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Approximate Euler-Lagrange quadratic mappings

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Abstract

For any fixed integer k with $k \neq 0, 1$, we prove the Hyers-Ulam stability of an Euler-Lagrange-type quadratic functional equation

$$f(kx+y)+f(kx-y)=kf(x+y)+kf(x-y)+2k(k-1)f(x)-2(k-1)f(y)$$

in normed spaces and in non-Archimedean normed spaces.

1 Introduction

The problem of stability of functional equations was originally stated by Ulam [1]. In 1941, Hyers [2] gave an affirmative answer to Ulam's problem for the case of approximate additive mappings on Banach spaces. In 1950, Aoki discussed the Hyers-Ulam stability theorem in [3]. His result was further generalized and rediscovered by Rassias [4] in 1978. The stability problem for functional equation has extensively been investigated by a number of mathematicians [5-9].

The quadratic function $f(x) = cx^2$ satisfies the functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1}$$

and therefore Equation (1) is called the quadratic functional equation. Every solution of Equation (1) is said to be a quadratic mapping. The Hyers-Ulam stability theorem for the quadratic functional equation (1) was by Skof [9] for the functions $f: E_1 \to E_2$ where E_1 is a normed space and E_2 is a Banach space. The result of Skof is still true if the relevant domain E_1 is replaced by an Abelian group and this was dealt with by Cholewa [10]. Czerwik [11] proved the Hyers-Ulam stability of the quadratic functional equation (1). This result was further generalized by Rassias [12], Borelli and Forti [13]. During the last three decades, a number of papers and research monographs have been published on various generalizations and applications of the Hyers-Ulam stability of several functional equations, and there are many interesting results concerning this problem [14-20]. In particular, Rassias investigated the Hyers-Ulam stability for the relative Euler-Lagrange functional equation

$$f(ax+by)+f(bx-ay)=(a^2+b^2)[f(x)+f(y)] \tag{2}$$

in [21-23].

In 2008, Ravi et al. [24] investigated the Hyers-Ulam stability of a quadratic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+4f(x)-2f(y). \tag{3}$$

In this article, we generalize the functional equation (3) to a more general form

$$f(kx+y)+f(kx-y)=kf(x+y)+kf(x-y)+2k(k-1)f(x)-2(k-1)f(y) \tag{4}$$

and investigate the Hyers-Ulam stability of the equation for any fixed integer k with $k \neq 0, 1$. As results, we improve the generalized stability results given in [24] in normed spaces and in non-Archimedean normed spaces.

2 General solution of (4)

First of all, if $k = -1$ in Equation (4), then it is easy to see that

$$f(-x+y)+f(-x-y)=-f(x+y)-f(x-y)+4f(x)+4f(y)$$

is equivalent to Equation (1), and so the solution of Equation (4) with $k = -1$ is a quadratic mapping. Thus, we consider general solutions of Equation (4) for any fixed integer k with $|k| > 1$ in the following theorem. The following lemma can be found in [25-27].

Lemma 2.1. A mapping $f: X \to Y$ between linear spaces satisfies the functional equation

$$f(2x+y)+f(2x-y)=4[f(x+y)+f(x-y)]+2[f(2x)-4f(x)]-6f(y)$$

if and only if f is quadratic and quartic.

Theorem 2.2. A mapping $f: X \to Y$ between linear spaces satisfies the functional equation (4) with $|k| > 1$ if and only if f is quadratic.

Proof. Let f be a solution of Equation (4). Letting $x = y = 0$ in (4), we have $f(0) = 0$. Putting $y = 0$ in (4), we get $f(kx) = k^2 f(x)$. Putting $x = 0$ in (4), we get $f(-y) = f(y)$. Thus, the mapping f is even. Therefore, it suffices to prove that if a mapping f satisfies Equation (4) for any fixed integer k with $|k| > 1$, then f is quadratic. Now, replacing y by $x + y$ in (4), we have

$$f((k+1)x+y)+f((k-1)x-y)=kf(2x+y)+kf(y)+2k(k-1)f(x)-2(k-1)f(x+y) \tag{5}$$

for all $x, y \in X$. Replacing y by $-y$ in (5), we obtain

$$f((k+1)x-y)+f((k-1)x+y)=kf(2x-y)+kf(y)+2k(k-1)f(x)-2(k-1)f(x-y) \tag{6}$$

for all $x, y \in X$. Adding (5) to (6), we get

$$\begin{aligned} f((k+1)x+y)+f((k+1)x-y)+f((k-1)x+y)+f((k-1)x-y) \\ =k[f(2x+y)+f(2x-y)]-2(k-1)[f(x+y)+f(x-y)] \\ =-2(k-1)[f(x+y)+f(x-y)]+4k(k-1)f(x)+2kf(y) \end{aligned} \tag{7}$$

for all $x, y \in X$. From the substitution $y = kx + y$ in (4), we have

$$\begin{aligned} f(2kx+y)+f(y)=k[f((k+1)x+y)+f((k-1)x+y)] \\ +2k(k-1)f(x)-2(k-1)f(kx+y) \end{aligned} \tag{8}$$

for all $x, y \in X$. Replacing y by $-y$ in (8), we get

$$\begin{aligned} f(2kx-y)+f(y)=k[f((k+1)x-y)+f((k-1)x-y)] \\ +2k(k-1)f(x)-2(k-1)f(kx-y) \end{aligned} \tag{9}$$

for all $x, y \in X$. Adding (8) to (9), we get

$$\begin{aligned} f(2kx+y) + (2kx-y) = k[&f((k+1)x+y) + f((k+1)x-y)] \\ &+ k[f((k-1)x+y) + f((k-1)x-y)] \\ &- 2(k-1)[f(kx+y) + f(kx-y)] \\ &+ 4k(k-1)f(x) - 2f(y) \end{aligned} \tag{10}$$

for all $x, y \in X$. It follows from (10), by using (4) and (7), that

$$\begin{aligned} f(2kx+y) + f(2kx-y) = k^2&[f(2x+y) + f(2x-y)] \\ &- 4x(k-1)[f(x+y) + f(x-y)] \\ &+ 8k(k-1)f(x) + 2(k-1)(3k-1)f(y) \end{aligned} \tag{11}$$

for all $x, y \in X$. If we replace x by $2x$ in (4), then we obtain that

$$\begin{aligned} f(2kx+y) + f(2kx-y) = k[&f(2x+y) + f(2x-y)] \\ &+ 2k(k-1)f(2x) - 2(k-1)f(y) \end{aligned} \tag{12}$$

for all $x, y \in X$. Associating (11) with (12), we conclude that the mapping f satisfies the equation

$$f(2x+y) + f(2x-y) = 4[f(x+y) + f(x-y)] + 2[f(2x) - 4f(x)] - 6f(y)$$

for all $x, y \in X$. Therefore, it follows from Lemma 2.1 that f is quadratic because of the property $f(kx) = k^2 f(x)$.

Conversely, if a mapping f is quadratic, then it is obvious that f satisfies (4).

3 Hyers-Ulam stability of (4) in banach spaces

In this section, let X be a normed space and Y a Banach space. We will investigate the Hyers-Ulam stability problem for the functional equation (4). For notational convenience, we define an operator $D_k f(x, y)$ as

$$\begin{aligned} D_k f(x,y) := &f(kx+y) + f(kx-y) - kf(x+y) - kf(x-y) \\ &- 2k(k-1)f(x) + 2(k-1)f(y) \end{aligned}$$

for all $x, y \in X$, where k is a fixed integer with $|k| > 1$.

Theorem 3.1. Let $\psi : X^2 \to [0, \infty)$ be a function such that

$$\sum_{i=0}^{\infty} \frac{\psi(k^i x, k^i y)}{k^{2i}} < \infty \tag{13}$$

for all $x, y \in X$. If a mapping $f : X \to Y$ satisfies the inequality

$$\left\| D_k f(x,y) \right\| \le \psi(x,y) \tag{14}$$

for all $x, y \in X$, then there exists a unique quadratic mapping $Q_1 : X \to Y$ which satisfies Equation (4) and the inequality

$$\left\| f(x) - \frac{f(0)}{k+1} - Q_1(x) \right\| \le \frac{1}{2k^2} \sum_{i=0}^{\infty} \frac{\psi(k^i x, 0)}{k^{2i}} \tag{15}$$

for all $x \in X$, where $\|f(0)\| \leq \frac{\psi(0,0)}{2k(k-1)}$. The mapping Q_1 is defined by

$$Q_1(x) = \lim_{n\to\infty} \frac{f(k^n x)}{k^{2n}} \tag{16}$$

for all $x \in X$.

Proof. Letting $x = y := 0$ in (14), we get

$$\|f(0)\| \leq \frac{\psi(0,0)}{2k(k-1)}.$$

Putting $y := 0$ in (14) and dividing by $2k^2$, we obtain

$$\left\|\frac{f(kx)}{k^2} - f(x) + \frac{k-1}{k^2} f(0)\right\| \leq \frac{\psi(x,0)}{2k^2} \tag{17}$$

for all $x \in X$. Setting $\bar{f}(x) := f(x) - \frac{f(0)}{k+1}$, we lead to a functional inequality

$$\left\|\frac{\bar{f}(kx)}{k^2} - \bar{f}(x)\right\| \leq \frac{\psi(x,0)}{2k^2} \tag{18}$$

for all $x \in X$. Using the induction argument on a positive integer n we may obtain that

$$\left\|\bar{f}(x) - \frac{\bar{f}(k^n x)}{k^{2n}}\right\| \leq \frac{1}{2k^2} \sum_{i-0}^{n-1} \frac{\psi(k^i x, 0)}{k^{2i}} \tag{19}$$

for all $x \in X$. Now, it follows from (19) that for $m > n > 0$,

$$\begin{aligned}\left\|\frac{\bar{f}(k^m x)}{k^{2m}} - \frac{\bar{f}(k^n x)}{k^{2n}}\right\| &= \left\|\frac{\bar{f}(k^{m-n+n} x)}{k^{2(m-n+n)}} - \frac{\bar{f}(k^n x)}{k^{2n}}\right\| \\ &= \frac{1}{k^{2n}} \left\|\bar{f}(k^n x) - \frac{\bar{f}(k^{m-n} k^n x)}{k^{2(m-n)}}\right\| \\ &\leq \frac{1}{2k^2} \sum_{i=0}^{m-n-1} \frac{\psi(k^{i+n} x, 0)}{k^{2(i+n)}}\end{aligned} \tag{20}$$

for all $x \in X$. Since the right-hand side of the inequality (20) tends to 0 as $n \to \infty$, a sequence $\left\{\frac{\bar{f}(k^n x)}{k^{2n}}\right\}$is Cauchy. Therefore, we may define a mapping $Q_1 : X \to Y$ as

$$Q_1(x) = \lim_{n\to\infty} \frac{\bar{f}(k^n x)}{k^{2n}} = \lim_{n\to\infty} \frac{f(k^n x)}{k^{2n}}$$

for all $x \in X$. Letting $n \to \infty$ in (19), we lead to the approximation (15).

Next, we have to show that Q_1 satisfies Equation (4). Replacing x, y by $k^n x$, $k^n y$ in (14) and dividing by k^{2n}, we obtain that

$$\frac{1}{k^{2n}} \left\| f(k^n(kx+y)) + f(k^n(kx-y)) - kf(k^n(x+y)) - kf(k^n(x-y)) \right.$$
$$\left. -2k(k-1)f(k^n x) + 2(k-1)f(x^n y) \right\| \le \frac{1}{k^{2n}} \psi(k^n x, k^n y)$$

for all $x, y \in X$. Taking the limit as $n \to \infty$, we see from (13) and (16) that the mapping Q_1 satisfies Equation (4) and so it is quadratic by Theorem 2.2.

To prove the uniqueness of the quadratic mapping Q_1 satisfying the inequality (15), let us assume that there exists a quadratic mapping $Q_1' : X \to Y$ which satisfies the inequality (15). Then, we have $Q_1(k^n x) = k^{2n} Q_1(x)$ and $Q_1'(k^n x) = k^{2n} Q_1'(x)$ for all $x \in X$ and all $n \in$ N. Hence, it follows from (15) that

$$\begin{aligned}\|Q_1(x) - Q_1'(x)\| &= \frac{1}{k^{2n}} \|Q_1(k^n x) - Q_1'(k^n x)\| \\ &\le \frac{1}{k^{2n}} \left\|Q_1(k^n x) - \bar{f}(k^n x)\right\| + \left\|\bar{f}(k^n x) - Q_1'(k^n x)\right\| \\ &\le \frac{1}{k^2} \sum_{i=n}^{\infty} \frac{\psi(k^i x, 0)}{k^{2i}}\end{aligned}$$

which tends to zero as $n \to \infty$. This completes the proof of the theorem.

The following theorem is an alternative stability result concerning the stability of functional equation (4).

Theorem 3.2. Let ψ: $X^2 \to [0, \infty)$ be a function such that

$$\sum_{i=1}^{\infty} k^{2i} \psi\left(\frac{x}{k^i}, \frac{y}{k^i}\right) < \infty \tag{21}$$

for all $x, y \in X$. If a mapping $f : X \to Y$ satisfies the inequality

$$\left\|D_k f(x, y)\right\| \le \psi(x, y) \tag{22}$$

for all $x, y \in X$, then there exists a unique quadratic mapping $Q_2 : X \to Y$ which satisfies Equation (4) and the inequality

$$\left\|f(x) - Q_2(x)\right\| \le \frac{1}{2k^2} \sum_{i=1}^{\infty} k^{2i} \psi\left(\frac{x}{k^i}, 0\right) \tag{23}$$

for all $x \in X$. The mapping Q_2 is defined by

$$Q_2(x) = \lim_{n \to \infty} k^{2n} f\left(\frac{x}{k^n}\right) \tag{24}$$

for all $x \in X$.

Corollary 3.3. Let $\varepsilon_1 \ge 0$, $\varepsilon_2 \ge 0$ and p, q be real numbers such that either $0 < p, q < 2$ or $p, q > 2$. If a mapping $f : X \to Y$ satisfies the inequality

$$\left\|D_k f(x, y)\right\| \le \varepsilon_1 \|x\|^p + \varepsilon_2 \left\|y\right\|^q$$

for all $x, y \in X$, then there exists a unique quadratic mapping $Q_i : X \to Y$ (i = 1,2) which satisfies (4) and inequality

$$\begin{cases} \|f(x) - Q_1(x)\| \le \dfrac{\varepsilon_1 \|x\|^p}{2(k^2 - k^p)}, \text{if} \quad 0 < p, q < 2; \\ \|f(x) - Q_2(x)\| \le \dfrac{\varepsilon_1 \|x\|^p}{2(k^p - k^2)}, \text{if} \quad p, q > 2. \end{cases}$$

for all $x \in X$. Furthermore, for each fixed $x \in X$ if $f(tx)$ is continuous for all $t \in \mathbf{R}$, then $f(tx) = t^2 f(x)$ for all $t \in \mathbf{R}$.

Proof. Taking $\psi(x, y) = \varepsilon_1 \|x\|^p + \varepsilon_2 \|y\|^q$ and applying Theorems 3.1 and 3.2, we obtain the desired approximations.

The following is a simple example that the quadratic functional equation $D_k f(x, y) = 0$, $k \ge 2$ is not stable for $p = 2 = q$ in Corollary 3.3.

Example 3.4. Let φ: $\mathbf{R} \to \mathbf{R}$ be defined by

$$\phi(x) = \begin{cases} \mu x^2 \text{ if } |x| < 1, \\ \mu \quad \text{otherwise.} \end{cases}$$

where $\mu > 0$ is a positive constant, and define $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \sum_{i=0}^{\infty} \frac{\phi(2^i x)}{4^i}, \quad \text{for all } x \in \mathbf{R}.$$

Then, f satisfies the functional inequality

$$\begin{aligned} &\left| f(2x + y) + f(2x - y)x - 2f(x + y) - 2f(x - y) - 4f(x) + 2f(y) \right| \\ &\le 4^4 \mu (|x|^2 + |y|^2) \end{aligned} \tag{25}$$

for all $x, y \in \mathbf{R}$, but there do not exist a quadratic function $Q : \mathbf{R} \to \mathbf{R}$ and a constant $\beta > 0$ such that

$$\left| f(x) - Q(x) \right| \le \beta |x|^2 \textit{ for all } x \in \mathbf{R}. \tag{26}$$

Proof. It is easy to see that f is bounded by $\frac{4\mu}{3}$on $\mathbf{R}$. If $|x|^2 + |y|^2 \ge \frac{1}{4}$ or 0, then the left side of (25) is less than 16μ, and thus (25) is true. Now suppose that $0 < |x|^2 + |y|^2 < \frac{1}{4}$. Then there exists a positive integer k such that

$$\frac{1}{4^{k+2}} \le |x|^2 + |y|^2 < \frac{1}{4^{k+1}}, \tag{27}$$

so that $4^4|x|^2 < \frac{1}{4}$, $4^k|y|^2 < \frac{1}{4}$ and $2^{k-1}(2x \pm y)$, $2^{k-1}(x \pm y)$, $2^{k-1}x$, $2^{k-1}y$ all belong to the interval (-1, 1). Hence, for $i = 0, 1, \ldots, k - 1$,

$$\phi(2^i(2x + y)) + \phi(2^i(2x - y)) - 2\phi(2^i(x + y)) + 2\phi(2^i(x - y)) - 4\phi(2^i x) + 2\phi(2^i y) = 0$$

Therefore, it follows from the definition of f and the inequality (27) that

$$\begin{aligned}
&\left|f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-4f(x)+2f(y)\right| \\
&\leq \sum_{i=0}^{\infty}\frac{1}{4^i}\left|\phi(2^i(2x+y))+\phi(2^i(2x-y))\right. \\
&\qquad\left.-2\phi(2^i(x+y))-2\phi(2^i(2x-y))-4\phi(2^ix)+2\phi(2^iy)\right| \\
&\leq \sum_{i=k}^{\infty}\frac{1}{4^i}12\mu = 16\mu 4^{-k} \leq 4^4\mu(|x|^2+|y|^2)
\end{aligned} \tag{28}$$

for all $x, y \in \mathbf{R}$ with $0 < |x|^2 + |y|^2 < \frac{1}{4}$. Thus f satisfies (25) for all $x, y \in \mathbf{R}$.

We claim that the quadratic functional equation $D_2f(x, y) = 0$ is not stable for $p = 2 = q$ in Corollary 3.3. Suppose on the contrary that there exist a quadratic function $Q : \mathbf{R} \to \mathbf{R}$ and a constant $\beta > 0$ satisfying (26). Since f is bounded and continuous for all $x \in \mathbf{R}$, Q is bounded on any open interval containing the origin and continuous at the origin. Therefore, Q must have the form $Q(x) = \eta x^2$ for any x in $\mathbf{R}$. Thus, we obtain that

$$\left|f(x)\right| \leq (\beta + |\eta|)|x|^2 \quad \text{for all } x \in \mathrm{R}. \tag{29}$$

However, we can choose a positive integer m with $m\mu > \beta + |\eta|$. If $x \in \left(0, \frac{1}{2^{m-1}}\right)$, then $2^ix \in (0, 1)$ for all $i = 0, 1,..., m - 1$, and for this x we get

$$f(x) = \sum_{i=0}^{\infty}\frac{\phi(2^ix)}{4^n} \geq \sum_{i=0}^{m-1}\frac{\mu(2^ix)^2}{4^i} = m\mu x^2 > (\beta + |\eta|)x^2,$$

which contradicts (29). Therefore, the quadratic functional equation $D_2f(x, y) = 0$ is not stable if $p = 2 = q$ in Corollary 3.3.

Corollary 3.5. Let ε be a nonnegative real number. If a mapping $f : X \to Y$ satisfies the inequality

$$\left\|D_kf(x,y)\right\| \leq \varepsilon$$

for all $x, y \in X$, then there exists a unique quadratic mapping $Q_1 : X \to Y$ which satisfies Equation (4) and the inequality

$$\left\|f(x) - \frac{f(0)}{k+1} - Q_1(x)\right\| \leq \frac{\varepsilon}{2(k^2-1)}$$

for all $x \in X$. Furthermore, for each fixed $x \in X$ if $f(tx)$ is continuous for all $t \in \mathbf{R}$, then $f(tx) = t^2f(x)$ for all $t \in \mathbf{R}$.

Proof. Taking $\psi(x, y) = \varepsilon$ and applying Theorem 3.1, we lead to the approximation.

In the last part, we consider *a singular case* $k = -1$, which is not investigated in Theorems 3.1 and 3.2 concerning the stability of the functional equation (4).

Theorem 3.6. Let $\psi : X^2 \to [0, \infty)$ be a function such that

$$\sum_{i=0}^{\infty}\frac{\psi(2^ix, 2^iy)}{4^i} < \infty \tag{30}$$

for all $x, y \in X$. If a mapping $f : X \to Y$ satisfies the inequality

$$\|D_{-1}f(x,y)\| \leq \psi(x,y) \tag{31}$$

for all $x, y \in X$, then there exists a unique quadratic mapping $Q_1 : X \to Y$ which satisfies the equation $D_{-1}Q_1(x, y) = 0$ and the inequality

$$\left\|f(x) - \frac{2f(0)}{3} - Q_1(x)\right\| \leq \frac{1}{8}\sum_{i=0}^{\infty}\frac{\psi(2^i x, 2^i x)}{4^i} + \frac{1}{4}\sum_{i=0}^{\infty}\frac{\psi(2^{i+1}x, 0)}{4^{i+1}} \tag{32}$$

for all $x \in X$, where $\|f(0)\| \leq \frac{\psi(0,0)}{4}$. The mapping Q_1 is defined by

$$Q_1(x) = \lim_{n\to\infty}\frac{f(2^n x)}{4^n} \tag{33}$$

for all $x \in X$.

Proof. Letting $x = y := 0$ in (31), we get $\|f(0)\| \leq \frac{\psi(0,0)}{4}$. Putting $y := 0$ in (31) and dividing by 2, we obtain

$$\|f(-x) - f(x) - 2f(0)\| \leq \frac{\psi(x,0)}{2}$$

for all $x \in X$. Setting $y := x$ in (31), one has

$$\|f(-2x) + f(2x) - 8f(x) + 2f(0)\| \leq \psi(x,x)$$

for all $x \in X$. Combining two inequalities above and then letting $\bar{f}(x) := f(x) - \frac{2f(0)}{3}$, we lead to a functional inequality

$$\left\|\frac{\bar{f}(2x)}{4} - \bar{f}(x)\right\| \leq \frac{1}{8}\psi(x,x) + \frac{1}{4^2}\psi(2x,0) \tag{34}$$

for all $x \in X$. Using the induction argument on a positive integer n, we may obtain that

$$\left\|\bar{f}(x) - \frac{\bar{f}(2^n x)}{4^n}\right\| \leq \frac{1}{8}\sum_{i=0}^{n-1}\frac{\psi(2^i x, 2^i x)}{4^i} + \frac{1}{4}\sum_{i=0}^{n-1}\frac{\psi(2^{i+1}x, 0)}{4^{i+1}} \tag{35}$$

for all $x \in X$. The remaining proof of this theorem follows similarly from the corresponding part of Theorem 3.1.

The following theorem is an alternative stability result concerning the stability of the functional equation (4) with k = -1.

Theorem 3.7. Let $\psi : X^2 \to [0, \infty)$ be a function such that

$$\sum_{i=1}^{\infty} 4^i \psi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty \tag{36}$$

for all $x, y \in X$. If a mapping $f : X \to Y$ satisfies the inequality

$$\|D_{-1}f(x,y)\| \leq \psi(x,y) \tag{37}$$

for all $x, y \in X$, then there exists a unique quadratic mapping $Q_2 : X \to Y$ which satisfies the equation $D_{-1}Q_2(x, y) = 0$ and the inequality

$$\|f(x) - Q_2(x)\| \leq \frac{1}{8}\sum_{i=1}^{\infty} 4^i \psi\left(\frac{x}{2^i}, \frac{x}{2^i}\right) + \frac{1}{4}\sum_{i=1}^{\infty} 4^{i-1}\psi\left(\frac{x}{2^{i-1}}, 0\right) \tag{38}$$

for all $x \in X$. The mapping Q_2 is defined by

$$Q_2(x) = \lim_{n\to\infty} 4^n f(\frac{x}{2^n}) \tag{39}$$

for all $x \in X$.

Corollary 3.8. Let $\varepsilon_1 \geq 0$, $\varepsilon_2 \geq 0$ and p, q be real numbers such that either $0 < p, q < 2$ or $p, q > 2$. If a mapping $f : X \to Y$ satisfies the inequality

$$\|D_{-1}f(x, y)\| \leq \varepsilon_1\|x\|^p + \varepsilon_2\|y\|^q$$

for all $x, y \in X$, then there exists a unique quadratic mapping $Q_i : X \to Y$ (i = 1,2) which satisfies the equation $D_{-1}Q_i(x, y) = 0$ and inequality

$$\begin{cases} \|f(x) - Q_1(x)\| \leq \dfrac{\varepsilon_1(2+2^p)\|x\|^p}{4(2^2-2^p)} + \dfrac{\varepsilon_2\|x\|^q}{2(2^2-2^q)}, \text{if} \quad 0 < p, q < 2; \\ \|f(x) - Q_2(x)\| \leq \dfrac{\varepsilon_1(2+2^p)\|x\|^p}{4(2^p-2^2)} + \dfrac{\varepsilon_2\|x\|^q}{2(2^q-2^2)}, \text{if} \quad p, q > 2. \end{cases}$$

for all $x \in X$. Furthermore, for each fixed $x \in X$ if $f(tx)$ is continuous for all $t \in \mathbf{R}$, then $f(tx) = t^2f(x)$ for all $t \in \mathbf{R}$.

4 Hyers-Ulam stability of (4) in non-archimedean spaces

In this section, let X be a vector space and Y a complete non-Archimedean space. We recall that a field $\mathbf{K}$, equipped with a function (non-Archimedean absolute value, valuation) $|\cdot|$ from $\mathbf{K}$ into $[0, \infty)$, is called a non-Archimedean field if the function $|\cdot|: \mathbf{K} \to [0, \infty)$ satisfies the following conditions:

(1) $|r| = 0$ if and only if $r = 0$;
(2) $|rs| = |r||s|$;
(3) the strong triangle inequality, namely, $|r + s| \leq \max\{|r|, |s|\}$ for all $r, s \in \mathbf{K}$.

Clearly, $|1| = 1 = |-1|$ and $|n| \leq 1$ for all nonzero integer n.

Let Y be a vector space over the non-Archimedean field $\mathbf{K}$ with a non-trivial non-Archimedean valuation $|\cdot|$. A function $\|\cdot\|: Y \to [0, \infty)$ is called a non-Archimedean norm (valuation) if it satisfies the following conditions:

(1) $\|x\| = 0$ if and only if $x = 0$;
(2) $|rx| = |r|\ \|x\|$ for all $x \in Y$ and all $r \in \mathbf{K}$;
(3) the strong triangle inequality, namely,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}$$

for all $x, y \in Y$.

In this case, the pair $(Y, \|\cdot\|)$ is called a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent. It follows from the strong triangle inequality that

$$\|x_n - x_m\| \leq \max\left\{\left\|x_{j+1} - x_j\right\| : m \leq j < n-1\right\}$$

for all $x_n, x_m \in Y$ and all $m,n \in \mathbf{N}$ with $n > m$. Therefore, a sequence $\{x_n\}$ is a Cauchy sequence in non-Archimedean space $(Y, \|\cdot\|)$ if and only if the sequence $\{x_{n+1}\text{-}x_n\}$ converges to zero in the space $(Y, \|\cdot\|)$. Now, we will investigate the generalized the Hyers-Ulam stability problem for the functional equation (4) in a complete non-Archimedean space Y.

Theorem 4.1. Let $\psi : X^2 \to [0, \infty)$ be a function such that

$$\begin{gathered}\Psi_1(x) := \lim_{n\to\infty} \max\left\{\frac{\psi(k^i x, 0)}{|k|^{2i}} : 0 \leq i < n\right\} < \infty, \\ \lim_{n\to\infty} \frac{\psi(k^n x, k^n y)}{|k|^{2n}} = 0\end{gathered} \tag{40}$$

for all $x, y \in X$. If a mapping $f : X \to Y$ satisfies the inequality

$$\left\|D_k f(x, y)\right\| \leq \psi(x, y) \tag{41}$$

for all $x, y \in X$, then there exists a quadratic mapping $Q_1 : X \to Y$ which satisfies Equation (4) and the inequality

$$\left\|f(x) - Q_1(x)\right\| \leq \frac{1}{|2|\,|k|^2}\Psi_1(x) \tag{42}$$

for all $x \in X$, where $\left\|f(0)\right\| \leq \dfrac{\psi(0,0)}{\left|2k(k-1)\right|}$. The mapping Q_1 is defined by

$$Q_1(x) = \lim_{n\to\infty} \frac{f(k^n x)}{k^{2n}} \tag{43}$$

for all $x \in X$. Moreover, if

$$\lim_{l\to\infty} \frac{\Psi_1(k^l x)}{|k|^{2l}} = \lim_{l\to\infty}\lim_{n\to\infty} \max\left\{\frac{\psi(k^j x, 0)}{|k|^{2j}} : l \leq j < l+n\right\} = 0$$

for all $x \in X$, then Q_1 is a unique quadratic mapping satisfying (42).

Proof. Letting $x = y := 0$ in (41), we get $f(0) = 0$ because $\psi(0,0) = 0$ by the condition (40). Putting $y := 0$ in (41) and dividing by $2|k|^2$, we obtain

$$\left\|\frac{f(kx)}{k^2} - f(x)\right\| \leq \frac{\psi(x, 0)}{2|k|^2} \tag{44}$$

for all $x \in X$, where $|k| \leq 1$ is a non-Archimedean valuation. Replacing x by $k^n x$ in (44) and dividing by $|k|^{2n}$,

$$\left\|\frac{f(k^{n+1}x)}{k^{2n+2}} - \frac{f(k^n x)}{k^{2n}}\right\| \leq \frac{\psi(k^n x, 0)}{2|k|^{2n+2}} \tag{45}$$

for all $x \in X$. Since the right-hand side of the inequality (45) tends to 0 as $n \to \infty$, a sequence $\{\dfrac{f(k^n x)}{k^{2n}}\}$ is Cauchy in the complete non-Archimedean space Y. Therefore, we may define a mapping $Q_1 : X \to Y$ as

$$Q_1(x) = \lim_{n\to\infty} \frac{f(k^n x)}{k^{2n}}$$

for all $x \in X$. Using the induction argument and the strong triangle inequality, we may obtain that

$$\left\| f(x) - \frac{f(k^n x)}{k^{2n}} \right\| \le \frac{1}{2|k|^2} \max\left\{ \frac{\psi(k^i x, 0)}{|k|^{2i}} : 0 \le i < n \right\} \tag{46}$$

for all $x \in X$. Letting $n \to \infty$ in (46), we lead to the approximation (42).

Next, we have to show that Q_1 satisfies Equation (4). Replacing x, y by $k^n x$, $k^n y$ in (41) and dividing by $|k|^{2n}$, it then follows that

$$\begin{aligned} \frac{1}{|k|^{2n}} \| f(k^n(kx+y)) + f(k^n(kx-y)) - kf(k^n(x+y)) - kf(k^n(x-y)) \\ -2k(k-1)f(k^n x) + 2(k-1)f(k^n y) \| \le \frac{1}{|k|^{2n}} \psi(k^n x, k^n y) \end{aligned}$$

for all $x, y \in X$. Taking the limit as $n \to \infty$, we see from (40) and (43) that

$$\begin{aligned} &Q_1(kx+y) + Q_1(kx-y) \\ &= kQ_1(x+y) + kQ_1(x-y) + 2k(k-1)Q_1(x) - 2(k-1)Q_1(y) \end{aligned}$$

for all $x, y \in X$. Therefore, the mapping Q_1 satisfies Equation (4) and so it is quadratic by Theorem 2.2.

Moreover, to prove the uniqueness of the quadratic mapping Q_1 satisfying the inequality (42), let us assume that there exists a quadratic mapping $Q_1' := X \to Y$ which satisfies the inequality (42). Then, we have $Q_1(k^l x) = k^{2l} Q_1(x)$ and $Q_1'(k^l x) = k^{2l} Q_1'(x)$ for all $x \in X$ and all $l \in$ N. Hence, it follows from (42) that for all $x \in X$

$$\begin{aligned} \| Q_1(x) - Q_1'(x) \| &= \frac{1}{|k|^{2l}} \left\| Q_1(k^l x) - Q_1'(k^l x) \right\| \\ &\le \frac{1}{|k|^{2l}} \max\left\{ \left\| Q_1(k^l x) - f(k^l x) \right\|, \left\| f(k^l x) - Q_1'(k^l x) \right\| \right\} \\ &\le \frac{1}{|k|^2} \lim_{n\to\infty} \max\left\{ \frac{\psi(k^{l+i} x, 0)}{|k|^{2(l+1)}} : 0 \le i < n \right\} \\ &= \frac{1}{|k|^2} \lim_{n\to\infty} \max\left\{ \frac{\psi(k^j x, 0)}{|k|^{2j}} : l \le j < l+n \right\} \\ &= \frac{1}{|k|^2} \frac{\Psi_1(k^l x)}{|k|^{2l}}, \ \forall l \in \mathrm{N}, \end{aligned}$$

which tends to zero as $l \to \infty$. This completes the proof of the theorem.

Theorem 4.2. Let $\psi : X^2 \to [0, \infty)$ be a function such that

$$\begin{aligned} \Psi_2(x) := \lim_{n\to\infty} \max\left\{ |k|^{2i} \psi\left(\frac{x}{k^i}, 0\right) : 1 \le i \le n \right\} < \infty \\ \lim_{n\to\infty} |k|^{2n} \psi\left(\frac{x}{k^n}, \frac{y}{k^n}\right) = 0 \end{aligned} \tag{47}$$

for all $x, y \in X$. If a mapping $f : X \to Y$ satisfies the inequality

$$\|D_k f(x, y)\| \le \psi(x, y) \tag{48}$$

for all $x, y \in X$, then there exists a quadratic mapping $Q_2 : X \to Y$ which satisfies Equation (4) and the inequality

$$\left\|f(x) - \frac{f(0)}{k+1} - Q_2(x)\right\| \le \frac{1}{|2|\,|k|^2}\Psi_2(x) \tag{49}$$

for all $x \in X$, where $\|f(0)\| \le \frac{\psi(0,0)}{|2k(k-1)|}$. The mapping Q_2 is defined by

$$Q_2(x) = \lim_{n\to\infty} k^{2n} f\left(\frac{x}{k^n}\right) \tag{50}$$

for all $x \in X$. Moreover, if $\lim_{l\to\infty}|k|^{2l}\Psi_2\left(\frac{x}{k^l}\right) = 0$ for all $x \in X$, then Q_2 is a unique quadratic mapping satisfying (49).

Proof. Letting $x = y := 0$ in (48), we get

$$\|f(0)\| \le \frac{\psi(0,0)}{|2k(k-1)|},$$

where $|2k(k-1)| \le 1$ is a non-Archimedean valuation. Putting $y = 0$ in (48), one has

$$\begin{aligned} &|2|\,\|f(kx) - k^2 f(x) + (k-1)f(0)\| \le \psi(x, 0), \\ &\text{i.e.,}\quad \left\|\bar{f}(x) - k^2\bar{f}\left(\frac{x}{k}\right)\right\| \le \frac{1}{|2|\,|k|^2}|k|^2\psi\left(\frac{x}{k}, 0\right) \end{aligned} \tag{51}$$

for all $x \in X$, where $\bar{f}(x) = f(x) - \frac{f(0)}{k+1}$. Replacing x by $\frac{x}{k^n}$in (51) and multiplying $|k|^{2n}$, we have

$$\left\|k^{2n} f\left(\frac{x}{k^n}\right) - k^{2n+2} f\left(\frac{x}{k^{n+1}}\right)\right\| \le \frac{1}{|2|}|k|^{2n+2}\psi\left(\frac{x}{k^{n+1}}, 0\right) \tag{52}$$

for all $x \in X$. Since the right-hand side of the inequality (52) tends to 0 as $n \to \infty$, the sequence $\{k^{2n} f(\frac{x}{k^n})\}$ is Cauchy. Therefore, one may define a mapping $Q_2 : X \to Y$ by

$$Q_2(x) = \lim_{n\to\infty} k^{2n} f\left(\frac{x}{k^n}\right)$$

for all $x \in X$. Using induction on positive integers n, we obtain that

$$\left\|f(x) - k^{2n} f\left(\frac{x}{k^n}\right)\right\| \le \frac{1}{|2|\,|k|^2}\max\left\{|k|^{2i}\psi\left(\frac{x}{k^i}, 0\right) : 1 \le i \le n\right\} \tag{53}$$

for all $x \in X$. Letting $n \to \infty$ in (53), we arrive at the estimation (49).

The remaining assertion is similar to that of Theorem 4.1.

Corollary 4.3. Let $\varepsilon_1 \ge 0$, $\varepsilon_2 \ge 0$ and p, q be real numbers such that either $0 < p, q < 2$ or $p, q > 2$. If a mapping $f : X \to Y$ satisfies the inequality

$$\|D_k f(x, y)\| \le \varepsilon_1\|x\|^p + \varepsilon_2\|y\|^q$$

for all $x,\ y \in X$, then there exists a unique quadratic mapping $Q_i : X \to Y$ (i = 1,2) which satisfies (4) and inequality

$$\begin{cases} \|f(x) - Q_1(x)\| \leq \dfrac{\varepsilon_1 \|x\|^p}{|2|\,|k|^2} \leq \dfrac{\varepsilon_1 \|x\|^p}{|2|\,|k|^p}, & \text{if} \quad p, q > 2; \\ \left\| f(x) - \dfrac{f(0)}{k+1} - Q_2(x) \right\| \leq \dfrac{\varepsilon_1 \|x\|^p}{|2|\,|k|^p} \leq \dfrac{\varepsilon_1 \|x\|^p}{|2|\,|k|^2}, & \text{if} \quad 0 < p, q < 2. \end{cases}$$

for all $x \in X$.

Proof. Taking $\psi(x,\ y) = \varepsilon_1 \|x\|^p + \varepsilon_2 \|y\|^q$ and applying Theorems 4.1 and 4.2, we obtain the desired approximations.

Corollary 4.4. Let ε be a nonnegative real number. If a mapping $f : X \to Y$ satisfies the inequality

$$\|D_k f(x, y)\| \leq \varepsilon$$

for all $x,\ y \in X$, then there exists a unique quadratic mapping $Q_1 : X \to Y$ which satisfies Equation (4) and the inequality

$$\left\| f(x) - \frac{f(0)}{k+1} - Q_1(x) \right\| \leq \frac{\varepsilon}{|2|}$$

for all $x \in X$.

We remark that stability results of the Euler-Lagrange-type quadratic equation (4) in normed spaces are very different from those of Equation (4) in non-Archimedean normed spaces, of which stability results in non-Archimedean normed spaces maybe come from the opposite direction to stability results in normed spaces in view of Corollaries 4.3 and 4.4.

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Authors' contributions
All authors carried out the proof. All authors conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

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