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Trace inequalities for positive semidefinite matrices with centrosymmetric structure

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Abstract

In this article, we present some results on the Hadamard product of positive semidefinite matrices with centrosymmetric structure. Based on these results, several trace inequalities on positive semidefinite centrosymmetric matrices are obtained.

1 Introduction and preliminaries

We will use the following notation. Let $\mathbb{C}^{n \times n}$ and $\mathbb{R}^{n \times n}$ be the space of $n \times n$ complex and real matrices, respectively. The identity matrix in $\mathbb{C}^{n \times n}$ is denoted by $I = I_n$. Let A^T, $\bar{A}$, A^H, and $tr(A)$ denote the transpose, the conjugate, the conjugate transpose, and the trace of a matrix A, respectively. Let $Re(a)$ represent the real part of a. The Frobenius inner product $< \cdot, \cdot >_F$ in $\mathbb{C}^{m \times n}$ over the complex field is defined as follows: $<A, B>_F = Re(tr(B^H A))$, for $A, B \in \mathbb{C}^{m \times n}$, i.e., $<A, B>$ is the real part of the trace of $B^H A$. The induced matrix norm is $\|A\|_F = \sqrt{<A, A>_F} = \sqrt{Re(tr(A^H A))} = \sqrt{tr(A^H A)}$, which is called the Frobenius (Euclidean) norm.

A matrix $A \in \mathbb{C}^{n \times n}$ is Hermitian if $A^H = A$, and A is called positive semidefinite, written as $A \geq 0$ (see [1, p. 159]), if

$$x^H Ax \geq 0, \quad \forall x \in \mathbb{C}^n. \tag{1.1}$$

A is further called positive definite, symbolized $A > 0$, if the strict inequality in (1.1) holds for all non-zero $x \in \mathbb{C}^n$. An equivalent condition for $A \in \mathbb{C}^n$ to be positive definite is that A is Hermitian and all eigenvalues of A are positive.

Let A and B be two Hermitian matrices of the same size. If $A - B$ is positive semidefinite, we write

$$A \geq B.$$

Next, we introduce some basic definitions and lemmas.

Definition 1.1 (see [1]). *Let A be a square complex matrix partitioned as*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{1.2}$$

where A_{11} is a square submatrix of A. If A_{11} is nonsingular, we call $\tilde{A}_{11} = A_{22} - A_{21}A_{11}^{-1}A_{12}$ the Schur complement of A_{11} in A.

Note. If A is a positive definite matrix, then A_{11} is nonsingular and $A_{22} \geq \tilde{A}_{11} \geq 0$.

Definition 1.2 (see [2]). $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ *is called a centrosymmetric matrix, if*

$$a_{ij} = a_{n-i+1,n-j+1}, 1 \leq i \leq n, 1 \leq j \leq n, \text{ or } J_n A J_n = A,$$

where $J_n = (e_n, e_{n-1},..., e_1)$, e_i *denotes the unit vector with the ith entry 1.*

If a matrix is both positive semidefinite and centrosymmetric, we call this matrix positive semidefinite centrosymmetric.

Using the partition of matrix, the central symmetric character of a square centrosymmetric matrix can be described as follows [2]:

Lemma 1.1 (see [2]). *Let* $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ $(n = 2m)$ *be centrosymmetric. Then, A has the following form,*

$$A = \begin{pmatrix} B & J_m C J_m \\ C & J_m B J_m \end{pmatrix}, \quad \text{and } P^T A P = \begin{pmatrix} B - J_m C & 0 \\ 0 & B + J_m C \end{pmatrix}, \tag{1.3}$$

where $B \in \mathbb{C}^{m \times m}, C \in \mathbb{C}^{m \times m}, P = \frac{1}{\sqrt{2}} \begin{pmatrix} I_m & I_m \\ -J_m & J_m \end{pmatrix}$.

Note. In this article, we mainly discuss the case $n = 2m$. For n is odd, i.e., $n = 2m + 1$, similar results can be obtained by taking similar steps.

Recently, in [3], Ulukök and Türkmen proved some matrix trace inequalities for positive semidefinite matrices:

Lemma 1.2 (see [3]). *Let* $A, B \in \mathbb{C}^{n \times n}$. *Then,*

$$\left\| (A \circ B)^m \right\|_F^2 \leq \left\| \left(A^H A\right)^m \right\|_F \cdot \left\| \left(B^H B\right)^m \right\|_F \tag{1.4}$$

where m is an positive integer, $A \circ B$ *stands for the Hadamard product of A and B.*

Note. Particularly, if A and B in Lemma 1.2 are both semidefinite matrices, then $\left\| (A \circ B)^m \right\|_F^2 \leq \left\| A^{2m} \right\|_F \cdot \left\| B^{2m} \right\|_F$.

Lemma 1.3 (see [3]). *Let* $A_i \in \mathbb{C}^{n \times n}$, $(i = 1,2,...,k)$ *be semidefinite matrices. Then, for positive real numbers s, m, t*

$$\left(\sum_{i=1}^{k} \left\| A_i^{((s+t)/2)m} \right\|_F^2 \right) \leq \left(\sum_{i=1}^{k} \left\| A_i^{sm} \right\|_F^2 \right) \left(\sum_{i=1}^{k} \left\| A_i^{tm} \right\|_F^2 \right) \tag{1.5}$$

Lemma 1.4 (see [3]). *Let* $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \geq 0, B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \geq 0$. Then,

$$tr\left[\left(\tilde{A}_{22}\right)^{1/2} B_{11}^{1/2} \right]^{2m} + tr\left[(A_{22})^{1/2} \tilde{B}_{11}^{1/2} \right]^{2m} \leq tr(AB)^m \leq tr(A^m B^m), \tag{1.6}$$

where m is an integer.

2 Main results

Lemma 2.1. *Let* $A = (a_{ij})_{n \times n}$, $B = (b_{ij})_{n \times n}$ $(n = 2m)$ *be two centrosymmetric matrices with the following form:*

$$A = \begin{pmatrix} A_1 & J_m A_2 J_m \\ A_2 & J_m A_1 J_m \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & J_m B_2 J_m \\ B_2 & J_m B_1 J_m \end{pmatrix}, A_1,\ A_2,\ B_1, B_2 \in \mathbb{C}^{m \times m} \tag{2.1}$$

Then, $A \circ B$ *is a centrosymmetric matrix.*

Proof. By the definition of Hadamard product,

$$A \circ B = \begin{pmatrix} A_1 & J_mA_2J_m \\ A_2 & J_mA_1J_m \end{pmatrix} \circ \begin{pmatrix} B_1 & J_mB_2J_m \\ B_2 & J_mB_1J_m \end{pmatrix} = \begin{pmatrix} A_1 \circ B_1 & (J_mA_2J_m) \circ (J_mB_2J_m) \\ A_2 \circ B_2 & (J_mA_1J_m) \circ (J_mB_1J_m) \end{pmatrix} \tag{2.2}$$

We shall prove the following

$$\begin{cases} (J_mA_1J_m) \circ (J_mB_1J_m) = J_m(A_1 \circ B_1)J_m, \\ (J_mA_2J_m) \circ (J_mB_2J_m) = J_m(A_1 \circ B_2)J_m. \end{cases} \tag{2.3}$$

From (2.1),

$$A_1 \circ B_1 = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{pmatrix} \circ \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mm} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & \cdots & a_{1m}b_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1}b_{m1} & \cdots & a_{mm}b_{mm} \end{pmatrix}, \tag{2.4}$$

and

$$J_m(A_1 \circ B_1)J_m = \begin{pmatrix} a_{mm}b_{mm} & \cdots & a_{m1}b_{m1} \\ \vdots & \ddots & \vdots \\ a_{1m}b_{1m} & \cdots & a_{11}b_{11} \end{pmatrix}. \tag{2.5}$$

Since

$$J_mA_1J_m = \begin{pmatrix} a_{mm} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1m} & \cdots & a_{11} \end{pmatrix}, \quad \text{and} \quad J_mB_1J_m = \begin{pmatrix} b_{mm} & \cdots & b_{m1} \\ \vdots & \ddots & \vdots \\ b_{1m} & \cdots & b_{11} \end{pmatrix}, \tag{2.6}$$

we have the following

$$(J_mA_1J_m) \circ (J_mB_1J_m) = \begin{pmatrix} a_{mm}b_{mm} & \cdots & a_{m1}b_{m1} \\ \vdots & \ddots & \vdots \\ a_{1m}b_{1m} & \cdots & a_{11}b_{11} \end{pmatrix}. \tag{2.7}$$

From (2.5) and (2.7), it is clear that

$$(J_mA_1J_m) \circ (J_mB_1J_m) = J_m\,(A_1 \circ B_1)\,J_m. \tag{2.8}$$

Similarly, we can prove that

$$(J_mA_2J_m) \circ (J_mB_2J_m) = J_m\,(A_2 \circ B_2)\,J_m. \tag{2.9}$$

From (2.8) and (2.9), we can see that (2.3) holds. By Lemma 1.1, $A \circ B$ is a centrosymmetric matrix.

Lemma 2.2 (see [4]) *Let* $A = (a_{ij})_{n \times n}$ $(n = 2m)$ *a positive semidefinite centrosymmetric matrix with the following form*

$$A = \begin{pmatrix} B & J_mCJ_m \\ C & J_mBJ_m \end{pmatrix}, \quad B,\ C \in \mathbb{C}^{m \times m}.$$

Let $M = B - J_mC$ *and* $N = B + J_mC$. *Then,* M, N *are positive semidefinite matrices.*

Theorem 2.1 *Let* A, $B \in \mathbb{C}^{n \times n}$ $(n = 2m)$ *be two positive semidefinite centrosymmetric matrices with the same form as in* (2.1). *Let*

$$M_A = A_1 - J_mA_2, \quad N_A = A_1 + J_mA_2, \quad M_B = B_1 - J_mB_2, \quad N_B = B_1 + J_mB_2,$$

and

$$X = A_1 \circ B_1 - J_m (A_2 \circ B_2), \quad Y = A_1 \circ B_1 + J_m (A_2 \circ B_2).$$

Then, the following inequality holds:

$$\|X^m\|_F^2 + \|Y^m\|_F^2 \le \sqrt{\left(\|M_A^{2m}\|_F^2 + \|N_A^{2m}\|_F^2\right) \cdot \left(\|M_B^{2m}\|_F^2 + \|N_B^{2m}\|_F^2\right)}. \qquad (2.10)$$

Proof. Since A, B are centrosymmetric matrices, by Lemma 2.1, $A \circ B$ is centrosymmetric and

$$A \circ B = \begin{pmatrix} A_1 \circ B_1 & (J_m A_2 J_m) \circ (J_m B_2 J_m) \\ A_2 \circ B_2 & (J_m A_1 J_m) \circ (J_m B_1 J_m) \end{pmatrix}.$$

From Lemma 1.1,

$$P^T (A \circ B) P = \begin{pmatrix} A_1 \circ B_1 - J_m (A_2 \circ B_2) & 0 \\ 0 & A_1 \circ B_1 - J_m (A_2 \circ B_2) \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}.$$

P is orthogonal, then

$$P^T (A \circ B)^m P = \begin{pmatrix} X^m & 0 \\ 0 & Y^m \end{pmatrix},$$

and

$$\|(A \circ B)^m\|_F^2 = \|P^T (A \circ B)^m P\|_F^2 = \left\| \begin{pmatrix} X^m & 0 \\ 0 & Y^m \end{pmatrix} \right\|_F^2 = \|X^m\|_F^2 + \|Y^m\|_F^2.$$

Similarly from Lemma 1.1,

$$P^T A P = \begin{pmatrix} A_1 - J_1 A_2 & 0 \\ 0 & A_1 + J_1 A_2 \end{pmatrix} = \begin{pmatrix} M_A & 0 \\ 0 & N_A \end{pmatrix},$$
$$P^T B P = \begin{pmatrix} B_1 - J_1 B_2 & 0 \\ 0 & B_1 + J_1 B_2 \end{pmatrix} = \begin{pmatrix} M_B & 0 \\ 0 & N_B \end{pmatrix}.$$

Then,

$$P^T A^{2m} P = \begin{pmatrix} M_A^{2m} & 0 \\ 0 & N_A^{2m} \end{pmatrix}, P^T B^{2m} P = \begin{pmatrix} M_B^{2m} & 0 \\ 0 & N_B^{2m} \end{pmatrix},$$

and

$$\begin{aligned} \|A^{2m}\|_F \cdot \|B^{2m}\|_F &= \left\| \begin{pmatrix} M_A^{2m} & 0 \\ 0 & N_A^{2m} \end{pmatrix} \right\|_F \cdot \left\| \begin{pmatrix} M_B^{2m} & 0 \\ 0 & N_B^{2m} \end{pmatrix} \right\|_F \\ &= \sqrt{\left(\|M_A^{2m}\|_F^2 + \|N_A^{2m}\|_F^2\right) \cdot \left(\|M_B^{2m}\|_F^2 + \|N_B^{2m}\|_F^2\right)}. \end{aligned}$$

Since both A and B are positive semidefinite matrices, by Lemma 1.2

$$\|(A \circ B)\|_F^2 \le \|A^{2m}\|_F \cdot \|B^{2m}\|_F.$$

Thus,

$$\|X^m\|_F^2 + \|Y^m\|_F^2 \le \sqrt{\left(\|M_A^{2m}\|_F^2 + \|N_A^{2m}\|_F^2\right) \cdot \left(\|M_B^{2m}\|_F^2 + \|N_B^{2m}\|_F^2\right)}.$$

Theorem 2.2. *Let $A \in \mathbb{C}^{n \times n}$ ($n = 2m$) be postive semidefinite centroysymmetric with the form:*

$$A = \begin{pmatrix} B \; J_m C J_m \\ C \; J_m B J_m \end{pmatrix}, \quad B,\ C \in \mathbb{C}^{m\times m}.$$

Let $M = B - J_mC$, $N = B + J_mC$. Then, for positive integers s, m, t, the following two equalities hold

$$\left\|A^{((s+t)/2)m}\right\|_F^2 = \left\|M^{((s+t)/2)m}\right\|_F^2 + \left\|N^{((s+t)/2)m}\right\|_F^2, \tag{2.11}$$

and

$$\left\|M^{((s+t)/2)m}\right\|_F^2 + \left\|N^{((s+t)/2)m}\right\|_F^2 \leq \left(\left\|M^{sm}\right\|_F^2 + \left\|N^{sm}\right\|_F^2\right) \cdot \left(\left\|M^{tm}\right\|_F^2 + \left\|N^{tm}\right\|_F^2\right). \tag{2.12}$$

Proof. By Lemma 1.1,

$$P^TAP = \begin{pmatrix} B - J_mC & 0 \\ 0 & B + J_mC \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}.$$

Since A is positive semidefinite, we have

$$P^TA^{((s+t)/2)m}P = \begin{pmatrix} M^{((s+t)/2)m} & 0 \\ 0 & N^{((s+t)/2)m} \end{pmatrix}.$$

Thus,

$$\begin{aligned}\left\|A^{((s+t)/2)m}\right\|_F^2 = \left\|P^TA^{((s+t)/2)m}\right\|_F^2 &= \left\|\begin{pmatrix} M^{((s+t)/2)m} & 0 \\ 0 & N^{((s+t)/2)m} \end{pmatrix}\right\|_F^2 \\ &= \left\|M^{((s+t)/2)m}\right\|_F^2 + \left\|N^{((s+t)/2)m}\right\|_F^2.\end{aligned}$$

From Lemma 2.2, M and N are positive semidefinite. Combining Lemma 1.3, we have

$$\left\|M^{((s+t)/2)m}\right\|_F^2 + \left\|N^{((s+t)/2)m}\right\|_F^2 \leq \left(\left\|M^{sm}\right\|_F^2 + \left\|N^{sm}\right\|_F^2\right) \cdot \left(\left\|M^{tm}\right\|_F^2 + \left\|N^{tm}\right\|_F^2\right).$$

Theorem 2.3 Let A, $B \in \mathbb{C}^{n \times n}$ ($n = 2m$) are positive semidefinite centrosymmetric matrices with the same form as in (2.1). Let

$$M_A = A_1 - J_mA_2, \quad N_A = A_1 + J_mA_2, \quad M_B = B_1 - J_mB_2, \quad N_B = B_1 + J_mB_2.$$

Then, the following inequality holds

$$tr\left(M_A^{1/2}M_B^{1/2}\right)^{2m} + tr\left(N_A^{1/2}N_B^{1/2}\right)^{2m} \leq tr(AB)^m \leq tr\left(A^mB^m\right). \tag{2.13}$$

Proof. From Lemma 1.1, there exists an orthogonal matrix P such that

$$P^TAP = \begin{pmatrix} A_1 - J_mA_2 & 0 \\ 0 & A_1 + J_mA_2 \end{pmatrix} = \begin{pmatrix} M_A & 0 \\ 0 & N_A \end{pmatrix},$$

$$P^TBP = \begin{pmatrix} B_1 - J_mB_2 & 0 \\ 0 & B_1 + J_mB_2 \end{pmatrix} = \begin{pmatrix} M_B & 0 \\ 0 & N_B \end{pmatrix}.$$

According to Definition 1.1,

$$\tilde{N}_A = M_A - 0 \cdot N_A^{-1} \cdot 0 = M_A, \tilde{M}_B = N_B - 0 \cdot M_B^{-1} \cdot 0 = N_B.$$

From Lemma 1.4,

$$tr\left(M_A^{1/2}M_B^{1/2}\right)^{2m} + tr\left(N_A^{1/2}N_B^{1/2}\right)^{2m} \leq tr(P^TAPP^TBP)^m = tr\left(P^TABP\right)^m.$$

Since P is orthogonal, we have $tr(P^TABP)^m = tr(AB)^m$. By Lemma 1.4, the following holds

$$tr\left(M_A^{1/2}M_B^{1/2}\right)^{2m} + tr\left(N_A^{1/2}N_B^{1/2}\right)^{2m} \leq tr(AB)^m \leq tr\left(A^mB^m\right).$$

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Authors' contributions
DZ carried out studies on the linear algebra and matrix theory with applications, and drafted the manuscript. HL helped prove some lemmas and theorems. ZG read the manuscript carefully and gave valuable suggestions and comments. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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