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Stability analysis for parametric generalized vector quasi-variational-like inequality problems

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Abstract

In this article, we consider a class of parametric generalized vector quasi-variational-like inequality problem (for short, (PGVQVLIP)) in Hausdorff topological vector spaces, where the constraint set K and a set-valued mapping T are perturbed by different parameters, and establish the nonemptiness and upper semicontinuity of the solution mapping S for (PGVQVLIP) under some suitable conditions. By virtue of the gap function, sufficient conditions for the H-continuity and B-continuity of the solution mapping S of (PGVQVLIP) are also derived. Moreover, examples are provided for illustrating the presented results.

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1 Introduction

In 1980, Giannessi [1] first introduced vector variational inequality problems in finite dimensional Euclidean spaces. Since Giannessi, vector variational inequalities were investigated by many authors in abstract spaces and widely applied to transportation, finance, and economics, mathematical physics, engineering sciences and many others (see, for instance, [2-18] and the reference therein).

The stability analysis of solution mappings for vector variational inequality problems is an important topic in optimization theory and applications. Especially, some authors have tried to discuss the upper and lower semi-continuity of solution mappings (see, for instance, [19-28] and the reference therein). Khanh and Luu [29] studied a parametric multi-valued quasi-variational inequalities and obtained the semi-continuity of the solution sets and approximate solution sets. Zhong and Huang [30] studied the solution stability of parametric weak vector variational inequalities in reflexive Banach spaces and obtained the lower semi-continuity of the solution mapping for the parametric weak vector variational inequalities with strictly C-pseudo-mapping and also proved the lower semi-continuity of the solution mapping by degree-theoretic method. Aussel and Cotrina [31] discussed the continuity properties of the strict and star solution mapping of a scalar quasi-variational inequality in Banach spaces. Zhao [32] obtained a sufficient and necessary condition ($H1$) for the Hausdorff lower semi-continuity of the solution mapping to a parametric optimization problems. Under mild assumptions, Kien [33] also obtained the sufficient and necessary condition ($H1$) for the Hausdorff lower semi-continuity of the solution mapping to a parametric optimization problems. By using a condition (Hg) similar to it given in [32], Li and Chen [21] proved that (Hg) is also sufficient for the Hausdorff lower semi-continuity of the solution mapping to a class of weak vector variational inequality.

Very recently, Chen et al. [34] further studied the Hausdorff lower semi-continuity of the solution mapping to the parametric weak vector quasi-variational inequality in Hausdorff topological vector spaces. Zhong and Huang [35] also derived a sufficient and necessary condition (Hg)' for the Hausdorff lower semi-continuity and Hausdorff continuity of the solution mapping to a parametric weak vector variational inequalities in reflexive Banach spaces. Lalitha and Bhatia [36] presented various sufficient conditions for the upper and lower semi-continuity of solution sets as well as the approximate solution sets to a parametric quasi-variational inequality of the Minty type.

Motivated and inspired by the studies reported in [29-37], the aim of this article is to investigate a class of (PGVQVLIP) in Hausdorff topological vector spaces, where the constraint set K and a set-valued mapping T are perturbed by different parameters. We establish the nonemptiness and upper semi-continuity of the solution mapping for (PGVQVLIP) under some suitable conditions. By virtue of the gap function, sufficient conditions for the H-continuity and B-continuity of the solution mapping of the (PGVQVLIP) are also derived. Moreover, some examples are provided for illustrating the presented results. The results presented in this article develop, extend and improve the some main results given in [29-35,37].

This article is organized as follows. In Section 2, we introduce the problem (PGVQVLIP), recall some basic definitions and some of their properties. In Section 3, we investigate the sufficient conditions for the upper semi-continuity nonemptiness and continuity of the solution mapping for (PGVQVLIP) in Hausdorff topological vector spaces.

2 Preliminaries

Throughout this article, let $\vee$ and $\wedge$ (the spaces of parameters) be two Hausdorff topological vector spaces and X, Y be two locally convex Hausdorff topological vector spaces. Let $L(X, Y)$ be the set of all linear continuous operators from X into Y, denoted $\langle t, x\rangle$ by the value of a linear operator $t \in L(X, Y)$ at $x \in X$, and $C : X \to 2^Y$ be a set-valued mapping such that $C(x)$ is a proper closed convex cone for all $x \in X$ with $\text{int } C(x) \neq \emptyset$. Let $T : X \times \vee \to 2^{L(X,Y)}$ and $K : \wedge \to 2^X$ be two set-valued mappings, $\eta : X \times X \to X$ and $\varphi : X \times X \to Y$ be two vector-valued mappings. We always assume that $\langle\cdot,\cdot\rangle$ is continuous and 2^X denotes the family of all nonempty subsets of X.

We consider the following *parametric generalized vector quasi-variational-like inequality problem* (for short, (PGVQVLIP)): Find $x \in K(\lambda)$ such that

$$\langle T(x,\mu), \eta(y,x)\rangle + \phi(y,x) \nsubseteq -\text{int } C(x), \quad \forall y \in K(\lambda), \tag{2.1}$$

where $\langle T(x, \mu), \eta(y, x)\rangle + \varphi(y, x) = \cup_{\xi \in T(x,\mu)} \langle \xi, \eta(y, x)\rangle + \varphi(y, x)$.

It is easy to see that (PGVQVLIP) is equivalent to find $x \in K(\lambda)$ and $\xi \in T(x, \mu)$ such that

$$\langle \xi, \eta(y,x)\rangle + \phi(y,x) \notin -\text{int } C(x), \quad \forall y \in K(\lambda). \tag{2.2}$$

Special cases of the problem (2.1) are as follows:

(I) If T is a single-valued mapping, then the problem (2.1) is reduced to the following *parametric vector quasi-variational-like inequality problem* (for short, (PVQVLIP)): Find $x \in K(\lambda)$ such that

$$\langle t(x, \mu), \eta(y, x)\rangle + \phi(y, x) \notin -\text{int } C(x), \quad \forall y \in K(\lambda), \tag{2.3}$$

where $t : X \times \vee \to L(X, Y)$ is a vector-valued mapping.

(II) If, for each pair of parameters $(\lambda, \mu) \in \wedge \times \vee$, $\eta(y, x) = y - x$, $T(x, \mu) = T(x)$ and $\varphi(y, x) = 0$ for all $x, y \in K(\lambda) = K$, then the problem (2.1) is reduced to the following *vector quasi-variational inequality problem*: Find $x \in K$ such that

$$\langle T(x), y - x\rangle \not\subseteq -\text{int } C(x), \quad \forall y \in K, \tag{2.4}$$

where K is a nonempty subset of X, $T : X \to 2^{L(X,Y)}$, which has been studied by Ansari et al. [4].

(III) If, for each pair of parameters $(\lambda, \mu) \in \wedge \times \vee$, $C(x) = C$, $\eta(y, x) = y - x$ and $\varphi(y, x) = 0$ for all $x, y \in K(\lambda)$, where C is a proper closed convex cone, then the problem (2.1) is reduced to the following *vector quasi-variational inequality problem:* Find $x \in K(\lambda)$ such that

$$\langle T(x, \mu), y - x\rangle \not\subseteq -\text{int } C, \quad \forall y \in K(\lambda), \tag{2.5}$$

which has been studied by Zhong and Huang [30].

(IV) If, for each pair of parameters $(\lambda, \mu) \in \vee \times \wedge$, $\eta(y, x) = 0$ for all $x, y \in K(\lambda)$, then the problem (2.1) is reduced to the following *vector equilibrium problem:* Find $x \in K(\lambda)$ such that

$$\phi(y, x) \notin -\text{int } C(x), \quad \forall y \in K(\lambda). \tag{2.6}$$

(V) If, for each pair of parameters $(\lambda, \mu) \in \vee \times \wedge$, $\eta(y, x) = 0$ and $\varphi(y, x) = f(y)-f(x)$ for all $x, y \in K(\lambda)$, where $f : X \to Y$, then the problem (2.1) is reduced to the following *vector optimization problem*:

$$\min_{y\in K(\lambda)} f(y). \tag{2.7}$$

For each pair of parameters $(\lambda, \mu) \in \wedge \times \vee$, we denote the solutions set of the problem (2.1) by $S(\lambda, \mu)$, i.e.,

$$S(\lambda, \mu) = \{x \in K(\lambda) : \langle T(x, \mu), \eta(y, x)\rangle + \phi(y, x) \not\subseteq -\text{int } C(x),\ \forall y \in K(\lambda)\}.$$

So, $S : \wedge \times \vee \to 2^X$ is a set-valued mapping, which is called the *solution mapping* of the problem (2.1).

We first recall some definitions and lemmas which are needed in our main results.

Definition 2.1 [9,10]. The *nonlinear scalarization function* $\xi_e : X \times Y \to R$ is defined by

$$\xi_e(x, y) = \inf\{z \in R : y \in ze(x) - C(x)\}, \quad \forall (x, y) \in X \times Y,$$

where $e : X \to Y$ is a vector-valued mapping and $e(x) \in \text{int}C(x)$ for all $x \in X$.

Example 2.1 [34]. If $Y = R^n$, $e(x) = e$ and $C(x) = R^n_+$ for any $x \in X$, where $e = (1, 1, \ldots, 1)^T \in \text{int } R^n_+$, then the function $\xi_e(x,y) = \max_{1\le i\le n}\{y_i\}$ is a nonlinear scalarization function for all $x \in X$, $y = (y_1, y_2,\ldots,y_n)^T \in Y$.

Definition 2.2 [34,35]. Let Γ be a Hausdorff topological space and X be a locally convex Hausdorff topological vector space. A set-valued mapping $F : \Gamma \to 2^X$ is said to be:

(1) *upper semi-continuous in the sense of Berge* (for short, (B-u.s.c)) at $\gamma_0 \in \Gamma$ if, for each open set V with $F(\gamma_0) \subset V$, there exists $\delta > 0$ such that

$$F(\gamma) \subset V, \quad \forall \gamma \in B(\gamma_0, \delta);$$

(2) *lower semi-continuous in the sense of Berge* (for short, (B-l.s.c)) at $\gamma_0 \in \Gamma$ if, for each open set V with $F(\gamma_0) \cap V \neq \emptyset$, there exists $\delta > 0$ such that

$$F(\gamma) \cap V \neq \emptyset, \quad \forall \gamma \in B(\gamma_0, \delta);$$

(3) *upper semi-continuous in the sense of Hausdorff* (for short, (H-u.s.c)) at $\gamma_0 \in \Gamma$ if, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$F(\gamma) \subset U(F(\gamma_0), \varepsilon), \quad \forall \gamma \in B(\gamma_0, \delta);$$

(4) *lower semi-continuous in the sense of Hausdorff* (for short, (H-l.s.c)) at $\gamma_0 \in \Gamma$ if, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$F(\gamma_0) \subset U(F(\gamma), \varepsilon), \quad \forall \gamma \in B(\gamma_0, \delta);$$

(5) *closed* if the graph of F is closed, i.e., the set $G(F) = \{(\gamma, x) \in \Gamma \times X : x \in F(\gamma)\}$ is closed in $\Gamma \times X$.

We say that F is H-l.s.c (resp., H-u.s.c, B-l.s.c, B-u.s.c) on Γ if it is H-l.s.c (resp., H-u.s.c, B-l.s.c, B-u.s.c) at each $\gamma \in \Gamma$. F is called continuous (resp., H-continuous) on Γ if it is both B-l.s.c (resp., H-l.s.c) and B-u.s.c (resp., H-u.s.c) on Γ.

By [9, Theorem 2.1], [34, Propositions 2.2 and 2.3], and [35, Lemma 2.3], the nonlinear scalarization function $\xi_e(\cdot,\cdot)$ has the following properties.

Proposition 2.1. Let $e : X \to Y$ be a continuous selection from the set-valued mapping int$C(\cdot)$. For any $x \in X$, $y \in Y$, and $r \in R$, the following hold:

(1) If the mappings $C(\cdot)$ and $Y \setminus \text{int}C(\cdot)$ are B-u.s.c on X, then $\xi_e(\cdot,\cdot)$ is continuous on$X \times Y$;
(2) The mapping $\xi_e(x, \cdot) : Y \to R$ is convex;
(3) $\xi_e(x, y) < r \Leftrightarrow y \in re(x)\text{-int}C(x)$;
(4) $\xi_e(x, y) \geq r \Leftrightarrow y \notin re(x)\text{-int}C(x)$;
(5) $\xi_e(x, re(x)) = r$, especially, $\xi_e(x, 0) = 0$.

Proposition 2.2 [9]. Let X, Y be two locally convex Hausdorff topological vector spaces and $C : X \to 2^Y$ be a set-valued mapping such that, for each $x \in X$, $C(x)$ is a proper closed convex cone in Y with $\text{int}\, C(x) \neq \emptyset$. Let $e : X \to Y$ be a continuous selection from the set-valued mapping int$C(\cdot)$. Define a set-valued mapping $V : X \to 2^Y$ by $V(x) = Y \setminus \text{int}C(x)$ for all $x \in X$. Then the following hold:

(1) if $V(\cdot)$ is B-u.s.c on X, then $\xi_e(\cdot,\cdot)$ is upper semicontinuous on $X \times Y$;
(2) if $C(\cdot)$ is B-u.s.c on X, then $\xi_e(\cdot,\cdot)$ is lower semicontinuous on $X \times Y$.

Definition 2.3 [38]. A set $B \subset X$ is said to be *balanced if* $\rho B \subset B$ for any $\rho \in R$ with $|\rho| \leq 1$.

Definition 2.4. Let $t : X \times \vee \to L(X, Y)$ be a vector-valued mapping and $T : X \times \vee \to 2^{L(X,Y)}$ be a set-valued mapping.

(1) The mapping t is called a *selection* of T on $X \times \vee$ if

$$t(x, \mu) \in T(x, \mu), \quad \forall (x, \mu) \in X \times \vee;$$

(2) The mapping t is called a *continuous selection* of T on $X \times \vee$ if t is a selection of T and continuous on $X \times \vee$.

Definition 2.5. For any pair $(\lambda, \mu) \in \vee \times \wedge$ of parameters and $x \in K(\lambda)$, the set-valued mapping $T : X \times \vee \to 2^{L(X,Y)}$ is said to be:

(1) *weakly* $(\eta, \varphi, C(x))$*-pseudo-mapping* on $K(\lambda)$ if, for any $y \in K(\lambda)$ and $\xi' \in T(x, \mu)$, $\xi'' \in T(y, \mu)$,

$$\langle \xi', \eta(y, x) \rangle + \phi(y, x) \notin -\text{int } C(x) \Rightarrow \langle \xi'', \eta(y, x) \rangle + \phi(y, x) \notin -\text{int } C(x);$$

(2) $(\eta, \varphi, C(x))$*-pseudo-mapping* on $K(\lambda)$ if, for any $y \in K(\lambda)$ and $\xi' \in T(x, \mu)$, $\xi'' \in T(y, \mu)$,

$$\langle \xi', \eta(y, x) \rangle + \phi(y, x) \notin -\text{int } C(x) \Rightarrow \langle \xi'', \eta(y, x) \rangle + \phi(y, x) \in C(x);$$

(3) *strictly* $(\eta, \varphi, C(x))$*-pseudo-mapping* on $K(\lambda)$ if, for any $y \in K(\lambda)$ and $\xi' \in T(x, \mu)$, $\xi'' \in T(y, \mu)$,

$$\langle \xi', \eta(y, x) \rangle + \phi(y, x) \notin -\text{int } C(x) \Rightarrow \langle \xi'', \eta(y, x) \rangle + \phi(y, x) \in \text{int } C(x).$$

Remark 2.1. If $\eta(y, x) = y - x$, $C(x) = C$, and $\varphi(y, x) = 0$ for all $x, y \in K(\lambda)$, then $(\eta, \varphi, C(x))$-pseudo-mapping (resp., strictly $(\eta, \varphi, C(x))$-pseudo-mapping) is reduced to C-pseudo-mapping (resp., strictly C-pseudo-mapping) in [30].

Remark 2.2. It is easy to see that every strictly $(\eta, \varphi, C(x))$-pseudo-mapping is an $(\eta, \varphi, C(x))$-pseudo-mapping and weakly $(\eta, \varphi, C(x))$-pseudo-mapping. Moreover, every $(\eta, \varphi, C(x))$-pseudo-mapping is also a weakly $(\eta, \varphi, C(x))$-pseudo-mapping.

Definition 2.6. Let K be a nonempty subset of a Hausdorff topological vector space X. A set-valued mapping $F : K \to 2^X$ is called a *KKM mapping* if, for each finite subset $\{x_1, x_2, \ldots, x_m\}$ of K, $\text{co}\{x_1, x_2, \ldots, x_m\} \subseteq \cup_{i=1}^{m} F(x_i)$, where co denotes the convex hull.

Definition 2.7. Let $\eta : X \times X \to X$ and $\varphi : X \times X \to Y$ be two vector-valued mapping.

(1) $\eta(x, y)$ is said to be *affine* with respect to the first argument if, for any $y \in X$,

$$\eta(\iota x_1 + (1 - \iota)x_2, y) = \iota\eta(x_1, y) + (1 - \iota)\eta(x_2, y), \quad \forall x_1, x_2 \in X, \iota \in R;$$

(2) $\varphi(x, y)$ is said to be $C(x)$*-convex* with respect to the first argument if, for any $y \in X$,

$$\phi(\iota x_1 + (1-\iota)x_2, y) \in \iota\phi(x_1, y) + (1-\iota)\phi(x_2, y) - C(y), \quad \forall x_1, x_2 \in X, \iota \in [0,1].$$

Lemma 2.1 [37]. Let K be a nonempty subset of a Hausdorff topological vector space X and $F: K \to 2^X$ be a KKM mapping such that, for all $y \in K$, $F(y)$ is closed and $F(y^*)$ is compact for some $y^* \in K$. Then $\cap_{y\in K}F(y) \neq \emptyset$.

Lemma 2.2 [38]. For each neighborhood V of 0_X, there exists a balanced open neighborhood $\sqcup$ of 0_X such that $\sqcup + \sqcup + \sqcup \subset V$.

Lemma 2.3 [39]. Let Γ be a Hausdorff topological space, X be a locally convex Hausdorff topological vector space and $F: \Gamma \to 2^X$ be a set-valued mapping. Then the following hold:

(1) F is B-l.s.c at $\gamma_0 \in \Gamma$ if and only if, for any net $\{\gamma_\alpha\} \subseteq \Gamma$ with $\gamma_\alpha \to \gamma_0$ and $x_0 \in F(\gamma_0)$, there exists a net $\{x_\alpha\} \subseteq X$ with $x_\alpha \in F(\gamma_\alpha)$ for all α such that $x_\alpha \to x_0$;
(2) If F is compact-valued, then F is B-u.s.c at $\gamma_0 \in \Gamma$ if and only if, for any net $\{\gamma_\alpha\} \subseteq \Gamma$ with $\gamma_\alpha \to \gamma_0$ and $\{x_\alpha\} \subseteq X$ with $x_\alpha \in F(\gamma_\alpha)$ for all α, there exists $x_0 \in F(\gamma_0)$ and a subnet $\{x_\beta\}$ of $\{x_\alpha\}$ such that $x_\beta \to x_0$;
(3) If F is B-u.s.c and closed-valued, then F is closed. Conversely, if F is closed and X is compact, then F is B-u.s.c.

Lemma 2.4 [40]. Let Γ be a Hausdorff topological space, X be a locally convex Hausdorff topological vector space, $F: \Gamma \to 2^X$ be a set-valued mapping and $\gamma_0 \in \Gamma$ be a given point. Then the following hold:

(1) If F is B-u.s.c at γ_0, then F is H-u.s.c at γ_0. Conversely, if F is H-u.s.c at γ_0 and $F(\gamma_0)$ is compact, then F is B-u.s.c at γ_0;
(2) If F is H-l.s.c at γ_0, then F is B-l.s.c at γ_0. Conversely, if F is B-l.s.c at γ_0 and $cl(F(\gamma_0))$ is compact, then F is H-l.s.c at γ_0.

3 Main results

In this section, we investigate the stability of solutions of (PGVQVLIP), that is, the upper and lower semi-continuity of the solution mapping $S(\lambda, \mu)$ for (PGVQVLIP) corresponding to a pair (λ, μ) of parameters in Hausdorff topological vector spaces.

Theorem 3.1. Let $T: X \times \vee \to 2^{L(X,Y)}$ be a set-valued mapping with nonempty values, $C: X \to 2^Y$ be a set-valued mapping such that, for each $x \in X$, $C(x)$ is a pointed closed and convex cone in Y and $\text{int}\, C(x) \neq \emptyset$, $\eta: X \times X \to X$ and $\varphi: X \times X \to Y$ be two vector-valued mappings. Assume that the following conditions are satisfied:

(a) $\eta(x, x) = 0$ and $\varphi(x, x) = 0$ for all $x \in X$;
(b) $\eta(x, y)$ is continuous and affine with respect to the first argument;
(c) $\varphi(x, y)$ is continuous and $C(x)$-convex with respect to the first argument;
(d) $T(x, \mu)$ is weakly $(\eta, \varphi, C(x))$-pseudo-mapping with respect to the first argument and B-u.s.c with compact-values on $X \times \vee$;
(e) there is a continuous selection $t$of T on $X \times \vee$;
(f) the mapping $W(\cdot) = Y \backslash -\text{int}C(\cdot)$ such that the graph $Gr(W)$ of W is weakly closed in $X \times Y$;
(g) $K: \wedge \to 2^X$ is B-u.s.c and B-l.s.c with weakly compact and convex-values.

Then the following hold:

(1) The solution mapping $S(\cdot,\cdot)$ is nonempty and closed on $\wedge \times \vee$;
(2) The solution mapping $S(\cdot,\cdot)$ is B-u.s.c on $\wedge \times \vee$.

Proof. For any $(\lambda, \mu) \in \wedge \times \vee$, we first show that $S(\lambda, \mu)$ is nonempty. Since T has a continuous selection t and $T(x, \mu)$ is weakly $(\eta, \varphi, C(x))$-pseudo-mapping with respect to the first argument on $X \times \vee$, we know that $t(x, \mu)$ is also weakly $(\eta, \varphi, C(x))$-pseudo-mapping with respect to the first argument on $X \times \vee$.

Now, we define two set-valued mappings $\Upsilon_1, \Upsilon_2 : K(\lambda) \to 2^{K(\lambda)}$ as follows: for all $y \in K(\lambda)$,

$$\Upsilon_1(y) = \{x \in K(\lambda) : \langle t(x, \mu), \eta(y, x)\rangle + \phi(y, x) \notin -\text{int } C(x)\}$$

and

$$\Upsilon_2(y) = \{x \in K(\lambda) : \langle t(y, \mu), \eta(y, x)\rangle + \phi(y, x) \notin -\text{int } C(x)\}.$$

Since $\eta(x, x) = 0$ and $\varphi\,(x, x) = 0$ for all $x \in X$, we have $y \in \Upsilon_1(y)$ and $y \in \Upsilon_2(y)$ and so $\Upsilon_1(y)$ and $\Upsilon_2(y)$ are nonempty for any $y \in K(\lambda)$. By virtue of the weakly $(\eta, \varphi, C(x))$-pseudo-mapping of $t(x, \mu)$ with respect to the first argument, we have

$$\Upsilon_1(y) \subseteq \Upsilon_2(y), \quad \forall y \in K(\lambda). \tag{3.1}$$

First, we assert that Υ_1 is a KKM mapping. Suppose that there exists a finite subset $\{y_1, y_2,\ldots,y_m\} \subseteq K(\lambda)$ such that

$$\text{co}\{y_1, y_2, \ldots, y_m\} \not\subseteq \bigcup_{i=1}^{m} \Upsilon_1(y_i).$$

Then there exists $\bar{y} \in \text{co}\{y_1, y_2, \ldots, y_m\}$, i.e., $\bar{y} = \sum_{i=1}^{m} \iota_i y_i \in K(\lambda)$ for some nonnegative real number ι_i with $1 \le i \le m$ and $\sum_{i=1}^{m} \iota_i = 1$ such that $\bar{y} \notin \bigcap_{i=1}^{m} \Upsilon_1(y_i)$. Moreover, $\bar{y} \notin \Upsilon_1(y_i)$ for $1 \le i \le m$. This yields that

$$\langle t(\bar{y}, \mu), \eta(y_i, \bar{y})\rangle + \phi(y_i, \bar{y}) \in -\text{int } C(\bar{y})$$

and so

$$\sum_{i=1}^{n} \iota_i(\langle t(\bar{y}, \mu), \eta(y_i, \bar{y})\rangle + \phi(y_i, \bar{y})) \in -\text{int } C(\bar{y}).$$

Taking into account (b) and (c) that

$$\left\langle t(\bar{y}, \mu), \eta\left(\sum_{i=1}^{m} \iota_i y_i, \bar{y}\right)\right\rangle + \phi\left(\sum_{i=1}^{m} \iota_i y_i, \bar{y}\right) = \langle t(\bar{y}, \mu), \eta(\bar{y}, \bar{y})\rangle + \phi(\bar{y}, \bar{y}) \in -\text{int } C(\bar{y}). \tag{3.2}$$

Again, from (a) together with (3.2), we have $0 \in -\text{int } C(\bar{y})$, which is a contradiction. Hence Υ_1 is a KKM mapping. It follows from (3.1) that Υ_2 is also a KKM mapping.

Second, we show that $\bigcap_{y \in K(\lambda)} \Upsilon_2(y) \neq \emptyset$. Taking any net $\{x_\beta\}$ of $\Upsilon_2(y)$ such that $\{x_\beta\}$ is weakly convergent to a point $\tilde{x} \in K(\lambda)$. Then, for each β, one has

$$\langle t(y, \mu), \eta(y, x_\beta)\rangle + \phi(y, x_\beta) \notin -\text{int } C(x_\beta).$$

From (b)-(e), it follows that

$$(x_\beta, \langle t(y,\mu), \eta(y,x_\beta)\rangle + \phi(y,x_\beta)) \to (\tilde{x}, \langle t(y,\mu), \eta(y,\tilde{x})\rangle + \phi(y,\tilde{x})) \in \mathrm{Gr}(W).$$

Consequently, we get

$$\langle t(y,\mu), \eta(y,\tilde{x})\rangle + \phi(y,\tilde{x}) \in Y \backslash (-\mathrm{int}\ C(\tilde{x})),$$

that is,

$$\langle t(y,\mu), \eta(y,\tilde{x})\rangle + \phi(y,\tilde{x}) \notin -\mathrm{int}\ C(\tilde{x}).$$

Therefore, $\tilde{x} \in \Upsilon_2(y)$ and so $\Upsilon_2(y)$ is weakly closed set for any $y \in K(\lambda)$. By the compactness of $K(\lambda)$, $\Upsilon_2(y)$ is weakly compact subset of $K(\lambda)$. From Lemma 2.1, it follows that

$$\bigcap_{y\in K(\lambda)} \Upsilon_2(y) \neq \emptyset,$$

i.e., there exists $\bar{x} \in K(\lambda)$ such that

$$\langle t(y,\mu), \eta(y,\bar{x})\rangle + \phi(y,\bar{x}) \notin -\mathrm{int}\ C(\bar{x}), \quad \forall y \in K(\lambda). \tag{3.3}$$

Third, we prove that $\bar{x} \in \bigcap_{y\in K(\lambda)} \Upsilon_1(y)$. For any $y \in K(\lambda)$, set $x_r = (1-r)\bar{x} + ry$ for all $r \in (0,1)$.

Then $x_r \in K(\lambda)$. So, from (3.3), we have

$$\langle t(x_r,\mu), \eta(x_r,\bar{x})\rangle + \phi(x_r,\bar{x}) \notin -\mathrm{int}\ C(\bar{x}), \quad \forall y \in K(\lambda). \tag{3.4}$$

Note that

$$\begin{aligned}
&\langle t(x_r,\mu), \eta(x_r,\bar{x})\rangle + \phi(x_r,\bar{x}) - r(\langle t(x_r,\mu), \eta(y,\bar{x})\rangle + \phi(y,\bar{x}))\\
&\quad = \langle t(x_r,\mu), \eta(x_r,\bar{x})\rangle + \phi(x_r,\bar{x}) - r(\langle t(x_r,\mu), \eta(y,\bar{x})\rangle + \phi(y,\bar{x}))\\
&\qquad - (1-r)(\langle t(x_r,\mu), \eta(\bar{x},\bar{x})\rangle + \phi(\bar{x},\bar{x}))\\
&\quad \in -\mathrm{int}\ C(\bar{x}).
\end{aligned}$$

It follows from (3.4) that

$$r(\langle t(x_r,\mu), \eta(y,\bar{x})\rangle + \phi(y,\bar{x})) \notin -\mathrm{int}\ C(\bar{x}), \quad \forall y \in K(\lambda),$$

and so

$$\langle t(x_r,\mu), \eta(y,\bar{x})\rangle + \phi(y,\bar{x}) \notin -\mathrm{int}\ C(\bar{x}), \quad \forall y \in K(\lambda).$$

Since t is continuous, we have

$$(x_r, \langle t(x_r,\mu), \eta(y,\bar{x})\rangle + \phi(y,\bar{x})) \to (\bar{x}, \langle t(\bar{x},\mu), \eta(y,\bar{x})\rangle + \phi(y,\bar{x})) \in \mathrm{Gr}(W)$$

as $r \to 0$. Therefore, by the weak closedness of Gr (W), we have

$$\langle t(\bar{x},\mu), \eta(y,\bar{x})\rangle + \phi(y,\bar{x}) \in Y \backslash (-\mathrm{int}\ C(\bar{x})),$$

that is,

$$\langle t(\bar{x},\mu), \eta(y,\bar{x})\rangle + \phi(y,\bar{x}) \notin -\mathrm{int}\ C(\bar{x}), \quad \forall y \in K(\lambda). \tag{3.5}$$

By the condition (e) and (3.5), there exist $\bar{x} \in K(\lambda)$ and $\xi \in T(\bar{x},\mu)$ such that

$$\langle \xi, \eta(y,\bar{x})\rangle + \phi(y,\bar{x}) \notin -\mathrm{int}\ C(\bar{x}), \quad \forall y \in K(\lambda) \tag{3.6}$$

and so $S(\lambda, \mu)$ is nonempty for any $(\lambda, \mu) \in \wedge \times \vee$.

Fourth, we show that the solution mapping $S(\cdot,\cdot)$ is B-u.s.c on $\wedge \times \vee$. Suppose that there exist $(\lambda_0, \mu_0) \in \wedge \times \vee$ such that $S(\cdot,\cdot)$ is not B-u.s.c at (λ_0, μ_0). Then there exist an open set V with $S(\lambda_0, \mu_0) \subset V$, a net $\{(\lambda_\alpha, \mu_\alpha)\}$ and $x_\alpha \in S(\lambda_\alpha, \mu_\alpha)$ such that $(\lambda_\alpha, \mu_\alpha) \to (\lambda_0, \mu_0)$ and $x_\alpha \notin V$ for all α. Since $x_\alpha \in S(\lambda_\alpha, \mu_\alpha)$, it follows that $x_\alpha \in K(\lambda_\alpha)$. By the condition (g), $K(\cdot)$ is B-u.s.c with compact-values at λ_0. Then there exists $x_0 \in K(\lambda_0)$ such that $x_\alpha \to x_0$ (here we may take a subnet $\{x_\beta\}$ of $\{x_\alpha\}$ if necessary). Suppose that $x_0 \notin S(\lambda_0, \mu_0)$, that is, for any $\bar{\xi} \in T(x_0, \mu_0)$, there exists $\bar{y} \in K(\lambda_0)$ such that

$$\langle \bar{\xi}, \eta(\bar{y}, x_0)\rangle + \phi(\bar{y}, x_0) \in -\text{int } C(x_0). \tag{3.7}$$

Since $x_\alpha \in S(\lambda_\alpha, \mu_\alpha)$, there exist $\xi'_\alpha \in T(x_\alpha, \mu_\alpha)$ such that

$$\langle \xi'_\alpha, \eta(z_\alpha, x_\alpha)\rangle + \phi(z_\alpha, x_\alpha) \notin -\text{int } C(x_\alpha), \qquad \forall z_\alpha \in K(\lambda_\alpha).$$

Since K is B-l.s.c at λ_0, it follows that, for any net $\{\lambda_\alpha\} \subseteq \wedge$ with $\lambda_\alpha \to \lambda_0$ and $z_0 \in K(\lambda_0)$, there exists $z_\alpha \in K(\lambda_\alpha)$ such that $z_\alpha \to z_0$. Again, from the condition (d), T is B-u.s.c with compact-values at (x_0, μ_0) and, for any net $\{(x_\alpha, \mu_\alpha)\} \subseteq X \times \vee$ with $(x_\alpha, \mu_\alpha) \to (x_0, \mu_0)$, there exists $\xi_0 \in T(z_0, \mu_0)$ such that $\xi'_\alpha \to \xi_0$.

Therefore, from (b), (c) and (f), we have

$$(x_\alpha, \langle \xi'_\alpha, \eta(z_\alpha, x_\alpha)\rangle + \phi(z_\alpha, x_\alpha)) \to (x_0, \langle \xi_0, \eta(z_0, x_0)\rangle + \phi(z_0, x_0)) \in \text{Gr}(W).$$

Furthermore, we have

$$\langle \xi_0, \eta(z_0, x_0)\rangle + \phi(z_0, x_0) \notin -\text{int } C(x_0), \quad \forall z_0 \in K(\lambda_0),$$

which contradicts (3.7). So, $x_0 \in S(\lambda_0, \mu_0) \subset V$, which is a contradiction. Since $x_\alpha \notin V$ for all α, it follows that $x_\alpha \to x_0$ and V is open. Consequently, the solution mapping $S(\cdot,\cdot)$ is B-u.s.c at any $(\lambda_0, \mu_0) \in \wedge \times \vee$.

Finally, we show that $S(\cdot,\cdot)$ is closed at any $(\lambda_0, \mu_0) \in \wedge \times \vee$. Taking $x_\alpha \in S(\lambda_\alpha, \mu_\alpha)$ with $(\lambda_\alpha, \mu_\alpha) \to (\lambda_0, \mu_0)$ and $x_\alpha \to x_0$. Then $x_\alpha \in K(\lambda_\alpha)$. By (g), $x_0 \in K(\lambda_0)$. By the same proof as above, we have $x_0 \in S(\lambda_0, \mu_0)$, which implies that the solution mapping $S(\cdot,\cdot)$ is closed on $\wedge \times M$. This completes the proof.

Remark 3.1. From Lemma 2.4, we know that, if all the conditions of Theorem 3.1 are satisfied, then the solution mapping $S(\cdot,\cdot)$ is H-u.s.c on $\wedge \times \vee$.

From Theorem 3.1, we can conclude the following:

Corollary 3.2. Let $(\lambda_0, \mu_0) \in \wedge \times \vee$ be a point, $K(\lambda_0)$ be a compact set, $T : X \times \vee \to 2^{L(X,Y)}$ be a set-valued mapping with nonempty values, $C : X \to 2^Y$ be a set-valued mapping such that, for each $x \in X$, $C(x)$ is a pointed closed and convex cone in Y and $\text{int } C(x) \neq \emptyset$, $\eta : X \times X \to X$ and $\varphi : X \times X \to Y$ be two vector-valued mappings. Assume that the conditions (a)-(c) and (f) in Theorem 3.1 and the following conditions are satisfied:

(d)' $T(x, \mu)$ is weakly $(\eta, \varphi, C(x))$-pseudo-mapping with respect to the first argument and B-u.s.c with compact-values on $X \times \{\mu_0\}$;
(e)' there is a continuous selection t of T on $X \times \{\mu_0\}$.

Then the following hold:

(1) The solution mapping $S(\cdot,\cdot)$ is nonempty and weakly compact at (λ_0, μ_0);
(2) The solution mapping $S(\cdot,\cdot)$ is B-u.s.c at (λ_0, μ_0).

If the set-valued mapping $K : \wedge \to 2^X$ is unbounded-values, then we have the following:

Theorem 3.3. Let $T : X \times \vee \to 2^{L(X,Y)}$ be a set-valued mapping with nonempty values, $C : X \to 2^Y$ be a set-valued mapping such that, for each $x \in X$, $C(x)$ is a pointed closed and convex cone in Y and $\operatorname{int} C(x) \neq \emptyset$, $\eta : X \times X \to X$ and $\varphi : X \times X \to Y$ be two vector-valued mappings. Assume that conditions (a)-(f) in Theorem 3.1 and the following conditions are satisfied:

(g)' $K : \wedge \to 2^X$ is B-u.s.c and B-l.s.c with closed and convex-values;
(h) for any $(\lambda,\mu) \in \wedge \times \vee$, there exists a weakly compact subset $\Delta(\lambda)$ of X and $z_0 \in \Delta(\lambda) \cap K(\lambda)$ such that

$$\langle t(z_0, \mu), \eta(z_0, x)\rangle + \phi(z_0, x) \in -\text{int } C(x), \quad \forall x \in K(\lambda)\backslash\Delta(\lambda).$$

Then the following hold:

(1) The solution mapping $S(\cdot,\cdot)$ is nonempty and closed on $\wedge \times \vee$;
(2) The solution mapping $S(\cdot,\cdot)$ is B-u.s.c on $\wedge \times \vee$.

Proof. By the proof of Theorem 3.1, we only need to prove that $\Upsilon_2(z_0)$ is weakly compact. Since $\Upsilon_2(z_0) \subseteq \Delta(\lambda)$ and $\Upsilon_2(z_0)$ is closed, it follows that $\Upsilon_2(z_0)$ is weakly compact and $S(\lambda, \mu) \subseteq \Delta(\lambda)$ for each $(\lambda, \mu) \in \wedge \times \vee$. This completes the proof.

Remark 3.2. In Theorems 3.1 and 3.3, if the condition (d) is replaced by the condition that $T(x, \mu)$ is (strictly) $(\eta, \varphi, C(x))$-pseudo-mapping with respect to the first argument and B-u.s.c with compact-values on $X \times \vee$, then Theorems 3.1 and 3.3 still hold.

Inspired the results in Chen et al. [34], we introduce the following function by the nonlinear scalarization function ξ_e. Suppose that $K(\lambda)$ is a compact set for any $\lambda \in \wedge$, $T(x, \mu)$ is also a compact set for any $(x, \mu) \in X \times \vee$, $\eta(x, x) = \varphi(x, x) = 0$ for all $x \in X$, $V(\cdot) =: Y\backslash \operatorname{int}C(\cdot)$ and $C(\cdot)$ are B-u.s.c on X. We define a function g: $X \times \wedge \times \vee \to R$ as follows:

$$g(x, \lambda, \mu) =: \max_{\zeta \in T(x,\mu)} \min_{y \in K(\lambda)} \xi_e(x, \langle \zeta, \eta(y, x)\rangle + \phi(y, x)), \quad \forall x \in K(\lambda). \tag{3.8}$$

Since $K(\lambda)$ and $T(x, \mu)$ are compact sets and $\xi_e(\cdot,\cdot)$ is continuous, $g(x, \lambda, \mu)$ is well-defined. Forward, we use the function $g(x, \lambda, \mu)$ to discuss the continuity of the solution mapping of (PGVQVLIP).

First, we discuss the relations between $g(\cdot,\cdot,\cdot)$ and the solution mapping $S(\cdot,\cdot)$.

Lemma 3.1. (1) $g(x_0, \lambda_0, \mu_0) = 0$ if and only if $x_0 \in S(\lambda_0, \mu_0)$;

(2) $g(x, \lambda, \mu) \le 0$ for all $x \in K(\lambda)$.

Proof. The proof is similar to the proof of Proposition 4.1 [34] and so the proof is omitted.

Remark 3.3. We say that the function g is a *parametric gap function* for (PGVQVLIP) if and only if (1) and (2) of Lemma 3.1 are satisfied. In fact, the gap functions are widely applied in optimization problems, equation problems, variational inequalities problems and others. The minimization of the gap function is an effectively approach for solving variational inequalities. Many authors have investigated the gap functions and applied to construct some algorithms for variational inequalities and equilibrium problems (see, for instance, [8,14,41]).

Lemma 3.2. Let $K(\lambda)$ be nonempty compact for any $\lambda \in \wedge$. Assume that the following conditions are satisfied:

(a) $T(\cdot,\cdot)$ is B-l.s.c with compact-values on $X \times \vee$;
(b) $C(\cdot)$ is B-u.s.c on X, and $e(\cdot) \in \operatorname{int}C(\cdot)$ is continuous on X.

Then $g(\cdot,\cdot,\cdot)$ is a lower semi-continuous function.

Proof. The proof is similar to the proof of Lemma 4.2 [34] and so the proof is omitted.

If the conditions of Lemma 3.2 are strengthened, then we can get the continuity of g.

Lemma 3.3. Let $K(\lambda)$ be nonempty compact for any $\lambda \in \wedge$. Assume that the following conditions are satisfied:

(a) $T(\cdot,\cdot)$ is B-continuous with compact-values on $X \times \vee$;
(b) $C(\cdot)$ and $V(\cdot) = Y\backslash \operatorname{int}C(\cdot)$ are B-u.s.c on X and $e(\cdot) \in \operatorname{int}C(\cdot)$ is continuous on X.

Then $g(\cdot,\cdot,\cdot)$ is continuous.

Proof. By Lemma 3.2, we only need to prove that g is upper semi-continuous. We can show that $-g$ is lower semi-continuous. The proof method of the lower semi-continuity of $-g$ is similar to that of the upper semi-continuity of g and so the proof is omitted.

Motivated by the hypothesis (H_1) of [32,33], (Hg) of [21,34] and $(Hg)'$ of [35], by virtue of the parametric gap function g, we also introduce the following key assumption:

$(Hg)''$ For any $(\lambda_0, \mu_0) \in \wedge \times \vee$ and $\epsilon > 0$, there exist $\varrho > 0$ and $\delta > 0$ such that, for any $(\lambda, \mu) \in B((\lambda_0, \mu_0), \delta)$ and $x \in \Delta(\lambda, \mu, \epsilon) = K(\lambda) \setminus U(S(\lambda, \mu), \epsilon)$,

$$g(\lambda, \mu, x) \leq -\varrho.$$

Remark 3.4. It is easy to see that, if $\wedge$ and $\vee$ are the same spaces and $\wedge$ is a metric space, $C(x) \equiv C$ for all $x \in X$ and $\mu = \lambda$, then the hypothesis $(Hg)''$ is reduced to the hypothesis $(Hg)'$ of [35].

Remark 3.5. As pointed in [21,32,34,35], the hypothesis $(Hg)''$ can be explained by the geometric properties that, for any small positive number ϵ, one can take two small positive real number ϱ and δ such that, for all problems in the δ-neighborhood of a pair parameters (λ_0, μ_0), if a feasible point x is away from the solution set by a distance of at least ϵ, then a "gap" by an amount of at least $-\varrho$ will be generated. As mentioned out in [32], the above hypothesis $(Hg)''$ is characterized by a common theme used in mathematical analysis. Such a theme interprets a proposition associated with a set in terms of other propositions related with the complement set. Instead of looking for restrictions within the solution set, the hypothesis $(Hg)''$ puts restrictions on the behavior of the parametric gap function on the complement of solution set. As showed in

[34], the hypothesis (Hg)" seems to be reasonable in establishing the Hausdorff continuity of $S(\cdot,\cdot)$ because of the complexity of the problem structure.

Theorem 3.4. Assume that (Hg)" and all the conditions of Theorem 3.1 holds and the following conditions are satisfied:

(a) $T(\cdot,\cdot)$ is B-continuous mapping with compact-values on $X \times \vee$;
(b) $C(\cdot)$ is B-u.s.c on X and $e(\cdot) \in \mathrm{int}C(\cdot)$ is continuous on X.

Then the following hold:

(1) The solution mapping $S(\cdot,\cdot)$ is nonempty and closed on $\wedge \times \vee$;
(2) The solution mapping $S(\cdot,\cdot)$ is H-continuous on $\wedge \times \vee$.

Proof. By Theorem 3.1 and Lemma 2.2, we know that the solution mapping $S(\cdot,\cdot)$ is nonempty closed and H-u.s.c on $\wedge \times \vee$.

Now, we only need to prove that the solution mapping $S(\cdot,\cdot)$ is H-l.s.c on $\wedge \times \vee$. Suppose that there exists $(\lambda_0, \mu_0) \in \wedge \times \vee$ such that the solution mapping S is not H-l.s.c at (λ_0, μ_0). Then there exist a neighborhood V of 0_X, nets $\{(\lambda_\alpha, \mu_\alpha)\} \subset \wedge \times \vee$ with $(\lambda_\alpha, \mu_\alpha) \to (\lambda_0, \mu_0)$ and $\{x_\alpha\}$ such that

$$x_\alpha \in S(\lambda_0, \mu_0) \backslash (S(\lambda_\alpha, \mu_\alpha) + V). \tag{3.9}$$

By Corollary 3.2, $S(\lambda_0, \mu_0)$ is a compact set. Without loss of generality, assume that $x_\alpha \to x_0 \in S(\lambda_0, \mu_0)$. For V and any $\epsilon > 0$, there exists a balanced open neighborhood $V(\epsilon)$ of 0_X such that $V(\epsilon) + V(\epsilon) + V(\epsilon) \subset V$. It is easy to see that, for all $\epsilon > 0$,

$$(x_0 + V(\varepsilon)) \cap K(\lambda_0) \neq \emptyset.$$

Since $K(\cdot)$ is B-l.s.c at λ_0, there exists β_1 such that

$$(x_0 + V(\varepsilon)) \cap K(\lambda_\beta) \neq \emptyset, \quad \forall \beta \geq \beta_1.$$

For any $\epsilon \in (0,1]$, assume that $y_\beta \in (x_0 + V(\epsilon)) \cap K(\lambda_\beta)$. Then $y_\beta \to x_0$. We assert that $y_\beta \notin S(\lambda_\beta, \mu_\beta) + V(\epsilon)$. Suppose that $y_\beta \in S(\lambda_\beta, \mu_\beta) + V(\epsilon)$. Then there exists $z_\beta \in S(\lambda_\beta, \mu_\beta)$ such that $y_\beta - z_\beta \in V(\epsilon)$. Note that $x_\alpha \to x_0 \in S(\lambda_0, \mu_0)$. Without loss of generality, we may assume that $x_\beta - x_0 \in V(\epsilon)$. Therefore, one has

$$x_\beta - z_\beta = (x_\beta - x_0) + (x_0 - y_\beta) + (y_\beta - z_\beta) \in V(\varepsilon) + V(\varepsilon) + V(\varepsilon) \subset V.$$

This yields that $x_\beta \in S(\lambda_\beta, \mu_\beta) + V$, which contradicts (3.9). Thus $y_\beta \notin S(\lambda_\beta, \mu_\beta) + V(\epsilon)$. In the light of ($Hg$)", there exist two positive real numbers $\varrho > 0$ and $\delta > 0$ such that, for any $(\lambda_\beta, \mu_\beta) \in B((\lambda_0, \mu_0), \delta)$ and $y_\beta \notin S(\lambda_\beta, \mu_\beta) + V(\epsilon)$,

$$g(y_\beta, \lambda_\beta, \mu_\beta) \leq -\varrho. \tag{3.10}$$

By Lemma 3.2, g is lower semi-continuous. So, it follows that, for any real number $\sigma > 0$,

$$g(y_\beta, \lambda_\beta, \mu_\beta) \geq g(x_0, \lambda_0, \mu_0) - \sigma. \tag{3.11}$$

Without loss of generality, assume that $\sigma < \varrho$. Then, from (3.10) and (3.11), it follows that

$$g(x_0, \lambda_0, \mu_0) \le \sigma - \varrho < 0,$$

that is,

$$g(x_0, \lambda_0, \mu_0) = \max_{\zeta \in T(x_0, \mu_0)} \min_{y \in K(\lambda_0)} \xi_e(x_0, \langle \zeta, \eta(y, x_0) \rangle + \phi(y, x_0)) < 0.$$

Hence there exist $\hat{y}_0 \in K(\lambda_0)$ and $\zeta_0 \in T(x_0, \mu_0)$ such that

$$\xi_e(x_0, \langle \zeta_0, \eta(\hat{y}_0, x_0) \rangle + \phi(\hat{y}_0, x_0)) < 0.$$

From Proposition 2.1, it follows that

$$\langle \zeta_0, \eta(\hat{y}_0, x_0) \rangle + \phi(\hat{y}_0, x_0) \in -\text{int } C(y_0),$$

which contradicts $x_0 \in S(\lambda_0, \mu_0)$. Therefore, the solution mapping S is H-l.s.c on $\wedge \times \vee$. This completes the proof.

Now, we give two examples to validate Theorems 3.1 and 3.4.

Example 3.1. Let $\wedge = \vee = (-1,1), X = R, Y = R^2$ and let $C(x) = R^2_+$ and $e(x) = (1, 1)^T \in \text{int } R^2_+$ for all $x \in X$. Define the set-valued mappings $K : \wedge \to 2^X$ and $T : X \times \vee \to 2^Y$ as follows: for any $x \in X$, $\mu \in \vee$ and $\lambda \in \wedge$,

$$K(\lambda) =: \left[-\frac{\lambda}{2}, |\lambda|\right], \quad T(x, \mu) =: \{(0, \ell)^T : 1 \le \ell \le +\mu^2\}.$$

It is easy to see that the conditions (a)-(g) of Theorem 3.1 and the conditions (a) and (b) of Theorem 3.4 are satisfied. From simple computation, we get $S(\lambda, \mu) = K(\lambda) = \left[-\frac{\lambda}{2}, |\lambda|\right]$ for all $(\lambda, \mu) \in \wedge \times \vee$. Therefore, $S(\cdot,\cdot)$ is H-continuous on $\wedge \times \vee$.

The following example illustrate the assumption $(Hg)''$ in Theorem 3.4 is essential.

Example 3.2. Let $\wedge = \vee = [0,1]$, $X = R$, $Y = R^2$ and let $\eta(y, x) = y - x$, $\varphi(y, x) = 0$ and $C(x) = R^2_+$ for all $x, y \in X$. Define the set-valued mappings $K : \wedge \to 2^X$ and $T : X \times \vee \to 2^Y$ by

$$K(\lambda) =: [-1, 1], \quad T(x, \lambda) =: \{(4, x^2 + \lambda)^T\}, \quad \forall x \in X, \lambda \in \wedge.$$

It is easy to see that the conditions (a)-(g) of Theorem 3.1 and the conditions (a) and (b) of Theorem 3.4 are satisfied. From simple computation, one has

$$S(\lambda) = \begin{cases} \{-1, 0\}, & \text{if } \lambda = 0, \\ \{-1\}, & \text{otherwise.} \end{cases}$$

Therefore, $S(\cdot,\cdot)$ is not H-continuous at $\lambda = 0$. Let us show that the assumption $(Hg)''$ is not satisfied at 0. Put $e(x) = (1, 1)^T \in \text{int } R^2_+$. Then, from Example 2.1,

$$\begin{aligned} g(x, \lambda) &= \max_{\zeta \in T(x,\lambda)} \min_{y \in K(\lambda)} \xi_e(x, \langle \zeta, \eta(y, x) \rangle + \phi(y, x)) \\ &= \min_{y \in K(\lambda)} \max\{4(y - x), (x^2 + \lambda)(y - x)\} \\ &= (x^2 + \lambda)(-1 - x). \end{aligned}$$

It is easy to see that g is a parametric gap function for (PGVQVLIP). Take $\epsilon \in (0,1)$ and, for any $\varrho > 0$, set $\lambda_n \to 0$ with $0 < \lambda_n < \varrho$ and $x_n = 0 \in \Delta(\lambda_n, \epsilon) = K(\lambda_n) \setminus U(S(\lambda_n),\epsilon)$ for all $n \ge 1$. Then we have

$$g(x_n, \lambda_n) = -\lambda_n > -\varrho.$$

Hence the assumption (Hg)" fails to hold at 0.

From Lemma 2.4, Remark 3.1 and Theorems 3.1 and 3.4, we can get the following:

Corollary 3.5. Assume that all the conditions of Theorem 3.4 are satisfied. Then the solution mapping $S(\cdot,\cdot)$ is B-continuous.

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