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Optimal generalized Heronian mean bounds for the logarithmic mean

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Abstract

In this article, we establish a double inequality between the generalized Heronian and logarithmic means. The achieved result is inspired by the articles of Lin and Shi et al., and the methods from Janous. The inequalities we obtained improve the existing corresponding results and, in some sense, are optimal.
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Keywords: generalized Heronian mean, logarithmic mean, power mean

1 Introduction

The logarithmic mean $L(a, b)$ of two positive numbers a and b is defined by

$$L(a,\ b) = \begin{cases} \frac{a-b}{\log a - \log b}, & a \neq b, \\ a, & a = b. \end{cases} \tag{1.1}$$

Recently, the logarithmic mean has been the subject of intensive research. In particular, many remarkable inequalities for the logarithmic mean can be found in the literature [1-30]. It might be surprising that the logarithmic mean has applications in physics, economics, and even in meteorology [31-33]. In [31], Kahlig and Matkowski study a variant of Jensen's functional equation involving the logarithmic mean, which appears in a heat conduction problem. A representation of the logarithmic mean as an infinite product and an iterative algorithm for computing the logarithmic mean as the common limit of two sequences of special geometric and arithmetic means are given in [17]. In [34,35] it is shown that the logarithmic mean can be expressed in terms of Gauss's hypergeometric function ${}_2F_1$. And, in [35], Carlson and Gastafson prove that the reciprocal of the logarithmic mean is strictly totally positive, that is, every $n \times n$ determinant with elements $\frac{1}{L(x_i, y_i)}$, where $0 < x_1 < x_2 <\ldots < x_n$ and $0 < y_1 < y_2 <\ldots < y_n$ are positive for all $n \geq 1$.

The power mean of order r of two positive numbers a and b is defined by

$$M_r(a,\ b) = \begin{cases} \left(\frac{a^r+b^r}{2}\right)^{\frac{1}{r}} & r \neq 0, \\ \sqrt{ab}, & r = 0. \end{cases} \tag{1.2}$$

It is well-known that $M_r(a, b)$ is continuous and strictly increasing with respect to $r \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$.

Lin [18] presents the sharp power mean bounds for logarithmic mean as follows:

$$M_0(a,b) < L(a,b) < M_{\frac{1}{3}}(a,b) \tag{1.3}$$

for all a, $b > 0$ with $a \neq b$.

For $\omega \geq 0$ and $p \in \mathbb{R}$ the generalized Heronian mean $H_{p,\omega}(a, b)$ of two positive numbers a and b is introduced by Shi et al. [36] as follows:

$$H_{p,\omega}(a,b) = \begin{cases} \left[\frac{a^p+\omega(ab)^{\frac{p}{2}}+b^p}{\omega+2}\right]^{\frac{1}{p}}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases} \tag{1.4}$$

From (1.2) and (1.4) we clearly see that $H_{p,0}(a, b) = M_p(a, b)$ for all $p \in \mathbb{R}$ and a, $b > 0$. It easily follows from (1.4) that $H_{p,\omega}(a, b)$ is continuous with respect to $p \in \mathbb{R}$ for fixed a, $b > 0$ and $\omega \geq 0$, strictly increasing with respect to $p \in \mathbb{R}$ for fixed a, $b > 0$ with $a \neq b$ and $\omega \geq 0$, strictly decreasing with respect to $\omega \geq 0$ for fixed a, $b > 0$ with $a \neq b$ and $p > 0$, and strictly increasing with respect to $\omega \geq 0$ for fixed a, $b > 0$ with $a \neq b$ and $p < 0$.

In [37], Janous prove that

$$L(a,b) < H_{1,4}(a,b) \tag{1.5}$$

for all a, $b > 0$ with $a \neq b$.

The purpose of this article is to find the greatest value $p = p(\omega)$ and the least value $q = q(\omega)$ such that the double inequality $H_{p,\omega}(a, b) < L(a, b) < H_{q,\omega}(a, b)$ holds for fixed $\omega \geq 0$ and all a, $b > 0$ with $a \neq b$.

2 Main result

Theorem 2.1. For fixed $\omega \geq 0$ and all a, $b > 0$ with $a \neq b$ we have

$$H_{0,\omega}(a,b) < L(a,b) < H_{\frac{\omega+2}{6},\omega}(a,b), \tag{2.1}$$

and $H_{\frac{\omega+2}{6},\omega}(a,b)$ and $H_{0,\omega}(a, b)$ are the best possible upper and lower generalized Heronian mean bounds of the logarithmic mean $L(a, b)$, respectively.

Proof. Without loss of generality, we assume $a > b$ and put $t = \frac{a}{b} > 1$. Then from (1.1) and (1.4) we get

$$\begin{aligned} &\log[L(a,b)] - \log[H_{\frac{\omega+2}{6},\omega}(a,b)] \\ &= \log\left(\frac{t-1}{\log t}\right) - \frac{6}{\omega+2}\log\left(\frac{1+\omega t^{\frac{\omega+2}{12}}+t^{\frac{\omega+2}{6}}}{\omega+2}\right). \end{aligned} \tag{2.2}$$

Let

$$f(t) = \log\left(\frac{t-1}{\log t}\right) - \frac{6}{\omega+2}\log\left(\frac{1+\omega t^{\frac{\omega+2}{12}}+t^{\frac{\omega+2}{6}}}{\omega+2}\right). \tag{2.3}$$

Then simple computations lead to

$$\lim_{t\to 1+} f(t) = 0, \tag{2.4}$$

$$f'(t) = \frac{f_1(t)}{t(t-1)\left(1+\omega t^{\frac{\omega+2}{12}}+t^{\frac{\omega+2}{6}}\right)\log t}, \tag{2.5}$$

where $f_1(t) = (\frac{\omega}{2}t^{\frac{\omega+14}{12}} + t^{\frac{\omega+2}{6}} + \frac{\omega}{2}t^{\frac{\omega+2}{12}} + t)\log t - t^{\frac{\omega+8}{6}} + t^{\frac{\omega+2}{6}} - \omega t^{\frac{\omega+14}{12}} + \omega t^{\frac{\omega+2}{12}} - t + 1 \cdot$

$$f_1(1) = 0, \tag{2.6}$$

$$\begin{aligned} f'_1(t) = {} & \left[\frac{\omega(\omega+14)}{24}t^{\frac{\omega+2}{12}} + \frac{\omega(\omega+2)}{24}t^{\frac{\omega+10}{12}} + \frac{\omega+2}{6}t^{\frac{\omega+4}{6}} + 1\right]]\log t \\ & -\frac{\omega+8}{6}t^{\frac{\omega+2}{6}} + \frac{\omega+8}{6}t^{\frac{\omega+4}{6}} - \frac{\omega(\omega+8)}{12}t^{\frac{\omega+2}{12}} + \frac{\omega(\omega+8)}{12}t^{\frac{\omega-10}{12}}, \end{aligned} \tag{2.7}$$

$$f'_1(1) = 0, \tag{2.8}$$

$$f''_1(t) = \frac{1}{288}t^{\frac{\omega-10}{6}}f_2(t), \tag{2.9}$$

where $f_2(t) = [\omega(\omega+2)(\omega+14)t^{\frac{10-\omega}{12}} + \omega(\omega+2)(\omega-10)t^{\frac{\omega+2}{12}} + 8(\omega+2)(\omega-4)]\log t - 2\omega(\omega^2 + 4\omega - 68)t^{\frac{10-\omega}{12}} + 2\omega(\omega^2+4\omega-68)t^{-\frac{\omega+2}{12}} + 288t^{\frac{4-\omega}{6}} - 8(\omega+2)(\omega+8)t + 8(\omega^2+10\omega-20)$.

$$f_2(1) = 0, \tag{2.10}$$

$$f'_2(t) = \frac{1}{12}t^{-\frac{\omega+14}{12}}f_3(t), \tag{2.11}$$

where $f_3(t) = -[\omega(\omega+2)(\omega+14)(\omega-10)t - \omega(\omega+2)^2(\omega-10)]\log t + 2\omega(\omega^3-12\omega+848)t + 2\omega(\omega+2)^2(4-\omega) + 96(\omega+2)(\omega-4)t^{\frac{\omega+12}{12}} - 576(\omega-4)t^{\frac{10-\omega}{12}} - 96(\omega+2)(\omega+8)t^{\frac{\omega+14}{12}}$.

$$f_3(1) = 0, \tag{2.12}$$

$$\begin{aligned} f'_3(t) = {} & -\omega(\omega+2)(\omega+14)(\omega-10)\log t - \omega(\omega+2)^2(\omega-10)t^{-1} \\ & + 8(\omega+2)^2(\omega-4)t^{\frac{\omega+10}{12}} - 8(\omega+2)(\omega+8)(\omega+14)t^{\frac{\omega+2}{12}} \\ & + 48(\omega-4)(\omega-10)t^{\frac{\omega+2}{12}} + \omega(\omega^3-6\omega^2+108\omega+1976), \end{aligned} \tag{2.13}$$

$$f'_3(1) = 0, \tag{2.14}$$

$$f''_3(t) = t^{-2}f_4(t), \tag{2.15}$$

where $f_4(t) = -\omega(\omega+2)(\omega+14)(\omega-10)t + \omega(\omega+2)^2(\omega-10) + \frac{2}{3}(\omega+2)^2(\omega-4)(\omega-10)t^{\frac{\omega+2}{12}} - \frac{2}{3}(\omega+2)^2(\omega+8)(\omega+14)t^{\frac{\omega+14}{12}} - 4(\omega-2)(\omega-4)(\omega-10)t^{\frac{10-\omega}{12}}$.

$$f_4(1) = -8(\omega+2)(5\omega^2-10\omega+32) < 0, \tag{2.16}$$

$$\begin{aligned} f'_4(t) &= -\omega(\omega+2)(\omega+14)(\omega-10) + \frac{1}{18}(\omega+2)^3(\omega-4)(\omega-10)t^{\frac{\omega-10}{12}} \\ &\quad - \frac{1}{18}(\omega+2)^2(\omega+8)(\omega+14)^2 t^{\frac{\omega+2}{12}} \\ &\quad + \frac{1}{3}(\omega+2)(\omega-4)(\omega-10)^2 t^{-\frac{\omega+2}{12}}, \end{aligned} \tag{2.17}$$

$$f'_4(1) = -\frac{2}{3}(\omega+2)(5\omega^3+60\omega^2-108\omega+448) < 0, \tag{2.18}$$

$$f''_4(t) = \frac{1}{216}(\omega+2)^2 t^{-\frac{\omega+14}{12}} f_5(t), \tag{2.19}$$

where $f_5(t) = (\omega+2)(\omega-4)(\omega-10)^2 t^{\frac{\omega-4}{6}} - (\omega+2)(\omega+8)(\omega+14)^2 t^{\frac{\omega-2}{6}} - 6(\omega-4)(\omega-10)^2$.

$$f_5(1) = -6(11\omega^3+36\omega^2+588\omega+256) < 0, \tag{2.20}$$

$$f'_5(t) = \frac{\omega+2}{6} t^{\frac{\omega-10}{6}} f_6(t), \tag{2.21}$$

where

$$f_6(t) = (\omega-4)^2(\omega-10)^2 - (\omega+2)(\omega+8)(\omega+14)^2 t, \tag{2.22}$$

$$f_6(1) = -6(11\omega^3+36\omega^2+588\omega+256) < 0. \tag{2.23}$$

From (2.22) we clearly see that $f_6(t)$ is strictly decreasing in $[1, +\infty)$, then (2.23) leads to that

$$f_6(t) < 0 \tag{2.24}$$

for $t \in (1, +\infty)$.

It easily follows from (2.5)-(2.21) and (2.24) that

$$f'(t) < 0 \tag{2.25}$$

for $t \in (1, +\infty)$.

Therefore, $L(a, b) < H_{\frac{\omega+2}{6},\omega}(a, b)$ follows from (2.2)-(2.4) and (2.25).

On the other hand, $H_{0,\omega}(a, b) = M_0(a, b) < L(a, b)$ follows from (1.3).

Next, we prove that $H_{0,\omega}(a, b)$ and $H_{\frac{\omega+2}{6},\omega}(a, b)$ are the optimal lower and upper generalized Heronian mean bounds of the logarithmic mean $L(a, b)$.

For any $0 < \varepsilon < \frac{\omega+2}{6}$, $\omega \geq 0$ and $x > 0$, from (1.1) and (1.4) we have

$$\begin{aligned} &\lim_{x\to+\infty} \frac{H_{\varepsilon,\omega}(1,x)}{L(1,x)} \\ &= (\omega+2)^{-\frac{1}{\varepsilon}} \lim_{x\to+\infty} [\frac{(1+\omega x^{-\frac{\varepsilon}{2}}+x^{-\varepsilon})^{\frac{1}{\varepsilon}}}{1-\frac{1}{x}} \log x] \\ &= +\infty, \end{aligned} \tag{2.26}$$

$$\log[L(1,\ 1+x)] - \log[H_{\frac{\omega+2}{6}-\varepsilon,\omega}(1,\ 1+x)] = \frac{\varepsilon}{4(\omega+2)}x^2 + o(x^3) \quad (x \to 0). \tag{2.27}$$

Equations (2.26) and (2.27) imply that for any $\omega \geq 0$ and $0 < \varepsilon < \frac{\omega+2}{6}$ there exist sufficiently large $X = X(\varepsilon, \omega) > 1$ and sufficiently small $\delta = \delta\ (\varepsilon, \omega) > 0$; such that $H_{\varepsilon,\omega}(1, x) > L(1, x)$ for $x \in (X, +\infty)$ and $L(1,\ 1+x) > H_{\frac{\omega+2}{6}-\varepsilon,\omega}(1,\ 1+x)$ for $x \in (0, \delta)$.

Remark 2.1. If we take $\omega = 0$, then inequality (2.1) reduce to inequality (1.3).

Remark 2.2. If we take $\omega = 4$, then the upper bound in inequality (2.1) becomes the upper bound in inequality (1.5).

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Authors' contributions
H-XS provided the main idea in this article. B-YL carried out the proof of inequality (2.1) in this article. Y-MC carried out the optimality proof of inequality (2.2) in this article. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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