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The twisted *(h, q)*-Genocchi numbers and polynomials with weight α and q-bernstein polynomials with weight α

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Abstract

In this article, we give some identities on the twisted (h, q)-Genocchi numbers and polynomials and q-Bernstein polynomials with weighted α.

Keywords: Genocchi numbers and polynomials, twisted (h, q)-Genocchi numbers and polynomials with weight α, q-Bernstein polynomials

1 Introduction

Let p be a fixed odd prime number. The symbol, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. As well known definition, the p-adic absolute value is given by $|x|_p = p^{-r}$, where $x = p^r \frac{t}{s}$ with $(t, p) = (s, p) = (t, s) = 1$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. In this article, we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$.

For $f \in UD(\mathbb{Z}_p) = \{f/f : \mathbb{Z}_p \to \mathbb{C}_p$ is uniformly differentiable function$\}$, Kim defined the fermionic p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x. \tag{1.1}$$

For $n \in \mathbb{N}$, let $f_n(x) = f(x + n)$ be translation. As well known equation, by (1.1), we have

$$q^n I_{-q}(f_n) = (-1)^n I_{-q}(f) + [2]_q \sum_{l-0}^{n-1} (-1)^{n-1-l} q^l f(l),. \tag{1.2}$$

Throughout this article we use the notation:

$$[x]_q = \frac{1-q^x}{1-q}.$$

$\lim_{q\to 1}|x|_q = x$ for any x with $|x|_p \leq 1$ in the present p-adic case. To investigate relation of the twisted (h, q)-Genocchi numbers and polynomials with weight α and the q-Bernstein polynomials with weight α, we will use useful property for $[x]_{q^\alpha}$ as following;

$$\begin{aligned} [x]_{q^\alpha} &= 1 - [1-x]_{q^{-\alpha}} \\ [1-x]_{q^{-\alpha}} &= 1 - [x]_{q^\alpha} \end{aligned} \tag{1.3}$$

The twisted (h, q)-Genocchi numbers and polynomials with weight α are defined by the generating function, respectively:

$$G_{n,q,w}^{(h,\alpha)} = n \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [x]_{q^\alpha}^{n-1} d\mu_{-q}(x). \tag{1.4}$$

$$G_{n,q,w}^{(h,\alpha)}(x) = n \int_{\mathbb{Z}_p} q^{y(h-1)} \phi_w(y) [y+x]_{q^\alpha}^{n-1} d\mu_{-q}(y). \tag{1.5}$$

In the special case, $x = 0$, $G_{n,q,w}^{(h,\alpha)}(0) = G_{n,q,w}^{(h,\alpha)}$ are called the nth twisted (h, q)-Genocchi numbers with weight α (see [1]).

Let $C_{p^n} = \{w | w^{p^n} = 1\}$ be the cyclic group of order p^n and let

$$T_p = \lim_{n\to\infty} C_{p^n} = \cup_{n\geq 1} C_{p^n}$$

see [1-5].

Kim defined the q-Bernstein polynomials with weight α of degree n as follows:

$$B_{k,n}^{(\alpha)}(x, q) = \binom{n}{k} [x]_{q^\alpha}^k [1-x]_{q^{-\alpha}}^{n-k}, \text{ where } x \in [0, 1], n, k \in \mathbb{Z}_+ \tag{1.6}$$

cf [6-12].

In this article, we investigate some properties for the twisted (h, q)-Genocchi numbers and polynomials with weight α. By using these properties, we give some interesting identities on the twisted (h, q)-Genocchi polynomials with weight α and q-Bernstein polynomials with weight α.

2 Twisted (*h, q*)-genocchi numbrs and polynomials with weight α and *q*-bernstein polynomials with weight α

From (1.2), we can get the following form for the twisted (h, q)-Genocchi numbers with weight α:

$$G_{0,q,w}^{(h,\alpha)} = 0, \text{ and } q^h w G_{n,q,w}^{(h,\alpha)}(1) + G_{n,q,w}^{(h,\alpha)} = \begin{cases} [2]_q, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \tag{2.1}$$

$$G_{0,q,w}^{(h,\alpha)} = 0, \text{ and } q^h w (1 + q^\alpha G_{q,w}^{(h,\alpha)})^n + q^\alpha G_{n,q,w}^{(h,\alpha)} = \begin{cases} q^\alpha [2]_q, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \tag{2.2}$$

$$q^{\alpha x} G_{n+1,q,w}^{(h,\alpha)}(x) = \left([x]_q^\alpha + q^{\alpha x} G_{q,w}^{(h,\alpha)} \right)^{n+1} \tag{2.3}$$

with usual convention about replacing $(G_{q,w}^{(h,\alpha)})^n$ by $G_{n,q,w}^{(h,\alpha)}$.

By (1.4), we can obtain

$$G_{n,q,w}^{(h,\alpha)}(x) = n[2]_q\left(\frac{1}{1-q^{\alpha}}\right)^{n-1}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l\frac{1}{1+wq^{\alpha l+h}} \tag{2.4}$$

By (2.4), we can get

$$\begin{aligned}
G_{n,q^{-1},w^{-1}}^{(h,\alpha)}(1-x) &= n[2]_{q^{-1}}\left(\frac{1}{1-q^{-\alpha}}\right)^{n-1}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l(q^{-1})^{\alpha l(1-x)}\frac{1}{1+w^{-1}(q^{-1})^{\alpha l+h}}\\
&= n\frac{1}{q}[2]_q\left(\frac{1}{1-q^{\alpha}}\right)^{n-1}(-1)^{n-1}q^{\alpha n+\alpha}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l q^{\alpha lx}\frac{wq^h}{1+wq^{\alpha l+h}}\\
&= n[2]_q\left(\frac{1}{1-q^{\alpha}}\right)^{n-1}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l q^{\alpha lx}\frac{1}{1+wq^{\alpha l+h}}\frac{1}{q}q^{\alpha n-\alpha}(-1)^{n-1}wq^h\\
&= (-1)^{n-1}wq^{\alpha(n-1)+(h-1)}G_{n,q,w}^{(h,\alpha)}(x).
\end{aligned}$$

So, we get the following theorem.

Theorem 1. Let $n \in \mathbb{Z}_+$. For $w \in T_p$, we have

$$G_{n,q,w}^{(h,\alpha)}(x) = (-1)^{n-1}w^{-1}q^{\alpha(1-n)+(1-h)}G_{n,q^{-1},w^{-1}}^{(h,\alpha)}(1-x).$$

By (2.1), (2.2), and (2.3), we note that

$$\begin{aligned}
G_{n,q,w}^{(h,\alpha)} &= -wq^h G_{n,q,w}^{(h,\alpha)}(1)\\
&= -wq^h(q^{-\alpha}(1+g^{\alpha}G_{q,w}^{(h,\alpha)})^n)\\
&= -wq^{h-\alpha}\sum_{l=0}^{n}\binom{n}{l}(q^{\alpha})^l G_{l,q,w}^{(h,\alpha)}\\
&= -wq^h\binom{n}{1}G_{1,q,w}^{(h,\alpha)} - wq^{h-\alpha}\sum_{l=2}^{n}\binom{n}{l}q^{\alpha l}\left(-wq^h G_{l,q,w}^{(h,\alpha)}(1)\right)\\
&= -wq^h\binom{n}{1}G_{1,q,w}^{(h,\alpha)} - wq^{h-\alpha}\sum_{l=2}^{n}\binom{n}{l}q^{\alpha l}(-wq^h q^{-\alpha}(1+q^{\alpha}G_{q,w}^{(h,\alpha)})^l)\\
&= -nwq^h G_{1,q,w}^{(h,\alpha)} + w^2q^{2h-2\alpha}\sum_{l=2}^{n}\binom{n}{l}q^{\alpha l}(1+q^{\alpha}G_{q,w}^{(h,\alpha)})^l\\
&= -nwq^h G_{1,q,w}^{(h,\alpha)} + w^2q^{2h-2\alpha}(1+q^{\alpha}(1+g^{\alpha}G_{q,w}^{(h,\alpha)}))^n - nw^2q^{2h-2\alpha}q^{\alpha}(1+q^{\alpha}G_{q,w}^{(h,\alpha)})^1\\
&= -nwq^h G_{1,q,w}^{(h,\alpha)} + w^2q^{2h-2\alpha}([2]_{q^{\alpha}}+q^{2\alpha}G_{q,w}^{(h,\alpha)})^n - nw^2q^{2h-\alpha}(q^{\alpha}G_{1,q,w}^{(h,\alpha)})\\
&= -nwq^h G_{1,q,w}^{(h,\alpha)} + w^2q^{2h-2\alpha}([2]_{q^{\alpha}}+q^{2\alpha}G_{q,w}^{(h,\alpha)})^n - nw^2q^{2h}G_{1,q,w}^{(h,\alpha)}\\
&= -nwq^h G_{1,q,w}^{(h,\alpha)} + w^2q^{2h}G_{n,q,w}^{(h,\alpha)}(2) - nw^2q^{2h}G_{1,q,w}^{(h,\alpha)}
\end{aligned} \tag{2.5}$$

Therefore, by (2.5), we obtain the theorem below.

Theorem 2. For $n \in \mathbb{N}$ with $n > 1$, we have

$$G_{n,q,w}^{(h,\alpha)}(2) = w^{-2}q^{-2h}G_{n,q,w}^{(h,\alpha)} + w^{-1}q^{-h}\frac{n[2]_q}{1+wq^h} + \frac{n[2]_q}{1+wq^h}.$$

From Theorem 2,

$$\frac{G^{(h,\alpha)}_{n+1,q,w}(2)}{n+1} = \frac{1}{n+1}\left(\frac{(n+1)[2]_q}{1+wq^h} + \frac{(n+1)w^{-1}q^{-h}[2]_q}{1+wq^h}\right) + w^{-2}q^{-2h}\frac{G^{(h,\alpha)}_{n+1,q,w}}{n+1}$$
$$= \frac{[2]_q}{1+wq^h} + w^{-1}q^{-h}\frac{[2]_q}{1+wq^h} + w^{-2}q^{-2h}\frac{G^{(h,\alpha)}_{n+1,q,w}}{n+1}$$

Therefore, we obtain the Corollary 3 by (1.5) and Theorem 2.

Corollary 3. For $n \in \mathbb{N}$, we have

$$\int_{\mathbb{Z}_p} q^{y(h-1)}\phi_w(y)[y+2]^n_{q^\alpha} q\mu_{-q}(y) = \frac{[2]_q}{1+wq^h} + w^{-1}q^{-h}\frac{[2]_q}{1+wq^h} + w^{-2}q^{-2h}\frac{G^{(h,\alpha)}_{n+1,q,w}}{n+1}$$

By Theorems 1, 2 and fermionic integral on $\mathbb{Z}_p$, we note that

$$\begin{aligned}\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)[1-x]^n_{q^{-\alpha}} q\mu_{-q}(x) &= (-1)^n q^{\alpha n}\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)[x-1]^n_{q^\alpha} d\mu_{-q}(x)\\ &= (-1)^n q^{\alpha n}\frac{G^{(h,\alpha)}_{n+1,q,w}(-1)}{n+1}\\ &= w^{-1}q^{1-h}\frac{G^{(h,\alpha)}_{n+1,q^{-1},w^{-1}}(2)}{n+1}\\ &= w^{-1}q^{1-h}\left(\frac{[2]_{q^{-1}}}{1+q^{-h}w^{-1}} + wq^h\frac{[2]_{q^{-1}}}{1+q^{-1}w^{-1}} + w^2q^{2h}\frac{G^{(h,\alpha)}_{n+1,q^{-1},w^{-1}}}{n+1}\right)\\ &= \frac{[2]_q}{1+wq^h} + wq^h\frac{[2]_q}{1+wq^h} + wq^{h+1}\frac{G^{(h,\alpha)}_{n+1,q^{-1},w^{-1}}}{n+1}\\ &= [2]_q + wq^{h+1}\frac{G^{(h,\alpha)}_{n+1,q^{-1},w^{-1}}}{n+1}.\end{aligned} \tag{2.6}$$

Hence, we get the following theorem.

Theorem 4. for $n \in \mathbb{N}$ with $n > 1$, we have

$$\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)[1-x]^n_{q^{-\alpha}} d\mu_{-q}(x) = [2]_q + wq^{h+1}\frac{G^{(h,\alpha)}_{n+1,q^{-1},w^{-1}}}{n+1}. \tag{2.7}$$

Corollary 5.

From (1.3) and Theorem 4, we take the fermionic p-adic invariant integral on $\mathbb{Z}_p$ for q-Bernstein polynomials as follows:

$$\begin{aligned}\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)B_{k,n}(x,q)d\mu_{-q}(x) &= \int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)\binom{n}{k}[x]^k_{q^\alpha}[1-x]^{n-k}_{q-\alpha}d\mu_{-q}(x)\\ &= \binom{n}{k}\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)[x]^k_{q^\alpha}(1-[x]_{q^\alpha})^{n-k}d\mu_{-q}(x)\\ &= \binom{n}{k}\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)[x]^k_{q^\alpha}(1-[x]_{q^\alpha})^{n-k}d\mu_{-q}(x)\\ &= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n}{n-k}(-1)^l\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)[x]^{k+l}_{q^\alpha}d\mu_{-q}(x)\\ &= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\frac{G^{(h,\alpha)}_{k+l+1,q,w}}{k+l+1}\end{aligned} \tag{2.8}$$

And we get the following formula;

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)B_{k,n}(x,q)d\mu_{-q}(x) \\
&= \int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)\binom{n}{k}[x]_{q^\alpha}^{n-k}[1-x]_{q^{-\alpha}}^{k}d\mu_{-q}(x) \\
&= \int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)\binom{n}{k}[1-x]_{q^{-\alpha}}^{k}(1-[1-x]_{q^{-\alpha}})^{n-k}d\mu_{-q}(x) \\
&= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^{n-k-l}[1-x]_{q^{-\alpha}}^{n-k-l}\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)[1-x]_{q^{-\alpha}}^{k}d\mu_{-q}(x) \\
&= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^{n-k-l}\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)[1-x]_{q^{-\alpha}}^{n-l}d\mu_{-q}(x) \\
&= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^{n-k-l}\left([2]_q + wq^{1+h}\frac{G_{n-l+1,q^{-1},w^{-1}}^{(h,\alpha)}}{n-l+1}\right)
\end{aligned} \tag{2.9}$$

Hence, we can get the following theorem by (2.8) and (2.9).

Theorem 5. for $n \in \mathbb{N}$ with $n > 1$, we have

$$\begin{aligned}
&\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^{l}\frac{G_{k+l+1,q,w}^{(h,\alpha)}}{k+l+1} \\
&= \sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^{n-k-l}\left([2] + wq^{1+h}\frac{G_{n-l+1,q^{-1},w^{-1}}^{(h,\alpha)}}{n-l+1}\right) \\
&= \sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^{n-k-l}\left(w^{-1}q^{1-h}\frac{G_{n-l+1,q^{-1},w^{-1}}^{(h,\alpha)}(2)}{n-l+1}\right)
\end{aligned} \tag{2.10}$$

Also, we can see that

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)B_{k,n}(x,q)d\mu_{-q}(x) \\
&= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^{l}\frac{G_{k+l+1,q,w}^{(h,\alpha)}}{k+l+1} \\
&= \binom{n}{k}\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)[1-x]_{q^{-\alpha}}^{n-k}[x]_{q^\alpha}^{k}d\mu_{-q}(x) \\
&= \binom{n}{k}\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)[1-x]_{q^{-\alpha}}^{n-k}\left(1-[1-x]_{q^{-\alpha}}\right)^{k}d\mu_{-q}(x) \\
&= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)[1-x]_{q^{-\alpha}}^{n-l}d\mu_{-q}(x) \\
&= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\left([2]_q + wq^{1+h}\frac{G_{n-l+1,q^{-1},w^{-1}}^{(h,\alpha)}}{n-l+1}\right).
\end{aligned} \tag{2.11}$$

Therefore, we have the theorem below.

Theorem 6. For $n, k \in \mathbb{Z}_+$ with $n > k + 1$, we have

$$\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)B_{k,n}(x,q)d\mu_{-q}(x) = \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\left([2]_q + wq^{1+h}\frac{G^{(h,\alpha)}_{n-l+1,q^{-1},w^{-1}}}{n-l+1}\right). \tag{2.12}$$

By (2.7) and Theorem 6, we can get the theorem below.

Theorem 7. Let $n, k \in \mathbb{Z}_+$ with $n > k + 1$. Then we have

$$\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^{l}\frac{G^{(h,\alpha)}_{k+l+1,q,w}}{k+l+1} = \sum_{l=0}^{k}\binom{k}{l}(-1)^{k-1}\left([2]_q + wq^{1+h}\frac{G^{(h,\alpha)}_{n-l+1,q^{-1},w^{-1}}}{n-l+1}\right).$$

Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k + 1$. Then we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)B^{(\alpha)}_{k,n_1}(x,q)B^{(\alpha)}_{k,n_2}(x,q)d\mu_{-q}(x)\\
&= \int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)\binom{n_1}{k}[x]^k_{q^\alpha}[1-x]^{n_1-k}_{q^{-\alpha}}\binom{n_2}{k}[x]^k_{q^\alpha}[1-x]^{n_2-k}_{q^{-\alpha}}d\mu_{-q}(x)\\
&= \binom{n_1}{k}\binom{n_2}{k}\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)[x]^{2k}_{q^\alpha}[1-x]^{n_1+n_2-k}_{q^{-\alpha}}d\mu_{-q}(x)\\
&= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{2k-l}\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)[1-x]^{n_1+n_2-l}_{q^{-\alpha}}d\mu_{-q}(x)\\
&= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{2k-l}\left([2]_q + wq^{1+h}\frac{G^{(h,\alpha)}_{n_1+n_2-l+1,q^{-1},w^{-1}}}{n_1+n_2-l+1}\right).
\end{aligned}$$

Therefore, we obtain the theorem below.

Theorem 8. For $n_1, n_2, k \in \mathbb{Z}_+$, we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)B^{(\alpha)}_{k,n_1}(x,q)B^{(\alpha)}_{k,n_2}(x,q)d\mu_{-q}(x)\\
&= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{2k-l}\left([2]_q + wq^{1+h}\frac{G^{(h,\alpha)}_{n_1+n_2-l+1,q^{-1},w^{-1}}}{n_1+n_2-l+1}\right)\\
&= \begin{cases}\left([2]_q + wq^{1+h}\frac{G^{(h,\alpha)}_{n_1+n_2-l+1,q^{-1},w^{-1}}}{n_1+n_2-l+1}\right), & \text{if } k=0,\\ wq^{1+h}\binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{2k-l}\frac{G^{(h,\alpha)}_{n_1+n_2-l+1,q^{-1},w^{-1}}}{n_1+n_2-l+1}, & \text{if } k>0,\end{cases}
\end{aligned} \tag{2.13}$$

And we can easily have that

$$
\begin{aligned}
&\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_\omega(x)B_{k,n_1}^{(\alpha)}(x,q)B_{k,n_2}^{(\alpha)}(x,q)d\mu_{-q}(x)\\
&=\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_\omega(x)\binom{n_1}{k}[x]_{q^\alpha}^{k}[1-x]_{q^{-\alpha}}^{n_1-k}\binom{n_2}{k}[x]_{q^\alpha}^{k}[1-x]_{q^{-\alpha}}^{n_2-k}d\mu_{-q}(x)\\
&=\binom{n_1}{k}\binom{n_2}{k}\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)[x]_{q^\alpha}^{2k}[1-x]_{q^{-\alpha}}^{n_1+n_2-k}d\mu_{-q}(x)\\
&=\binom{n_1}{k}\binom{n_2}{k}\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)[x]_{q^\alpha}^{2k}(1-[x]_{q^\alpha})^{n_1+n_2-2k}d\mu_{-q}(x)\\
&=\binom{n_1}{k}\binom{n_2}{k}\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)[x]_{q^\alpha}^{2k}\sum_{l=0}^{n_1+n_2-2k}\binom{n_1+n_2-2k}{l}(-1)^l[x]_{q^\alpha}^{l}d\mu_{-q}(x)\\
&=\binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{n_1+n_2-2k}(-1)^l\binom{n_1+n_2-2k}{l}\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)[x]_{q^\alpha}^{2k+1}d\mu_{-q}(x)\\
&=\binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{n_1+n_2-2k}(-1)^l\binom{n_1+n_2-2k}{l}\frac{G_{2k+l+1,q,w}^{(h,\alpha)}}{2k+l+1},\text{ where } n_1,n_2,k\in\mathbb{Z}_+.
\end{aligned}
\tag{2.14}
$$

Therefore, by (2.14) and Theorem 8, we obtain the theorem below.

Theorem 9. Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k + 1$. Then we have

$$
\begin{aligned}
&\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{2k-l}\left([2]_q+wq^{1+h}\frac{G_{n_1+n_2-l+1,q^{-1},w^{-1}}^{(h,\alpha)}}{n_1+n_2-l+1}\right)\\
&=\sum_{l=0}^{n_1+n_2-2k}(-1)^l\binom{n_1+n_2-2k}{l}\frac{G_{2k+l+1,q,w}^{(h,\alpha)}}{2k+l+1}.
\end{aligned}
$$

For $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+, n_1 + n_2 + \cdots + n_s > sk + 1$, then by the symmetry of q-Bernstein polynomials with weight α, we see that

$$
\begin{aligned}
&\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)\prod_{i=1}^{s}B_{k,n_i}^{(\alpha)}(x,q)d\mu_{-q}(x)\\
&=\prod_{i=1}^{s}\binom{n_i}{k}\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)[x]_{q^\alpha}^{sk}[1-x]_{q^{-\alpha}}^{n_1+n_2+\cdots+n_s-sk}d\mu_{-q}(x)\\
&=\prod_{i=1}^{s}\binom{n_i}{k}\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)(1-[1-x]_{q^{-\alpha}})^{sk}[1-x]_{q^{-\alpha}}^{n_1+n_2+\cdots+n_s-sk}d\mu_{-q}(x)\\
&=\prod_{i=1}^{s}\binom{n_i}{k}\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{sk-l}\int_{\mathbb{Z}_p} q^{x(h-1)}\phi_w(x)[1-x]_{q^{-\alpha}}^{n_1+n_2+\cdots+n_s-l}d\mu_{-q}(x)\\
&=\prod_{i=1}^{s}\binom{n_i}{k}\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{sk-l}\left([2]_q+wq^{1+h}\frac{G_{n_1+n_2+\cdots+n_s-l+1,q^{-1},w^{-1}}^{(h,\alpha)}}{n_1+n_2+\cdots+n_s-l+1}\right).
\end{aligned}
$$

Therefore, we have the theorem below.

Theorem 10. For $n_1, n_2, n_3, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + n_2 + \ldots + n_s > sk + 1$, we have

$$\int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) \prod_{i=1}^{s} B_{k,n_i}^{(\alpha)}(x,q) d\mu_{-q}(x)$$
$$= \prod_{i=1}^{s} \binom{n_i}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk-l} \left([2]_q + wq^{1+h} \frac{G_{n_1+n_2+\cdots+n_s-l+1,q^{-1},w^{-1}}^{(h,\alpha)}}{n_1+n_2+\cdots+n_s-l+1} \right).$$

In the same manner as in (2.11), we can get the following relation:

$$\int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) \prod_{i=1}^{s} B_{k,n_i}^{(\alpha)}(x,q) d\mu_{-q}(x)$$
$$= \prod_{i=1}^{s} \binom{n_i}{k} \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [x]_{q^\alpha}^{sk} (1-[x]_{q^\alpha})^{n_1+n_2+\cdots+n_s-sk} d\mu_{-q}(x)$$
$$= \prod_{i=1}^{s} \binom{n_i}{k} \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [x]_{q^\alpha}^{sk} \sum_{l=0}^{n_1+n_2+\cdots+n_s-sk} (-1)^l \binom{n_1+n_2+\cdots+n_s-sk}{l} (-1)^l [x]_{q^\alpha}^l d\mu_{-q}(x)$$
$$= \prod_{i=1}^{s} \binom{n_i}{k} \sum_{l=0}^{n_1+n_2+\cdots+n_s-sk} (-1)^l \binom{n_1+n_2+\cdots+n_s-sk}{l} \int_{\mathbb{Z}_p} q^{x(h-1)} \phi_w(x) [x]_{q^\alpha}^{sk+l} d\mu_{-q}(x)$$
$$= \prod_{i=1}^{s} \binom{n_i}{k} \sum_{l=0}^{n_1+n_2+\cdots+n_s-sk} (-1)^l \binom{n_1+n_2+\cdots+n_s-sk}{l} \frac{G_{sk+l+1,q,w}^{(h,\alpha)}}{sk+l+1},$$

where $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + n_2 + \ldots + n_s > sk + 1$.

By Theorem 11 and (2.9), we have the following corollary.

Corollary 11. Let $m \in \mathbb{N}$. For $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \ldots + n_s > sk + 1$, we have

$$\sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk-l} \left([2]_q + wq^{1+h} \frac{G_{n_1+n_2+\cdots+n_s-l+1,q^{-1},w^{-1}}^{(h,\alpha)}}{n_1+n_2+\cdots+n_s-l+1} \right)$$
$$= \sum_{l=0}^{n_1+n_2+\cdots+n_s-sk} (-1)^l \binom{n_1+n_2+\cdots+n_s-sk}{l} \frac{G_{sk+l+1,q,w}^{(h,\alpha)}}{sk+l+1},$$

Authors' contributions

All authors contributed equally to the manuscript and read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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