j,k}, \frac{\partial}{\partial x_{i_{e'}}}\widetilde{\psi}^{\Delta,I_0\setminus\{i_{e'}\}}_{e',m';j',k'} \right\rangle.$$

Then by the fact $\frac{d}{dx}\widetilde{\eta}^{-}_{m} = -2\widetilde{\eta}^{+}_{m}$ and the bi-orthogonality of $\psi^{\Delta,I_0}_{e,m}$, $\widetilde{\psi}^{\Delta,I_0}_{e,m}$, one obtains

$$\left\langle \overrightarrow{\psi}^{\Delta,c}_{e,m;j,k}, \overrightarrow{\widetilde{\psi}}^{\Delta,c}_{e',m';j',k'} \right\rangle = 2^{j'-j}\left\langle \psi^{\Delta,I_0}_{e,m;j,k}, \widetilde{\psi}^{\Delta,I_0}_{e',m';j',k'} \right\rangle = \delta_{e,e'}\delta_{m,m'}\delta_{j,j'}\delta_{k,k'}.$$

Now, it remains to prove (iv), which is equivalent to

$$\begin{aligned} 2^{j_1-j_2}\left\langle 2\varepsilon_1\psi^{\Delta,I_0\setminus\{i_1,i_{e_1}\}}_{e_1,m_1;j_1,k_1}\delta_{i'_1} + 2^{-j_1}\varepsilon_2\frac{\partial}{\partial x_{i'_1}}\psi^{\Delta,I_0\setminus\{i_1\}}_{e_1,m_1;j_1,k_1}\delta_{i_{e_1}}, \frac{1}{2}\varepsilon_1\widetilde{\psi}^{\Delta,I_0\setminus\{i_2,i_{e_2}\}}_{e_2,m_2;j_2,k_2}\delta_{i'_2} \right\rangle \\ = \delta_{e_1,e_2}\delta_{m_1,m_2}\delta_{i_1,i_2}\delta_{j_1,j_2}\delta_{k_1,k_2}. \end{aligned} \tag{2.3}$$

It is easily proved, when $e_1 = e_2$: In fact, since $i_{e_1} = i_{e_2}$, one can assume $i_1 = i_2$ and $i'_1 = i'_2$, because $i_1 = i'_2$ leads to (2.3) obviously. In that case, the left-hand side of (2.3) reduces to $2\varepsilon_1 \cdot \frac{\varepsilon_1}{2} \cdot 2^{j_1-j_2}\left\langle \psi^{I\setminus\{i_1,i_{e_1}\}}_{e_1,m_1;j_1,k_1}, \widetilde{\psi}^{I\setminus\{i_2,i_{e_2}\}}_{e_2,m_2;j_2,k_2} \right\rangle = \delta_{m_1,m_2}\delta_{j_1,j_2}\delta_{k_1,k_2}$, which is the desired. To the end, it is sufficient to prove that for $e_1 \neq e_2$, that is

$$\left\langle 2\varepsilon_1\psi^{\Delta,I_0\setminus\{i_1,i_{e_1}\}}_{e_1,m_1;j_1,k_1}\delta_{i'_1} + 2^{-j_1}\varepsilon_2\frac{\partial}{\partial x_{i'_1}}\psi^{\Delta,I_0\setminus\{i_1\}}_{e_1,m_1;j_1,k_1}\delta_{i_{e_1}}, \frac{1}{2}\varepsilon_1\widetilde{\psi}^{\Delta,I_0\setminus\{i_2,i_{e_2}\}}_{e_2,m_2;j_2,k_2}\delta_{i'_2} \right\rangle = 0. \tag{2.4}$$

Note that $i'_2 \in \{i_1, i'_1, i_{e_1}\}$. Then the conclusion is obvious when $i'_2 = i_1$. When $i'_2 = i'_1$, then $\{i_1, i_{e_1}\} = \{i_2, i_{e_2}\}$ and the left-hand side of (2.4) reduces to $2\varepsilon_1 \cdot \frac{\varepsilon_1}{2}\left\langle \psi^{\Delta,I_0\setminus\{i_1,i_{e_1}\}}_{e_1,m_1;j_1,k_1}, \widetilde{\psi}^{\Delta,I_0\setminus\{i_2,i_{e_2}\}}_{e_2,m_2;j_2,k_2} \right\rangle = 0$. Hence one only need to show (2.4), when $i'_2 = i_{e_1}$. However, (2.4) becomes

$$\left\langle \frac{\partial}{\partial x_{i'_1}}\psi^{\Delta,I_0\setminus\{i_1\}}_{e_1,m_1;j_1,k_1}, \widetilde{\psi}^{\Delta,I_0\setminus\{i_2,i_{e_2}\}}_{e_2,m_2;j_2,k_2} \right\rangle = 0. \tag{2.5}$$

in that case. Since $\{i_1, i'_1, i_{e_1}\} = \{i_2, i'_2, i_{e_2}\} = I$, two cases should be considered: $i_2 = i_1, i'_2 = i_{e_1}, i_{e_2} = i'_1$ and $i_2 = i'_1, i'_2 = i_{e_1}, i_{e_2} = i_1$. Using $\frac{d}{dx}\widetilde{\eta}^{-}_{m} = -2\widetilde{\eta}^{+}_{m}$, the left-hand side of (2.5) is

$$-\left\langle \psi^{\Delta,I_0\setminus\{i_1\}}_{e_1,m_1,j_1,k_1}, \frac{\partial}{\partial x_{i'_1}}\widetilde{\psi}^{\Delta,I_0\setminus\{i_2,i_{e_2}\}}_{e_2,m_2,j_2,k_2} \right\rangle = 2^{j_2+1}\left\langle \psi^{\Delta,I_0\setminus\{i_1\}}_{e_1,m_1,j_1,k_1}, \widetilde{\psi}^{\Delta,I_0\setminus\{i_2\}}_{e_2,m_2;j_2,k_2} \right\rangle = 0$$

in the first case; In the second one, the left-hand side of (2.5) becomes $\left\langle \frac{\partial}{\partial x_{i_2}}\psi^{\Delta,I_0\setminus\{i_1\}}_{e_1,m_1;j_1,k_1} \widetilde{\psi}^{\Delta,I_0\setminus\{i_2,i_2\}}_{e_2,m_2;j_2,k_2} \right\rangle$. According to the differential relation (1.1), $\frac{\partial}{\partial x_{i_2}}\psi^{\Delta,I_0\setminus\{i_1\}}_{e_1,m_1;j_1,k_1}$ is a linear combination of $\psi_{e_1,m_1;j_1,k_1}^{\Delta,I_0\backslash\{i_1,i_2\}}$. By the bi-orthogonality of $\psi_{e_1,m_1;j_1,k_1}^{\Delta,I_0\backslash\{i_1,i_2\}}$ and $\widetilde{\psi}_{e_2,m_2;j_2,k_2}^{\Delta,I_0\backslash\{i_1,i_2\}}$, one receives the desired conclusion.

3 Characterization for Besov spaces

We shall characterize a class of vector-valued Besov spaces in this section. For $0 < p, q \le \infty$ and $s > 0$, the Besov space $B_q^s(L^p(\Omega))$ is the set of all $f \in L^p(\Omega)$ such that

$$|f|_{B_q^s(L^p(\Omega))} =: \left\|\{2^{sj}\omega_m(f, 2^{-j}, L^p(\Omega))\}\right\|_{\ell^q} < +\infty$$

with $m = [s] + 1$ and $\omega_m(f, 2^{-j}, L^p(\Omega))$ the classical m-order modulus of smoothness. The corresponding norm is defined by

$$\|f\|_{B_q^s(L^p(\Omega))} =: \|f\|_{L^p(\Omega)} + |f|_{B_q^s(L^p(\Omega))}.$$

Our Besov space is defined as

$$\widehat{B}_q^s(L^p([0,1]^3)) =: \{\overrightarrow{f} \in (B_q^s(L^p([0,1]^3)))^3 : \frac{\partial}{\partial x_j} f_i \in B_q^s(L^p([0,1]^3)), i = 1, 2, 3; j \neq i\}$$

with the norm

$$\left\|\overrightarrow{f}\right\|_{\widehat{B}_q^s(L^p([0,1]^3))} =: \sum_{i=1}^{3} \|f_i\|_{B_q^s(L^p([0,1]^3))} + \sum_{i=1}^{3} \sum_{\substack{1\le j\le 3\\ j\neq i}} \left\|\frac{\partial f_i}{\partial x_j}\right\|_{B_q^s(L^p([0,1]^3))}.$$

Clearly, $\operatorname{curl} \overrightarrow{f} \in (B_q^s(L^p([0,1]^3)))^3$, when $\overrightarrow{f} \in \widehat{B}_q^s(L^p([0,1]^3))$.

The following lemma is easily proved by the definition of modulus of smoothness:

Lemma 3.1. If $f(x), g(x) \in B_q^s(L^p(R))$, then $f(x_1)g(x_2) \in B_q^s(L^p(R^2))$.

For $\alpha = (\alpha_j)_{j\ge j_0}$ and $\alpha_j = (\alpha_{j,k})_k$, define

$$\|\alpha\|_{\ell_{p,q}^s} =: \left\|\left(2^{j(s-\frac{n}{p})}\|\alpha_j\|_{\ell^p}\right)\right\|_{\ell^q}.$$

Lemma 3.2 [8]. If $\phi \in B_\infty^\sigma(L^p(R^n))$ is compactly supported, $0 < p, q \le \infty$ and $0 < s < \sigma$, then

$$\left\|\sum_{k\in\nabla_j} \beta_k \phi(2^j \cdot - k)\right\|_{B_q^s(L^p([0,1]^n))} \lesssim 2^{(s-\frac{n}{p})j}\|\beta\|_{\ell^p},$$

$$\left\|\sum_{j\ge j_0}\sum_{k\in\nabla_j} \alpha_{j,k}\phi(2^j \cdot - k)\right\|_{B_q^s(L^p([0,1]^n))} \lesssim \|\alpha\|_{\ell_{p,q}^s},$$

where $\nabla_j =: \{k : \operatorname{supp}\varphi(2^j \cdot - k) \subseteq [0,1]^n\}$.

Theorem 3.1. Let $\overrightarrow{\varphi}_{m,i;j,k}^{\Delta}$, $\overrightarrow{\psi}_{e,m;j,k}^{\Delta,c}$, and $\overrightarrow{\psi}_{e,m,i;j,k}^{\Delta,\mathrm{non}}$ be defined in Section 2. If $0 < s < 1 + \frac{1}{p}$ and $0 < p, q \le \infty$, then one has

$$\left\| \sum_{m,i} \sum_{k\in\nabla_{j_0}} \beta_{m,i,k} \overrightarrow{\varphi}^{\Delta}_{m,i;j_0,k} + \sum_{j\geq j_0} \sum_{e,m,i\neq i_e} \sum_{k\in\nabla_j^1} \alpha^c_{e,m;j,k} \overrightarrow{\psi}^{\Delta,c}_{e,m;j,k} + \sum_{j\geq j_0} \sum_{e,m} \sum_{k\in\nabla_j^2} \alpha^{\text{non}}_{e,m,i;j,k} \overrightarrow{\psi}^{non}_{e,m,i;j,k} \right\|_{\widehat{B}^s_q(L^p([0,1]^3))}$$
$$\lesssim \sum_{m,i} \|\beta_{m,i}\|_{\ell^p} + \sum_{e,m} \left(\sum_{i\neq i_e} \|\alpha^{\text{non}}_{e,m,i}\|_{\ell^{s+1}_{p,q}} + \|\alpha^c_{e,m}\|_{\ell^{s+1}_{p,q}} \right).$$

Proof. It is enough to prove the following inequality:

(i) $\left\| \sum_{k\in\nabla_{j_0}} \beta_{m,i,k} \overrightarrow{\varphi}^{\Delta}_{m,i;j_0,k} \right\|_{\widehat{B}^s_q(L^p([0,1]^3))} \lesssim \|\beta_{m,i}\|_{\ell p};$

(ii) $\left\| \sum_{j\geq j_0} \sum_{k\in\nabla_j^1} \alpha^c_{e,m;j,k} \overrightarrow{\psi}^{\Delta,c}_{e,m;j,k} \right\|_{\widehat{B}^s_q(L^p([0,1]^3)} \lesssim \|\alpha^c_{e,m}\|_{\ell^{s+1}_{p,q}};$

(iii) $\left\| \sum_{j\geq j_0} \sum_{k\in\nabla_j^2} \alpha^{\text{non}}_{e,m,i;j,k} \overrightarrow{\psi}^{non}_{e,m,i;j,k} \right\|_{\widehat{B}^s_q(L^p([0,1]^3)} \lesssim \|\alpha^{\text{non}}_{e,m,i}\|_{\ell^{s+1}_{p,q}} (i \neq i_e).$

Let $\overrightarrow{h} =: \sum_{k\in\nabla_{j_0}} \beta_{m,i,k} \overrightarrow{\varphi}^{\Delta}_{m,i;j_0,k}$ and h_v be the vth component of $\overrightarrow{h}$. Then, for $\mu \neq i$,

$$h_i = \sum_{k\in\nabla_{j_0}} \beta_{m,i;k} \varphi_m^{I_0\setminus\{i\}}(2^{j_0}x - k),\ \frac{\partial}{\partial x_\mu} h_i = \sum_{k\in\nabla_{j_0}} \beta_{m,i;k} 2^{j_0} \left(\frac{\partial}{\partial x_\mu} \varphi_m^{I_0\setminus\{i\}} \right) (2^{j_0}x - k).$$

Since $\xi_m^+ \in B_\infty^{2+\frac{1}{p}}(L^p(R)) \subseteq B_\infty^{1+\frac{1}{p}}(L^p(R))$ and $\xi_m^- \in B_\infty^{1+\frac{1}{p}}(L^p(R))$, then both $\varphi_m^{I\setminus\{i\}}$ and $\frac{\partial}{\partial x_\mu}\varphi_m^{I\setminus\{i\}}$ are in the Besov space $B_\infty^{1+\frac{1}{p}}(L^p(R^3))$, due to Lemma 3.1. Moreover, Lemma 3.2 implies $\|h_i\|_{B^s_q(L^p([0,1]^3))} \lesssim \|\beta_{m,i}\|_{\ell^p}$ and $\left\| \frac{\partial}{\partial x_\mu} h_i \right\|_{B^s_q(L^p([0,1]^3))} \lesssim \|\beta_{m,i}\|_{\ell^p}$. Note that $h_v = 0$ for $v \neq i$. Finally, the first inequality follows from the definition.

Let $\overrightarrow{g} =: \sum_{j\geq j_0} \sum_{k\in\nabla_j^1} \alpha^c_{e,m;j,k} \overrightarrow{\psi}^{\Delta,c}_{e,m;j,k}$ and g_v be the vth component of $\overrightarrow{g}$ $(1 \leq v \leq 3)$. Then

$$g_v = \sum_{j\geq j_0} \sum_{k\in\nabla_j^1} \alpha^c_{e,m;j,k} \left(\frac{\partial}{\partial x_v} \psi^{I_0}_{e,m} \right) (2^j x - k),$$
$$\frac{\partial}{\partial x_\mu} g_v = \sum_{j\geq j_0} \sum_{k\in\nabla_j^1} 2^j \alpha^c_{e,m;j,k} \left(\frac{\partial^2}{\partial x_\mu \partial x_v} \psi^{I_0}_{e,m} \right) (2^j x - k).$$

Similar to the above, $\frac{\partial}{\partial x_v} \psi^{I_0}_{e,m}, \frac{\partial^2}{\partial x_v \partial x_\mu} \psi^{I_0}_{e,m} \in B_\infty^{1+\frac{1}{p}}(L^p(R^3))$. According to Lemma 3.2,

$$\|g_v\|_{B^s_q(L^p([0,1]^3))} \lesssim \|\alpha_{e,m}\|_{\ell^s_{p,q}} \lesssim \|\alpha_{e,m}\|_{\ell^{s+1}_{p,q}} \quad \text{and} \quad \left\| \frac{\partial}{\partial x_\mu} g_v \right\|_{B^s_q(L^p([0,1]^3))} \lesssim \|\alpha_{e,m}\|_{\ell^{s+1}_{p,q}}.$$

Finally, one receives the second inequality and the last one follows analogously.

Let $W_\tau^\mu(D)$ denotes the Sobolev space with regularity exponent μ and domain D. Moreover,

$$E_d\left(f, W_\tau^\mu(D)\right) =: \inf_{P\in\prod_{d-1}} \left\|f-P\right\|_{W_\tau^\mu(D)}.$$

Furthermore, let $\sigma_{j,k} = 2^{-j}([0,1]^3 + k)$ for $k \in (Z_j^2)^3$, the boundary cases are: when there is only one $k_i = 2^j (1 \le i \le 3)$, $\sigma_{j,k}$ is defined as replacing the ith position of $2^{-j}([k_1, k_1 + 1] \otimes [k_2, k_2 + 1] \otimes [k_3, k_3 + 1])$ by $[2^{j-1}, 2^j]$; when $k_i = k_{i'} = 2^i$ for $i, i' \in \{1, 2, 3\}$, both the positions i and i' are replaced by $[2^{j-1}, 2^j]$; finally, $\sigma_{j,(2^j,2^j,2^j)} =: 2^{-j}([2^{j-1},2^j]\otimes[2^{j-1},2^j]\otimes[2^{j-1},2^j])$.

Lemma 3.3 [8]. Let $\frac{n}{p} - \frac{n}{\tau} + \mu < s < d, s > 0, \mu \in N_0, 0 < p, q \le \infty, 1 \le \tau \le \infty, j_0 \in N_0$.

Then

$$\left\| \left(2^{j(s-\frac{n}{p}+\frac{n}{\tau}-\mu)} \left\| \left(E_d\left(f, W_\tau^\mu\left(\sigma_{j,k}\right)\right)\right)_{k\in Z^n} \right\|_{\ell^p} \right)_{j\ge j_0} \right\|_{\ell^q} \lesssim |f|_{B_q^s(L^p([0,1]^n))}.$$

The following lemma can be easily proved, but it is important for proving Theorem 3.2:

Lemma 3.4. The following relations hold:

$$\left\langle f, \tilde{\eta}_{1;j,k}^{\Delta,+} \right\rangle = f\left(2^{-j}\left(k+\frac{1}{2}\right)\right) - \frac{1}{2}f(2^{-j}k) - \frac{1}{2}(2^{-j}(k+1)) - \frac{1}{8}\cdot 2^{-j}f'(2^{-j}k) + \frac{1}{8}\cdot 2^{-j}f'(2^{-j}(k+1));$$

$$\left\langle f, \tilde{\eta}_{2;j,k}^{\Delta,+} \right\rangle = 2^{-(j+1)}f'\left(2^{-j}\left(k+\frac{1}{2}\right)\right) - \frac{3}{4}f(2^{-j}(k+1)) + \frac{3}{4}f(2^{-j}k) + \frac{1}{4}\cdot 2^{-(j+1)}[f'(2^{-j}k) + f'(2^{-j}(k+1))];$$

$$\left\langle f, \tilde{\eta}_{1;j,k}^{\Delta,-} \right\rangle = 2^j \int_{2^{-j}k}^{2^{-j}(k+\frac{1}{2})} f(t)dt - 2^j \int_{2^{-j}(k+\frac{1}{2})}^{2^{-j}(k+1)} f(t)dt - \frac{1}{4}f(2^{-j}k) + \frac{1}{4}f(2^{-j}(k+1));$$

$$\left\langle f, \tilde{\eta}_{2;j,k}^{\Delta,-} \right\rangle = -\frac{3}{2}\cdot 2^j \int_{2^{-j}k}^{2^{-j}(k+1)} f(t)dt + \frac{1}{4}f(2^{-j}k) + \frac{1}{4}f(2^{-j}(k+1)) + f(2^{-j}(k+\frac{1}{2})).$$

Theorem 3.2. Let $1 + \frac{3}{p} < s < 3$ and $0 < p, q \le \infty$. Then $\beta_{m,i} = \left(\left\langle \overrightarrow{f}, \overrightarrow{\tilde{\varphi}}_{m,i;j_0,k}^{\Delta} \right\rangle\right)_{k\in\nabla_{j_0}}$, $\alpha_{e,m}^c = \left(\left\langle \overrightarrow{f}, \overrightarrow{\tilde{\psi}}_{e,m;j,k}^{\Delta,c} \right\rangle\right)_{j\ge j_0, k\in\nabla_j^1}$, $\alpha_{e,m,i}^{\text{non}} = \left(\left\langle \overrightarrow{f}, \overrightarrow{\tilde{\psi}}_{e,m,i;j,k}^{\Delta,\text{non}} \right\rangle\right)_{j\ge j_0, k\in\nabla_j^2}$, $j_0 \ge 1, i \ne i_e$ satisfy

$$\sum_{m,i} \left\|\beta_{m,i}\right\|_{\ell^p} + \sum_{e,m} \left(\sum_{i\ne i_e} \left\|\alpha_{e,m,i}^{\text{non}}\right\|_{\ell_{p,q}^{s+1}} + \left\|\alpha_{e,m}^c\right\|_{\ell_{p,q}^{s+1}} \right) \lesssim \left\|\overrightarrow{f}\right\|_{\widehat{B}_q^s(L^p([0,1]^3))}.$$

Proof. One only need to show the following inequality:

(i) $\left\|\beta_{m,i}\right\|_{\ell^p} \lesssim \left\|\overrightarrow{f}\right\|_{\widehat{B}_q^s(L^p([0,1]^3))}$;

(ii) $\left\|\alpha_{e,m,i}^{\text{non}}\right\|_{\ell_{p,q}^{s+1}} \lesssim \left\|\overrightarrow{f}\right\|_{\widehat{B}_q^s(L^p([0,1]^3))}$;

(iii) $\left\|\alpha_{e,m}^c\right\|_{\ell_{p,q}^{s+1}} \lesssim \left\|\overrightarrow{f}\right\|_{\widehat{B}_q^s(L^p([0,1]^3))}$.

Note that $\beta_{m,i;k} = \left\langle \overrightarrow{f}, \overrightarrow{\tilde{\varphi}}^{\Delta}_{m,i;j_0,k} \right\rangle = \left\langle f_i, \tilde{\varphi}^{\Delta, I_0 \backslash \{i\}}_{m;j_0,k} \right\rangle$. Then one assumes $i = 1$ without loss of generality and proves first

$$\|\beta_{m,1}\|_{\ell^p} = \left\| \left(\left\langle f_1, \tilde{\varphi}^{\Delta,\{2,3\}}_{m;j_0,k} \right\rangle \right)_{k \in \nabla'_{j_0}} \right\|_{\ell^p} \lesssim \left\| \overrightarrow{f} \right\|_{\widehat{B}^s_q(L^p([0,1]^3))}.$$

By the embedding property, $B^s_q(L^p(D)) \subseteq W^{\mu}_{\tau}(D)$ for $s > \dfrac{1}{p} - \dfrac{1}{\tau} + \mu$, and

$$\|f\|_{W^{\mu}_{\tau}(D)} \lesssim \|f\|_{B^s_q(L^p(D))}. \tag{3.1}$$

When $m_2 = m_3 = 1$, $\left| \left\langle f_1, \tilde{\varphi}^{\Delta,\{2,3\}}_{m;j_0,k} \right\rangle \right| \le \|f_1\|_{L^\infty(\sigma_{j_0,k-\delta_1})}$ for $m_1 = 1$ or $\|f_1\|_{L^\infty(\sigma_{j_0,k})}$ for $m_1 = 2$.

Using (3.1), one obtains

$$\|\beta_{m,1}\|_{\ell^p} \le \begin{cases} \left(\sum\limits_{k \in \nabla'_{j_0}} \|f_1\|^p_{L^\infty(\sigma_{j_0,k-\delta_1})} \right)^{\frac{1}{p}}, m_1 = 1 \\ \left(\sum\limits_{k \in \nabla'_{j_0}} \|f_1\|^p_{L^\infty(\sigma_{j_0,k})} \right)^{\frac{1}{p}}, m_1 = 2 \end{cases} \lesssim \|f_1\|_{B^s_q(L^p([0,1]^3))} \lesssim \left\| \overrightarrow{f} \right\|_{\widehat{B}^s_q(L^p([0,1]))}.$$

When $m_3 = 2$ (similarly for $m_2 = 2$), we obtain

$$\left| \left\langle f_1, \tilde{\varphi}^{\Delta,\{2,3\}}_{m;j_0,k} \right\rangle \right| = 2^{-j_0} \left| \left\langle \frac{\partial f_1}{\partial x_3}, \tilde{\varphi}^{\Delta,\{2\}}_{m;j_0,k} \right\rangle \right| \le \begin{cases} 2^{-j_0} \left\| \dfrac{\partial f_1}{\partial x_3} \right\|_{W^1_\infty(\sigma_{j_0,k-\delta_1})}, m_1 = 1 \\ 2^{-j_0} \left\| \dfrac{\partial f_1}{\partial x_3} \right\|_{W^1_\infty(\sigma_{j_0,k})}, m_1 = 2. \end{cases}$$

Note that $1 + \dfrac{1}{p} < s < 3$, then the same arguments as above lead to (i).

For $h \in B^s_q(L^p([0,1]^3))$, we first claim that there are only the following two cases:

(a) $\left| \left\langle h, \tilde{\psi}^{\Delta,\{i'\}}_{e,m;j,k} \right\rangle \right| \lesssim 2^{-j} E_3(h, W^1_\infty(\sigma_{j,k}))$ or $2^{-j} E_3(h, W^1_\infty(\sigma_{j,k-\delta_i}))$;

(b) $\left| \left\langle h, \tilde{\psi}^{\Delta,\{i'\}}_{e,m;j,k} \right\rangle \right| \lesssim E_3(h, L^\infty(\sigma_{j,k}))$ or $E_3(h, L^\infty(\sigma_{j,k-\delta_i}))$.

In fact, by the vanishing moment property of the dual wavelets, that is,

$$\tilde{\eta}^{+}_{m;j,k}(P_1) = 0, P_1 \in \prod\nolimits_3; \tilde{\eta}^{-}_{m;j,k}(P_2) = 0, P_2 \in \prod\nolimits_2.$$

Then $\left\langle h, \tilde{\psi}^{\Delta,\{i'\}}_{e,m;j,k} \right\rangle = \left\langle h - P, \tilde{\psi}^{\Delta,\{i'\}}_{e,m;j,k} \right\rangle$ for each $P \in \Pi_2$. Hence, if $e_{i'} = 1$ or $e_{i'} = 0$ but $m_{i'} = 2$, the differential relation (1.1) implies that

$$\left| \left\langle h, \tilde{\psi}^{\Delta,\{i'\}}_{e,m;j,k} \right\rangle \right| \lesssim 2^{-j} \left| \left\langle \frac{\partial}{\partial x_{i'}}(h - P), \tilde{\psi}^{\Delta,\emptyset}_{e,m;j,k} \right\rangle \right| \lesssim \begin{cases} 2^{-j} \|h - P\|_{W^1_\infty(\sigma_{j,k-\delta_i})}, & e_i = 0, m_i = 1; \\ 2^{-j} \|h - P\|_{W^1_\infty(\sigma_{j,k})}, & o.w. \end{cases}$$

Moreover, the (a) part follows from the definition of $E_3(h, W^1_\infty(D))$; If $e_{i'} = 0$ and $m_{i'} = 1$,

$$\left|\left\langle h, \widetilde{\psi}^{\Delta,\{i'\}}_{e,m;j,k}\right\rangle\right| = \left|\left\langle h-P, \widetilde{\psi}^{\Delta,\{i'\}}_{e,m;j,k}\right\rangle\right| \lesssim \begin{cases} \|h-P\|_{L^\infty(\sigma_{j,k-\delta_i})}, & e_i=0, m_i=1; \\ \|h-P\|_{L^\infty(\sigma_{j,k})}, & o.w. \end{cases}$$

and the (b) part follows.

Now, one is ready to estimate $\alpha^{\text{non}}_{e,m,i;j,k}$ and $\alpha^{c}_{e,m;j,k}$. By the definition of $\overrightarrow{\widetilde{\psi}}^{\Delta,\text{non}}_{e,m,i}$, one knows

$$\left|\alpha^{\text{non}}_{e,m,i;j,k}\right| =: \left|\left\langle \overrightarrow{f}, \overrightarrow{\widetilde{\psi}}^{\Delta,\text{non}}_{e,m,i;j,k}\right\rangle\right| = 2^{-j-1}\left|\left\langle \operatorname{curl}\overrightarrow{f}, \widetilde{\psi}^{\Delta,I_0\backslash\{i,i_e\}}_{e,m;j,k}\delta_{i'}\right\rangle\right|$$
$$= 2^{-j-1}\left|\left\langle \frac{\partial f_i}{\partial x_{i_e}} - \frac{\partial f_{i_e}}{\partial x_i}, \widetilde{\psi}^{\Delta,\{i'\}}_{e,m;j,k}\right\rangle\right| \le 2^{-j-1}\left(\left|\left\langle \frac{\partial f_i}{\partial x_{i_e}}, \widetilde{\psi}^{\Delta,\{i'\}}_{e,m;j,k}\right\rangle\right| + \left|\left\langle \frac{\partial f_{i_e}}{\partial x_i}, \widetilde{\psi}^{\Delta,\{i'\}}_{e,m;j,k}\right\rangle\right|\right).$$

Define $\beta^{\text{non}}_{e,m,i;j,k} =: 2^{-j-1}\left|\left\langle \frac{\partial f_i}{\partial x_{i_e}}, \widetilde{\psi}^{\Delta,\{i'\}}_{e,m;j,k}\right\rangle\right|$ and $\gamma^{\text{non}}_{e,m,i;j,k} =: 2^{-j-1}\left|\left\langle \frac{\partial f_{i_e}}{\partial x_i}, \widetilde{\psi}^{\Delta,\{i'\}}_{e,m;j,k}\right\rangle\right|$. Then, it is sufficient to show

$$\|\beta^{\text{non}}_{e,m,i}\|_{\ell^{s+1}_{p,q}} \lesssim \left\|\overrightarrow{f}\right\|_{\widehat{B}^s_q(L^p([0,1]^3))} \quad \text{and} \quad \|\gamma^{\text{non}}_{e,m,i}\|_{\ell^{s+1}_{p,q}} \lesssim \left\|\overrightarrow{f}\right\|_{\widehat{B}^s_q(L^p([0,1]^3))}.$$

Let $h =: \frac{\partial f_i}{\partial x_{i_e}}$ in our claim. Then $\beta^{\text{non}}_{e,m,i;j,k} \lesssim 2^{-2j}E_3(h, W^1_\infty(\sigma_{j,k})), 2^{-2j}E_3(h, W^1_\infty(\sigma_{j,k-\delta_i}))$ or $\beta^{\text{non}}_{e,m,i;j,k} \lesssim 2^{-j}E_3(h, L^\infty(\sigma_{j,k})), 2^{-j}E_3(h, L^\infty(\sigma_{j,k-\delta_i}))$. By $\|\alpha\|_{\ell^{s+1}_{p,q}} =: \left\|2^{j(s+1-\frac{n}{p})}\right\|\alpha_j\|_{\ell^p})\|_{\ell^q}$ and Lemma 3.3, one receives that

$$\|\beta^{\text{non}}_{e,m,i}\|_{\ell^{s+1}_{p,q}} \lesssim \left|\frac{\partial f_i}{\partial x_{i_e}}\right|_{B^s_q(L^p([0,1]^3))} \lesssim \left\|\overrightarrow{f}\right\|_{\widehat{B}^s_q(L^p([0,1]^3))}.$$

Similarly, $\|\gamma^{\text{non}}_{e,m,i}\|_{\ell^{s+1}_{p,q}} \lesssim \left\|\overrightarrow{f}\right\|_{\widehat{B}^s_q(L^p([0,1]^3))}$ holds and $\|\alpha^{\text{non}}_{e,m,i}\|_{\ell^{s+1}_{p,q}} \lesssim \left\|\overrightarrow{f}\right\|_{\widehat{B}^s_q(L^p([0,1]^3))}$.

Finally, to estimate $\alpha^{c}_{e,m;j,k} =: \left\langle \overrightarrow{f}, \overrightarrow{\widetilde{\psi}}^{\Delta,c}_{e,m;j,k}\right\rangle = \frac{1}{2}\left\langle f_{i_e}, \widetilde{\psi}^{\Delta,I_0\backslash\{i_e\}}_{e,m;j,k}\right\rangle$, one assumes without loss of generality that $i_e = 1$ and $\alpha^{c}_{e,m;j,k} = \frac{1}{2}\left\langle f_1, \widetilde{\psi}^{\Delta,\{2,3\}}_{e,m;j,k}\right\rangle$. Note that $\frac{d}{dx}\widetilde{\eta}^-_m = -2\widetilde{\eta}^+_m$ and $\frac{d}{dx}\widetilde{\xi}^-_2 = -\widetilde{\xi}^+_2$. Then $\left|\alpha^{c}_{e,m;j,k}\right| \lesssim 2^{-j}\left|\left\langle \frac{\partial f_1}{\partial x_i}, \widetilde{\psi}^{\Delta,\{l\}}_{e,m;j,k}\right\rangle\right|$ with $i, l \in \{2, 3\}$ and $i \ne l$, when $e_2 = 1$ or $e_3 = 1$ or $m_2 = 2$ or $m_3 = 2$. Similar to the last case, one obtains

$$\left|\alpha^{c}_{e,m;j,k}\right| \lesssim 2^{-2j}E_3\left(\frac{\partial f_1}{\partial x_i}, W^1_\infty(\sigma_{j,k})\right) \quad or \quad 2^{-j}E_3\left(\frac{\partial f_1}{\partial x_i}, L^\infty(\sigma_{j,k})\right)$$

and (iii) is proved in these cases. Now it remains to show (iii), when $e_2 = e_3 = 0$ and $m_2 = m_3 = 1$:

For each $P \in \Pi_2$, let $g(x_1,\cdot, x_3)$ be a primitive of $P(x_1,\cdot,x_3)$, i.e.

$$g(x_1, x_2, x_3) =: \int P(x_1, x_2, x_3)dx_2,$$

moreover, $g\big|_{x_2=2^{-j}(k_2-1)} =: f_1\big|_{x_2=2^{-j}(k_2-1)}$ if $k_2 = 1,2,\ldots,2^j$ and $g\big|_{x_2=2^{-j}} =: f_1\big|_{x_2=2^{-j}}$ if $k_2 = 0$. Since $e_1 = 1$ and $\widetilde{\eta}^-_m$ has vanishing moments of order 2, then we obtain

$$\left|\left\langle f_1 - g, \widetilde{\psi}^{\Delta,\{2,3\}}_{e,m;j,k}\right\rangle\right| = \begin{cases} \left|-2^{-j}\left\langle (f_1 - g), \dfrac{\partial}{\partial x_2}\widetilde{\psi}^{\Delta,\{3\}}_{e,m;j,k}\right\rangle + \left\langle f_1 - g, \widetilde{\psi}^{\Delta,\{2,3\}}_{e,m;j,k-\delta_2}\right\rangle\right|, & k_2 = 1, 2, \ldots, 2^j; \\ \left|2^{-j}\left\langle (f_1 - g), \dfrac{\partial}{\partial x_2}\widetilde{\psi}^{\Delta,\{3\}}_{e,m;j,k+\delta_2}\right\rangle + \left\langle f_1 - g, \widetilde{\psi}^{\Delta,\{2,3\}}_{e,m;j,k+\delta_2}\right\rangle\right|, & k_2 = 0 \end{cases}$$
$$= \begin{cases} \left|-2^{-j}\left\langle (f_1 - g), \dfrac{\partial}{\partial x_2}\widetilde{\psi}^{\Delta,\{3\}}_{e,m;j,k}\right\rangle\right|, & k_2 = 1, 2, \ldots, 2^j; \\ \left|2^{-j}\left\langle (f_1 - g), \dfrac{\partial}{\partial x_2}\widetilde{\psi}^{\Delta,\{3\}}_{e,m;j,k+\delta_2}\right\rangle\right|, & k_2 = 0 \end{cases}$$
$$\lesssim \begin{cases} 2^{-j}\left\|\dfrac{\partial f_1}{\partial x_2} - \dfrac{\partial g}{\partial x_2}\right\|_{L^\infty(\sigma_{j,k-\delta_2})}, & k_2 = 1, 2, \ldots, 2^j; \\ 2^{-j}\left\|\dfrac{\partial f_1}{\partial x_2} - \dfrac{\partial g}{\partial x_2}\right\|_{L^\infty(\sigma_{j,k})}, & k_2 = 0 \end{cases}$$
$$= \begin{cases} 2^{-j}\left\|\dfrac{\partial f_1}{\partial x_2} - P\right\|_{L^\infty(\sigma_{j,k-\delta_2})}, & k_2 = 1, 2, \ldots, 2^j; \\ 2^{-j}\left\|\dfrac{\partial f_1}{\partial x_2} - P\right\|_{L^\infty(\sigma_{j,k})}, & k_2 = 0. \end{cases}$$

Therefore, we have

$$\left|\left\langle f_1, \widetilde{\psi}^{\Delta,\{2,3\}}_{e,m;j,k}\right\rangle\right| = \left|\left\langle f_1 - g, \widetilde{\psi}^{\Delta,\{2,3\}}_{e,m;j,k}\right\rangle\right| \lesssim \begin{cases} 2^{-j}E_3\left(\dfrac{\partial f_1}{\partial x_2}, L^\infty(\sigma_{j,k-\delta_2})\right), & k_2 = 1, 2, \ldots, 2^j; \\ 2^{-j}E_3\left(\dfrac{\partial f_1}{\partial x_2}, L^\infty(\sigma_{j,k})\right), & k_2 = 0. \end{cases}$$

The desired result follows from Lemma 3.3.

It should be pointed out that there is no common range for s in Theorems 3.1 and 3.2. Indeed, this is a big shortcoming. However, we need only one estimate in many cases.

4 The stability of curl-free wavelet bases

In this part, we shall prove that the single-scale wavelet bases that we have constructed in Section 2 are stable. The following lemma is the classical result of functional analysis:

Lemma 4.1. Let X be a Banach space and $x_1, x_2, \ldots, x_n \subseteq X$ be linearly independent. Then there exists a constant $C > 0$ such that for any scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, one has

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n\| \geq C\left(|\alpha_1| + |\alpha_2| + \ldots + |\alpha_n|\right).$$

Lemma 4.2 [8]. Let X be a Banach space and $f_{i1}, f_{i2}, \ldots, in_i \subseteq X$ be linearly independent for each $i = 1, 2, \ldots, m$, then the tensor products $\{f_{1j_1}(x_1)f_{2j_2}(x_2)\ldots f_{mj_m}(x_m)\}j_i \in \{1, 2, \ldots, n_i\}, i = 1, 2, \ldots, m$ are also linearly independent.

Theorem 4.1. The function system $\left\{2^{\frac{3j}{2}}\overrightarrow{\psi}^{\Delta,c}_{e,m;j,k'}, 2^{\frac{3j}{2}}\overrightarrow{\psi}^{\Delta,\mathrm{non}}_{e,m,i;j,k'}, e \in E_3^*, m \in \{1,2\}^3, k \in \nabla_j^1, k' \in \nabla_j^2, i \neq i_e\right\}$ generates a Riesz basis for $\overrightarrow{W}_j^\Delta$ with Riesz bounds independent of j.

Proof. By Proposition 2.2, one need only show the stability of the function system. Let

$$\overrightarrow{\omega} =: \sum_{e,m,k} d^c_{e,m;j,k}\overrightarrow{\psi}^{\Delta,c}_{e,m;j,k} + \sum_{e,m,k}\sum_{i\neq i_e} d^{\mathrm{non}}_{e,m,i;j,k}\overrightarrow{\psi}^{\Delta,\mathrm{non}}_{e,m,i;j,k} \in \overrightarrow{W}_j^\Delta.$$

Then $\|\overrightarrow{\omega}\|_{L^2([0,1]^3)^3} \le \|\overrightarrow{\omega}\|_{H^s([0,1]^3)^3}$ for $s > 0$. Since $H^s([0,1]^3) = B_2^s(L^2([0,1]^3))$, then $\|\overrightarrow{\omega}\|_{L^2([0,1]^3)^3} \le \|\overrightarrow{\omega}\|_{\widehat{B}_2^s(L^2([0,1]^3)^3)}$. Moreover, one receives

$$\|\overrightarrow{\omega}\|_{L^2([0,1]^3)^3} \lesssim \left(\sum_{e,m,k}\sum_{i\neq i_e}\left|d^{\text{non}}_{e,m,i;j,k}\right|^2 + \sum_{e,m,k}\left|d^c_{e,m;j,k}\right|^2\right)^{\frac{1}{2}},$$

due to Theorem 3.1. Now, it remains to prove the lower bound. Let $\sigma_{j,k} =: 2^{-j}([0,1]^3 + k), k \in (Z_j^2)^3$, then $\bigcup_{k\in(Z_j^2)^3}\sigma_{j,k} = [0,1]^3$. We take an example for $e = (0,0,1)$ and $m = (2,1,1)$,

$$\overrightarrow{\psi}^{\Delta,c}_{e,m;j,k} = \left(\xi^{\Delta,-}_{2;j,k_1}\xi^{\Delta,+}_{1;j,k_2}\eta^{\Delta,+}_{1;j,k_3}, \xi^{\Delta,+}_{2;j,k_1}\left(\xi^{\Delta,-}_{1;j,k_2} - \xi^{\Delta,-}_{1;j,k_2+1}\right)\eta^{\Delta,+}_{1;j,k_3}, \xi^{\Delta,+}_{2;j,k_1}\xi^{\Delta,+}_{1;j,k_2}\eta^{\Delta,-}_{1;j,k_3}\right),$$
$$\overrightarrow{\psi}^{\Delta,\text{non}}_{e,m,1;j,k} = \left(\xi^{\Delta,-}_{2;j,k_1}\xi^{\Delta,+}_{1;j,k_2}\eta^{\Delta,+}_{1;j,k_3}, 0, 0\right), \overrightarrow{\psi}^{\Delta,\text{non}}_{e,m,2;j,k} = \left(0, \xi^{\Delta,+}_{2;j,k_1}\xi^{\Delta,-}_{1;j,k_2}\eta^{\Delta,+}_{1;j,k_3}, 0\right).$$

For each fixed $k \in (Z_j^2)^3$, by the characteristics of supports, Lemma 4.1 and 4.2, one has

$$\begin{aligned}\int_{\sigma_{j,k}}|\overrightarrow{\omega}|^2dx = \int_{\sigma_{j,k}}\Bigg|&d^c_{e,m;j,k}\overrightarrow{\psi}^{\Delta,c}_{e,m;j,k} + \sum_{i=1}^{2}d^c_{e,m;j,k+\delta_i}\overrightarrow{\psi}^{\Delta,c}_{e,m;j,k} + d^c_{e,m;j,k+\delta_1+\delta_2}\overrightarrow{\psi}^{\Delta,c}_{e,m;j,k+\delta_1+\delta_2}\\
&+d^{\text{non}}_{e,m,1;j,k}\overrightarrow{\psi}^{\Delta,\text{non}}_{e,m,1;j,k} + d^{\text{non}}_{e,m,1;j+\delta_1}\overrightarrow{\psi}^{\Delta,\text{non}}_{e,m,1;j,k+\delta_1} + \sum_{\mu=1}^{2}\Big(d^{\text{non}}_{e,m,\mu;j,k+\delta_2}\overrightarrow{\psi}^{\Delta,\text{non}}_{e,m,\mu;j,k+\delta_2}+\\
&d^{\text{non}}_{e,m,\mu;j,k+\delta_1+\delta_2}\overrightarrow{\psi}^{\Delta,\text{non}}_{e,m,\mu;j,k+\delta_1+\delta_2}\Big) + \sum_{e\neq(0,0,1),m\neq(2,1,1),i\neq i_e,k'}\left(d^c_{e,m;j,k'}\overrightarrow{\psi}^{\Delta,c}_{e,m;j,k'} + d^{\text{non}}_{e,m,i;j,k'}\overrightarrow{\psi}^{\Delta,\text{non}}_{e,m,i;j,k'}\right)\Bigg|^2dx\\
\geq C\Bigg(&\left|d^c_{e,m;j,k}\right|^2 + \sum_{i=1}^{2}\left|d^c_{e,m;j,k+\delta_i}\right|^2 + \left|d^c_{e,m;j,k+\delta_1+\delta_2}\right|^2 + \left|d^{\text{non}}_{e,m,1;j,k}\right|^2 + \left|d^{\text{non}}_{e,m,1;j,k+\delta_1}\right|^2\\
&+\sum_{\mu=1}^{2}\left(\left|d^{\text{non}}_{e,m,\mu;j,k+\delta_2}\right|^2 + \left|d^{\text{non}}_{e,m,\mu;j,k+\delta_1+\delta_2}\right|^2\right) + \sum_{e\neq(0,0,1),m\neq(2,1,1),i\neq i_e,k'}\left(\left|d^c_{e,m;j,k'}\right|^2 + \left|d^{\text{non}}_{e,m,i;j,k'}\right|^2\right)\Bigg).\end{aligned}$$

Finally, the lower estimation follows from

$$\|\overrightarrow{\omega}\|^2_{L^2([0,1]^3)^3} = \sum_{k\in(Z_j^2)^3}\int_{\sigma_{j,k}}|\overrightarrow{\omega}|^2dx \gtrsim C\left(\sum_{e,m,k}\sum_{i\neq i_e}\left|d^{\text{non}}_{e,m,i;j,k}\right|^2 + \sum_{e,m,k}\left|d^c_{e,m;j,k}\right|^2\right).$$

Corollary 4.2. The system $\left\{2^{\frac{3j}{2}}\overrightarrow{\psi}^{\Delta,c}_{e,m;j,k}, e \in E_3^*, m \in \{1,2\}^3, k \in \nabla_j^1\right\}$ is a Riesz basis for $\overrightarrow{W}_j^{\Delta,c} =: \text{span}\left\{2^{\frac{3j}{2}}\overrightarrow{\psi}^{\Delta,c}_{e,m;j,k}, e \in E_3^*, m \in \{1,2\}^3, k \in \nabla_j^1\right\}$ with bounds independent of j.

Proof. Note that $\overrightarrow{\omega} = \sum_{e,m,k}d^c_{e,m;j,k}\overrightarrow{\psi}^{\Delta,c}_{e,m;j,k} + \sum_{e,m,k}\sum_{i\neq i_e}d^{\text{non}}_{e,m,i;j,k}\overrightarrow{\psi}^{\Delta,\text{non}}_{e,m,i;j,k} \in \overrightarrow{W}_j^{\Delta}$ and curl·grad = 0. Then the desired result follows from the fact that $\overrightarrow{\omega}$ is curl-free if and only if for all $d^{\text{non}}_{e,m,i;j,k} = 0$.

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Competing interests
The authors declare that they have no competing interests.

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