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Some new identities of Frobenius-Euler numbers and polynomials

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Abstract

In this paper, we give some new and interesting identities which are derived from the basis of Frobenius-Euler. Recently, several authors have studied some identities of Frobenius-Euler polynomials. From the methods of our paper, we can also derive many interesting identities of Frobenius-Euler numbers and polynomials.

1 Introduction

Let $\lambda (\neq 1) \in \mathbf{C}$. As is well known, the Frobenius-Euler polynomials are defined by the generating function to be

$$\frac{1-\lambda}{e^t-\lambda} e^{xt} = e^{H(x|\lambda)t} = \sum_{n=0}^{\infty} H_n(x|\lambda) \frac{t^n}{n!}, \tag{1}$$

with the usual convention about replacing $H^n(x|\lambda)$ by $H_n(x|\lambda)$ (see [1–6]).

In the special case, $x = 0$, $H_n(0|\lambda) = H_n(\lambda)$ are called the n th Frobenius-Euler numbers. Thus, by (1), we get

$$(H(\lambda) + 1)^n - \lambda H_n(\lambda) = H_n(1|\lambda) - \lambda H_n(\lambda) = (1-\lambda)\delta_{0,n}, \tag{2}$$

where $\delta_{0,n}$ is the Kronecker symbol.

From (1), we can derive the following equation:

$$H_n(x|\lambda) = (H(\lambda) + x)^n = \sum_{0 \leq l \leq n} \binom{n}{l} H_{n-l}(\lambda) x^l \quad (\text{see [6–16]}). \tag{3}$$

Thus, by (3), we easily see that the leading coefficient of $H_n(x|\lambda)$ is $H_0(\lambda) = 1$. So, $H_n(x|\lambda)$ are monic polynomials of degree n with coefficients in $\mathbf{Q}(\lambda)$.

From (1), we have

$$\sum_{n=0}^{\infty} (H_n(x+1|\lambda) - \lambda H_n(x|\lambda)) \frac{t^n}{n!} = \frac{(1-\lambda)e^{(x+1)t}}{e^t-\lambda} - \lambda \frac{1-\lambda}{e^t-\lambda} e^{xt}. \tag{4}$$

Thus, by (4), we get

$$H_n(x+1|\lambda) - \lambda H_n(x|\lambda) = (1-\lambda)x^n, \quad \text{for } n \in \mathbf{Z}_+. \tag{5}$$

It is easy to show that

$$\frac{d}{dx}H_n(x|\lambda) = \frac{d}{dx}(H(\lambda) + x)^n = nH_{n-1}(x|\lambda) \quad (n \in \mathbf{N}). \tag{6}$$

From (6), we have

$$\int_0^1 H_n(x|\lambda) dx = \frac{1}{n+1}(H_{n+1}(1|\lambda) - H_{n+1}(\lambda)) = \frac{\lambda-1}{n+1}H_{n+1}(\lambda). \tag{7}$$

Let $\mathbb{P}_n(\lambda) = \{p(x) \in \mathbf{Q}(\lambda)[x] \mid \deg p(x) \leq n\}$ be a vector space over $\mathbf{Q}(\lambda)$. Then we note that $\{H_0(x|\lambda), H_1(x|\lambda), \dots, H_n(x|\lambda)\}$ is a good basis for $\mathbb{P}_n(\lambda)$.

In this paper, we develop some new methods to obtain some new identities and properties of Frobenius-Euler polynomials which are derived from the basis of Frobenius-Euler polynomials. Those methods are useful in studying the identities of Frobenius-Euler polynomials.

2 Some identities of Frobenius-Euler polynomials

Let us take $p(x) \in \mathbb{P}_n(\lambda)$. Then $p(x)$ can be expressed as a $\mathbf{Q}(\lambda)$ -linear combination of $H_0(x|\lambda), \dots, H_n(x|\lambda)$ as follows:

$$p(x) = b_0H_0(x|\lambda) + b_1H_1(x|\lambda) + \dots + b_nH_n(x|\lambda) = \sum_{0 \leq k \leq n} b_k H_k(x|\lambda). \tag{8}$$

Let us define the operator Δ_λ by

$$g(x) = \Delta_\lambda p(x) = p(x+1) - \lambda p(x). \tag{9}$$

From (9), we can derive the following equation (10):

$$g(x) = \Delta_\lambda p(x) = \sum_{0 \leq k \leq n} b_k (H_k(x+1|\lambda) - \lambda H_k(x|\lambda)) = (1-\lambda) \sum_{0 \leq k \leq n} b_k x^k. \tag{10}$$

For $r \in \mathbf{Z}_+$, let us take the r th derivative of $g(x)$ in (10) as follows:

$$g^{(r)}(x) = (1-\lambda) \sum_{r \leq k \leq n} k(k-1) \dots (k-r+1) b_k x^{k-r}, \quad \text{where } g^{(r)}(x) = \frac{d^r g(x)}{dx^r}. \tag{11}$$

Thus, by (11), we get

$$g^{(r)}(0) = \frac{d^r g(x)}{dx^r} \Big|_{x=0} = (1-\lambda)r!b_r. \tag{12}$$

From (12), we have

$$b_r = \frac{g^{(r)}(0)}{(1-\lambda)r!} = \frac{1}{(1-\lambda)r!} (p^{(r)}(1) - \lambda p^{(r)}(0)), \tag{13}$$

where $r \in \mathbf{Z}_+$ and $p^{(r)}(0) = \frac{d^r p(x)}{dx^r} \Big|_{x=0}$. Therefore, by (13), we obtain the following theorem.

Theorem 1 For $\lambda (\neq 1) \in \mathbf{C}$, $n \in \mathbf{Z}_+$, let $p(x) \in \mathbb{P}_n(\lambda)$ with $p(x) = \sum_{0 \leq k \leq n} b_k H_k(x|\lambda)$. Then we have

$$b_k = \frac{1}{(1-\lambda)k!} g^{(k)}(0) = \frac{1}{(1-\lambda)k!} (p^{(k)}(1) - \lambda p^{(k)}(0)).$$

Let us take $p(x) = H_n(x|\lambda^{-1})$. Then, by Theorem 1, we get

$$H_n(x|\lambda^{-1}) = \sum_{0 \leq k \leq n} b_k H_k(x|\lambda), \tag{14}$$

where

$$\begin{aligned} b_k &= \frac{1}{(1-\lambda)k!} \frac{n!}{(n-k)!} \{H_{n-k}(1|\lambda^{-1}) - \lambda H_{n-k}(\lambda^{-1})\} \\ &= \frac{1}{1-\lambda} \binom{n}{k} \{H_{n-k}(1|\lambda^{-1}) - \lambda H_{n-k}(\lambda^{-1})\} \\ &= \frac{1}{1-\lambda} \binom{n}{k} \left\{ (1-\lambda^{-1})0^{n-k} + \frac{1}{\lambda} H_{n-k}(\lambda^{-1}) - \lambda H_{n-k}(\lambda^{-1}) \right\}. \end{aligned} \tag{15}$$

By (14) and (15), we get

$$\begin{aligned} H_n(x|\lambda^{-1}) &= -\frac{1}{\lambda} H_n(x|\lambda) + \sum_{k=0}^n \left\{ \frac{\binom{n}{k}}{\lambda(1-\lambda)} H_{n-k}(\lambda^{-1}) - \frac{\lambda \binom{n}{k}}{1-\lambda} H_{n-k}(\lambda^{-1}) \right\} H_k(x|\lambda) \\ &= -\frac{1}{\lambda} H_n(x|\lambda) + \sum_{k=0}^n \binom{n}{k} \frac{1+\lambda}{\lambda} H_{n-k}(\lambda^{-1}) H_k(x|\lambda). \end{aligned} \tag{16}$$

Therefore, by (16), we obtain the following theorem.

Theorem 2 For $n \in \mathbf{Z}_+$, we have

$$\lambda H_n(x|\lambda^{-1}) + H_n(x|\lambda) = (1+\lambda) \sum_{0 \leq k \leq n} \binom{n}{k} H_{n-k}(\lambda^{-1}) H_k(x|\lambda).$$

Let

$$p(x) = \sum_{0 \leq k \leq n} H_k(x|\lambda) H_{n-k}(x|\lambda) \in \mathbb{P}_n(\lambda). \tag{17}$$

From Theorem 2, we note that $p(x)$ can be generated by $\{H_0(x|\lambda), H_1(x|\lambda), \dots, H_n(x|\lambda)\}$ as follows:

$$p(x) = \sum_{0 \leq k \leq n} H_k(x|\lambda) H_{n-k}(x|\lambda) = \sum_{0 \leq k \leq n} b_k H_k(x|\lambda). \tag{18}$$

By (17), we get

$$p^{(k)}(x) = \frac{(n+1)!}{(n-k+1)!} \sum_{k \leq l \leq n} H_{l-k}(x|\lambda) H_{n-k}(x|\lambda), \tag{19}$$

and

$$\begin{aligned}
 b_k &= \frac{1}{(1-\lambda)k!} \{p^{(k)}(1) - \lambda p^{(k)}(0)\} \\
 &= \frac{(n+1)!}{(1-\lambda)k!(n-k+1)!} \sum_{l=k}^n \{H_{l-k}(1|\lambda)H_{n-l}(1|\lambda) - \lambda H_{l-k}(\lambda)H_{n-l}(\lambda)\} \\
 &= \frac{n+1}{(1-\lambda)(n-k+1)} \binom{n}{k} \sum_{l=k}^n \{(\lambda H_{l-k}(\lambda) + (1-\lambda)\delta_{0,l-k})(\lambda H_{n-l} + (1-\lambda)\delta_{0,n-l}) \\
 &\quad - \lambda H_{l-k}(\lambda)H_{n-l}(\lambda)\} \\
 &= \frac{n+1}{(1-\lambda)(n-k+1)} \binom{n}{k} \sum_{l=k}^n \{\lambda(1-\lambda)\delta_{0,l-k}H_{n-l}(\lambda) + \lambda(1-\lambda) \\
 &\quad \times H_{l-k}(\lambda)\delta_{0,n-l} + (1-\lambda)^2\delta_{0,l-k}\delta_{0,n-l} + \lambda(\lambda-1)H_{l-k}(\lambda)H_{n-l}(\lambda)\} \\
 &= \frac{n+1}{(1-\lambda)(n-k+1)} \binom{n}{k} \sum_{l=k}^n \{\lambda(\lambda-1)H_{l-k}(\lambda)H_{n-l}(\lambda) + \lambda(1-\lambda) \\
 &\quad \times H_{n-k}(\lambda) + \lambda(1-\lambda)H_{n-k}(\lambda) + (1-\lambda)^2\delta_{n,k}\} \\
 &= \frac{n+1}{n-k+1} \binom{n}{k} \sum_{l=k}^n \{-\lambda H_{l-k}(\lambda)H_{n-l}(\lambda) + 2\lambda H_{n-k}(\lambda) + (1-\lambda)\delta_{n,k}\}. \tag{20}
 \end{aligned}$$

From (18) and (20), we have

$$\begin{aligned}
 \sum_{0 \leq k \leq n} H_k(x|\lambda)H_{n-k}(x|\lambda) &= (n+1) \sum_{0 \leq k \leq n-1} \frac{\binom{n}{k}}{n-k+1} \sum_{k \leq l \leq n} \{(-\lambda)H_{l-k}(\lambda)H_{n-l}(\lambda) \\
 &\quad + 2\lambda H_{n-k}(\lambda)\} H_k(x|\lambda) + (n+1)H_n(x|\lambda). \tag{21}
 \end{aligned}$$

Therefore, by (21), we obtain the following theorem.

Theorem 3 For $n \in \mathbf{Z}_+$, we have

$$\begin{aligned}
 &\frac{1}{n+1} \sum_{0 \leq k \leq n} H_k(x|\lambda)H_{n-k}(x|\lambda) \\
 &= \sum_{0 \leq k \leq n-1} \frac{\binom{n}{k}}{n-k+1} \sum_{k \leq l \leq n} \{(-\lambda)H_{l-k}(\lambda)H_{n-l}(\lambda) + 2\lambda H_{n-k}(\lambda)\} H_k(x|\lambda) + H_n(x|\lambda).
 \end{aligned}$$

Let us consider

$$p(x) = \sum_{k=0}^n \frac{1}{k!(n-k)!} H_k(x|\lambda)H_{n-k}(x|\lambda) \in \mathbb{P}_n(\lambda). \tag{22}$$

By Theorem 1, $p(x)$ can be expressed by

$$p(x) = \sum_{k=0}^n b_k H_k(x|\lambda). \tag{23}$$

From (22), we have

$$p^{(r)}(x) = 2^r \sum_{k=r}^n \frac{H_{k-r}(x|\lambda)H_{n-k}(x|\lambda)}{(k-r)!(n-k)!} \quad (r \in \mathbf{Z}_+). \tag{24}$$

By Theorem 1, we get

$$\begin{aligned} b_k &= \frac{1}{2k!} \{p^{(k)}(1) - p^{(k)}(0)\} \\ &= \frac{2^{k-1}}{k!} \sum_{l=k}^n \frac{1}{(l-k)!(n-l)!} \{H_{l-k}(1|\lambda)H_{n-l}(1|\lambda) - \lambda H_{l-k}(\lambda)H_{n-l}(\lambda)\} \\ &= \frac{2^{k-1}}{k!} \sum_{l=k}^n \frac{1}{(l-k)!(n-l)!} \{(\lambda H_{l-k}(\lambda) + (1-\lambda)\delta_{0,l-k})(\lambda H_{n-l}(\lambda) + (1-\lambda)\delta_{0,n-l}) \\ &\quad - \lambda H_{l-k}(\lambda)H_{n-l}(\lambda)\} \\ &= \frac{2^{k-1}}{k!} \left\{ \sum_{l=k}^n \frac{\lambda(\lambda-1)H_{l-k}(\lambda)H_{n-l}(\lambda)}{(l-k)!(n-l)!} + \frac{2\lambda(1-\lambda)H_{n-k}(\lambda)}{(n-k)!} + (1-\lambda)^2\delta_{n,k} \right\} \\ &= \begin{cases} \frac{2^{k-1}}{k!} \sum_{l=k}^n \left\{ \frac{\lambda(\lambda-1)H_{l-k}(\lambda)H_{n-l}(\lambda)}{(l-k)!(n-l)!} + \frac{2\lambda(1-\lambda)H_{n-k}(\lambda)}{(n-k)!} \right\}, & \text{if } k \neq n, \\ \frac{2^{n-1}(1-\lambda)}{n!}, & \text{if } k = n. \end{cases} \end{aligned} \tag{25}$$

Therefore, by (25), we obtain the following theorem.

Theorem 4 For $n \in \mathbf{Z}_+$, we have

$$\begin{aligned} &\sum_{0 \leq k \leq n} \frac{1}{k!(n-k)!} H_k(x|\lambda)H_{n-k}(x|\lambda) \\ &= \sum_{0 \leq k \leq n-1} \frac{2^{k-1}}{k!} \sum_{k \leq l \leq n} \left\{ \frac{\lambda(\lambda-1)H_{l-k}(\lambda)H_{n-l}(\lambda)}{(l-k)!(n-l)!} + \frac{2\lambda(1-\lambda)H_{n-k}(\lambda)}{(n-k)!} \right\} H_k(x|\lambda) \\ &\quad + \frac{2^{n-1}(1-\lambda)}{n!} H_n(x|\lambda). \end{aligned}$$

3 Higher-order Frobenius-Euler polynomials

For $n \in \mathbf{Z}_+$, the Frobenius-Euler polynomials of order r are defined by the generating function to be

$$\begin{aligned} \left(\frac{1-\lambda}{e^t - \lambda} \right)^r e^{xt} &= e^{H^{(r)}(x|\lambda)t} \\ &= \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda) \frac{t^n}{n!}, \end{aligned} \tag{26}$$

with the usual convention about replacing $(H^{(r)}(x|\lambda))^n$ by $H_n^{(r)}(x|\lambda)$ (see [1-10]). In the special case, $x = 0$, $H_n^{(r)}(0|\lambda) = H_n^{(r)}(\lambda)$ are called the n th Frobenius-Euler numbers of order r (see [8, 9]).

From (26), we have

$$H_n^{(r)}(x|\lambda) = (H^{(r)}(\lambda) + x)^n = \sum_{l=0}^n \binom{n}{l} H_{n-l}^{(r)}(\lambda) x^l, \tag{27}$$

with the usual convention about replacing $(H^{(r)}(\lambda))^n$ by $H_n^{(r)}(\lambda)$.

By (26), we get

$$H_n^{(r)}(\lambda) = \sum_{n_1+\dots+n_r=n} \binom{n}{n_1, n_2, \dots, n_r} H_{n_1}(\lambda) \cdots H_{n_r}(\lambda), \tag{28}$$

where $\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1!n_2!\cdots n_r!}$. From (27) and (28), we note that the leading coefficient of $H_n^{(r)}(x|\lambda)$ is given by

$$\begin{aligned} H_0^{(r)}(\lambda) &= \sum_{n_1+\dots+n_r=0} \binom{n}{n_1, n_2, \dots, n_r} H_{n_1}(\lambda) \cdots H_{n_r}(\lambda) \\ &= H_0(\lambda) \cdots H_0(\lambda) = 1. \end{aligned} \tag{29}$$

Thus, by (29), we see that $H_n^{(r)}$ is a monic polynomial of degree n with coefficients in $\mathbf{Q}(\lambda)$. From (26), we have

$$H_n^{(0)}(x|\lambda) = x^n, \quad \text{for } n \in \mathbf{Z}_+, \tag{30}$$

and

$$\frac{\partial}{\partial x} H_n^{(r)}(x|\lambda) = \frac{\partial}{\partial x} (H^{(r)}(\lambda) + x)^n = nH_{n-1}^{(r)}(x|\lambda) \quad (r \geq 0). \tag{31}$$

It is not difficult to show that

$$H_n^{(r)}(x+1|\lambda) - \lambda H_n^{(r)}(x|\lambda) = (1-\lambda)H_n^{(r-1)}(x|\lambda). \tag{32}$$

Now, we note that $\{H_0^{(r)}(x|\lambda), H_1^{(r)}(x|\lambda), \dots, H_n^{(r)}(x|\lambda)\}$ is also a good basis for $\mathbb{P}_n(\lambda)$.

Let us define the operator D as $Df(x) = \frac{df(x)}{dx}$ and let $p(x) \in \mathbb{P}_n(\lambda)$. Then $p(x)$ can be written as

$$p(x) = \sum_{k=0}^n C_k H_k^{(r)}(x|\lambda). \tag{33}$$

From (9) and (32), we have

$$\Delta_\lambda H_n^{(r)}(x|\lambda) = H_n^{(r)}(x+1|\lambda) - \lambda H_n^{(r)}(x|\lambda) = (1-\lambda)H_n^{(r-1)}(x|\lambda). \tag{34}$$

Thus, by (33) and (34), we get

$$\Delta_\lambda^r p(x) = (1-\lambda)^r \sum_{k=0}^n C_k H_k^{(0)}(x|\lambda) = (1-\lambda)^r \sum_{k=0}^n C_k x^k. \tag{35}$$

Let us take the k th derivative of $\Delta_\lambda^r p(x)$ in (35).

Then we have

$$D^k(\Delta_\lambda^r p(x)) = (1-\lambda)^r \sum_{l=k}^n \frac{l!}{(l-k)!} C_l x^{l-k}. \tag{36}$$

Thus, from (36), we have

$$D^k(\Delta_\lambda^r p(0)) = (1-\lambda)^r \sum_{l=k}^n \frac{l! C_l}{(l-k)!} 0^{l-k} = (1-\lambda)^r k! C_k. \tag{37}$$

Thus, by (37), we get

$$\begin{aligned} C_k &= \frac{D^k(\Delta_\lambda^r p(0))}{(1-\lambda)^r k!} \\ &= \frac{\Delta_\lambda^r(D^k p(0))}{(1-\lambda)^r k!} = \frac{1}{(1-\lambda)^r k!} \sum_{j=0}^r \binom{r}{j} (-\lambda)^{r-j} D^k p(j). \end{aligned} \tag{38}$$

Therefore, by (33) and (38), we obtain the following theorem.

Theorem 5 For $r \in \mathbf{Z}_+$, let $p(x) \in \mathbb{P}_n(\lambda)$ with

$$p(x) = \frac{1}{(1-\lambda)^r} \sum_{0 \leq k \leq n} C_k H_k^{(r)}(x|\lambda) \quad (C_k \in \mathbf{Q}(\lambda)).$$

Then we have

$$C_k = \frac{1}{(1-\lambda)^r k!} \sum_{0 \leq j \leq r} \binom{r}{j} (-\lambda)^{r-j} D^k p(j).$$

That is,

$$p(x) = \frac{1}{(1-\lambda)^r} \sum_{0 \leq k \leq n} \left(\sum_{0 \leq j \leq r} \frac{1}{k!} \binom{r}{j} (-\lambda)^{r-j} D^k p(j) \right) H_k^{(r)}(x|\lambda).$$

Let us take $p(x) = H_n(x|\lambda) \in \mathbf{P}_n(\lambda)$. Then, by Theorem 5, $p(x) = H_n(x|\lambda)$ can be generated by $\{H_0^{(r)}(x|\lambda), H_1^{(r)}(\lambda), \dots, H_n^{(r)}(x|\lambda)\}$ as follows:

$$H_n(x|\lambda) = \sum_{0 \leq k \leq n} C_k H_k^{(r)}(x|\lambda), \tag{39}$$

where

$$C_k = \frac{1}{(1-\lambda)^r} \frac{1}{k!} \sum_{0 \leq j \leq r} \binom{r}{j} (-\lambda)^{r-j} D^k p(j), \tag{40}$$

and

$$p^{(k)}(x) = D^k p(x) = n(n-1) \cdots (n-k+1) H_{n-k}(x|\lambda) = \frac{n!}{(n-k)!} H_{n-k}(x|\lambda). \tag{41}$$

By (40) and (41), we get

$$C_k = \frac{1}{(1-\lambda)^r} \binom{n}{k} \sum_{0 \leq j \leq r} \binom{r}{j} (-\lambda)^{r-j} H_{n-k}(j|\lambda). \tag{42}$$

Therefore, by (39) and (42), we obtain the following theorem.

Theorem 6 For $n \in \mathbf{Z}_+$, we have

$$H_n(x|\lambda) = \frac{1}{(1-\lambda)^r} \sum_{0 \leq k \leq n} \binom{n}{k} \left(\sum_{0 \leq j \leq r} \binom{r}{j} (-\lambda)^{r-j} H_{n-k}(j|\lambda) \right) H_k^{(r)}(x|\lambda).$$

Let us assume that $p(x) = H_n^{(r)}(x|\lambda)$.

Then we have

$$\begin{aligned} p^k(x) &= n(n-1) \cdots (n-k+1) H_{n-k}^{(r)}(x|\lambda) \\ &= \frac{n!}{(n-k)!} H_{n-k}^{(r)}(x|\lambda). \end{aligned} \tag{43}$$

From Theorem 1, we note that $p(x) = H_n^{(r)}(x|\lambda)$ can be expressed as a linear combination of $H_0(x|\lambda), H_1(x|\lambda), \dots, H_n(x|\lambda)$

$$H_n^{(r)}(x|\lambda) = \sum_{0 \leq k \leq n} b_k H_k(x|\lambda), \tag{44}$$

where

$$\begin{aligned} b_k &= \frac{1}{(1-\lambda)k!} \{p^k(1) - \lambda p^{(k)}(0)\} \\ &= \frac{n!}{(1-\lambda)k!(n-k)!} \{H_{n-k}^{(r)}(1|\lambda) - \lambda H_{n-k}^{(r)}(\lambda)\}. \end{aligned} \tag{45}$$

By (34) and (45), we get

$$b_k = \binom{n}{k} H_{n-k}^{(r-1)}(\lambda). \tag{46}$$

Therefore, by (44) and (46), we obtain the following theorem.

Theorem 7 For $n \in \mathbf{Z}_+$, we have

$$H_n^{(r)}(x|\lambda) = \sum_{0 \leq k \leq n} \binom{n}{k} H_{n-k}^{(r-1)}(\lambda) H_k(x|\lambda).$$

Remark From (2) and (37), we note that

$$\begin{aligned} \frac{d}{d\lambda} \left(\frac{1-\lambda}{e^t - \lambda} \right) &= \frac{1 - e^t}{(e^t - \lambda)^2} = \frac{1}{(1-\lambda)^2} \left(\frac{(1-\lambda)^2}{(e^t - \lambda)^2} - \frac{(1-\lambda)^2}{(e^t - \lambda)^2} e^t \right) \\ &= \frac{1}{(1-\lambda)^2} \left(\frac{(1-\lambda)^2}{(e^t - \lambda)^2} - \frac{(1-\lambda)^2}{(e^t - \lambda)^2} (e^t - \lambda + \lambda) \right) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{1-\lambda} \left(\frac{(1-\lambda)^2}{(e^t-\lambda)^2} - \frac{1-\lambda}{e^t-\lambda} \right) \\
 &= \frac{1}{1-\lambda} \sum_{n=0}^{\infty} (H_n^{(2)}(\lambda) - H_n(\lambda)) \frac{t^n}{n!}, \tag{47}
 \end{aligned}$$

and

$$\begin{aligned}
 \frac{d^2}{d\lambda^2} \left(\frac{1-\lambda}{e^t-\lambda} \right) &= 2! \frac{1-e^t}{(e^t-\lambda)^3} = \frac{2!}{(1-\lambda)^3} \left(\frac{(1-\lambda)^3}{(e^t-\lambda)^3} - \frac{(1-\lambda)^3}{(e^t-\lambda)^3} e^t \right) \\
 &= \frac{2!}{(1-\lambda)^3} \left(\frac{(1-\lambda)^3}{(e^t-\lambda)^3} - \frac{(1-\lambda)^3}{(e^t-\lambda)^3} (e^t - \lambda + \lambda) \right) \\
 &= \frac{2!}{(1-\lambda)^2} \left(\frac{(1-\lambda)^3}{(e^t-\lambda)^3} - \frac{(1-\lambda)^2}{(e^t-\lambda)^2} \right) \\
 &= \frac{2!}{(1-\lambda)^2} \sum_{n=0}^{\infty} (H_n^{(3)}(\lambda) - H_n^{(2)}(\lambda)) \frac{t^n}{n!}. \tag{48}
 \end{aligned}$$

Continuing this process, we obtain the following equation:

$$\begin{aligned}
 \frac{d^k}{d\lambda^k} \left(\frac{1-\lambda}{e^t-\lambda} \right) &= \frac{k!}{(1-\lambda)^k} \left(\frac{(1-\lambda)^{k+1}}{(e^t-\lambda)^{k+1}} - \frac{(1-\lambda)^k}{(e^t-\lambda)^k} \right) \\
 &= \frac{k!}{(1-\lambda)^k} \sum_{n=0}^{\infty} (H_n^{(k+1)}(\lambda) - H_n^{(k)}(\lambda)) \frac{t^n}{n!} \quad (\text{see [8]}). \tag{49}
 \end{aligned}$$

By (1), (2) and (49), we get

$$\frac{d^k}{d\lambda^k} H_n(\lambda) = \frac{k!}{(1-\lambda)^k} (H_n^{(k+1)}(\lambda) - H_n^{(k)}(\lambda)),$$

where k is a positive integer (see [7, 8]).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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