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Some new fractional q-integral Grüss-type inequalities and other inequalities

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Abstract

In this paper, we employ a fractional q-integral on the specific time scale, $\mathbb{T}_{t_0} = \{t : t = t_0 q^n, n \text{ a nonnegative integer}\} \cup \{0\}$, where $t_0 \in \mathbb{R}$ and $0 < q < 1$, to establish some new fractional q-integral Grüss-type inequalities by using one or two fractional parameters. Furthermore, other fractional q-integral inequalities are also obtained.

MSC: 26D10; 26A33

Keywords: fractional q-integral; integral inequalities; Grüss-type inequalities

1 Introduction

In the past several years, by using the Riemann-Liouville fractional integrals, the fractional integral inequalities and applications have been addressed extensively by several researchers. For example, we refer the reader to [1–6] and the references cited therein. Dahmani *et al.* [7] gave the following fractional integral inequalities by using the Riemann-Liouville fractional integrals. Let f and g be two integrable functions on $[0,\infty)$ satisfying the following conditions:

$$\varphi_1 \le f(x) \le \varphi_2, \qquad \psi_1 \le g(x) \le \psi_2, \qquad \varphi_1, \varphi_2, \psi_1, \psi_2 \in \mathbb{R}, \quad x \in [0,\infty).$$

For all $t > 0$, $\alpha > 0$ and $\beta > 0$, then

$$\left|\frac{t^\alpha}{\Gamma(\alpha+1)} J^\alpha(fg)(t) - J^\alpha f(t) J^\alpha g(t)\right| \le \left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)^2 (\varphi_2-\varphi_1)(\psi_2-\psi_1)$$

and

$$\begin{aligned}
&\left(\frac{t^\alpha}{\Gamma(\alpha+1)} J^\beta(fg)(t) + \frac{t^\beta}{\Gamma(\beta+1)} J^\alpha(fg)(t) - J^\alpha f(t) J^\beta g(t) - J^\beta f(t) J^\alpha g(t)\right)^2 \\
&\quad \le \left(\left(\varphi_2 \frac{t^\alpha}{\Gamma(\alpha+1)} - J^\alpha f(t)\right)\left(J^\beta f(t) - \varphi_1 \frac{t^\beta}{\Gamma(\beta+1)}\right) + \left(J^\alpha f(t) - \varphi_1 \frac{t^\alpha}{\Gamma(\alpha+1)}\right)\right. \\
&\qquad \times \left.\left(\psi_2 \frac{t^\beta}{\Gamma(\beta+1)} - J^\beta f(t)\right)\right)\left(\left(\psi_2 \frac{t^\alpha}{\Gamma(\alpha+1)} - J^\alpha g(t)\right)\left(J^\beta g(t) - \psi_1 \frac{t^\beta}{\Gamma(\beta+1)}\right)\right. \\
&\qquad + \left.\left(J^\alpha g(t) - \psi_1 \frac{t^\alpha}{\Gamma(\alpha+1)}\right)\left(\psi_2 \frac{t^\beta}{\Gamma(\beta+1)} - J^\beta g(t)\right)\right).
\end{aligned}$$

To the best of authors' knowledge, only some fractional q-integral inequalities have been established in recent years. That is, only Öğünmez and Özkan [8], Bohner and Ferreira [9] and Yang [10] obtained some fractional q-integral inequalities. With motivation from the papers [7, 11, 12], the main purpose of this article is to establish some new fractional q-integral inequalities. First of all, by using one or two fractional parameters, we establish some new fractional q-integral Grüss-type inequalities on the specific time scale $\mathbb{T}_{t_0} = \{t : t = t_0 q^n, n \text{ a nonnegative integer}\} \cup \{0\}$, where $t_0 \in \mathbb{R}$ and $0 < q < 1$. In general, a time scale is an arbitrary nonempty closed subset of real numbers [13]. Furthermore, other fractional q-integral inequalities are also obtained.

2 Description of fractional q-calculus

In this section, we introduce the basic definitions on fractional q-calculus. More results concerning fractional q-calculus can be found in [14–17].

Let $t_0 \in \mathbb{R}$ and define $\mathbb{T}_{t_0} = \{t : t = t_0 q^n, n \text{ a nonnegative integer}\} \cup \{0\}$, $0 < q < 1$. For a function $f : \mathbb{T}_{t_0} \to \mathbb{R}$, the nabla q-derivative of f

$$\nabla_q f(t) = \frac{f(qt) - f(t)}{(q-1)t}$$

for all $t \in \mathbb{T}_{t_0} \backslash \{0\}$. The q-integral of f is

$$\int_0^t f(s)\nabla s = (1-q)t \sum_{i=0}^{\infty} q^i f\bigl(tq^i\bigr).$$

The q-factorial function is defined in the following way: if n is a positive integer, then

$$(t-s)^{\underline{(n)}} = (t-s)(t-qs)\bigl(t-q^2 s\bigr)\cdots\bigl(t-q^{n-1}s\bigr).$$

If n is not a positive integer, then

$$(t-s)^{\underline{(n)}} = t^n \prod_{k=0}^{\infty} \frac{1-(s/t)q^k}{1-(s/t)q^{n+k}}.$$

The q-derivative of the q-factorial function with respect to t is

$$\nabla_q (t-s)^{\underline{(n)}} = \frac{1-q^n}{1-q}(t-s)^{\underline{(n-1)}},$$

and the q-derivative of the q-factorial function with respect to s is

$$\nabla_q (t-s)^{\underline{(n)}} = -\frac{1-q^n}{1-q}(t-qs)^{\underline{(n-1)}}.$$

The q-exponential function is defined as

$$e_q(t) = \prod_{k=0}^{\infty}\bigl(1-q^k t\bigr), \qquad e_q(0) = 1.$$

Define the q-gamma function by

$$\Gamma_q(\nu) = \frac{1}{1-q}\int_0^1 \left(\frac{t}{1-q}\right)^{\nu-1} e_q(qt)\nabla t, \quad \nu \in \mathbb{R}^+.$$

Note that

$$\Gamma_q(\nu+1) = [\nu]_q \Gamma_q(\nu), \quad \nu \in \mathbb{R}^+,$$

where $[\nu]_q := (1-q^\nu)/(1-q)$. The fractional q-integral is defined as

$$\nabla_q^{-\nu} f(t) = \frac{1}{\Gamma_q(\nu)}\int_0^t (t-qs)^{\underline{(\nu-1)}} f(s)\nabla s.$$

Note that

$$\nabla_q^{-\nu}(1) = \frac{1}{\Gamma_q(\nu)}\frac{q-1}{q^\nu-1} t^{\underline{(\nu)}} = \frac{1}{\Gamma_q(\nu+1)} t^{\underline{(\nu)}}.$$

3 Fractional q-integral Grüss-type inequalities

To state the main results in this paper, we employ the following lemmas. For the sake of convenience, we use the following assumption (A) in this section:

$$\varphi_1 \le f(x) \le \varphi_2, \qquad \psi_1 \le g(x) \le \psi_2, \qquad \varphi_1, \varphi_2, \psi_1, \psi_2 \in \mathbb{R}, \quad x \in \mathbb{T}_{t_0}.$$

Lemma 1 *Let $\varphi_1, \varphi_2 \in \mathbb{R}$ and f be a function defined on $\mathbb{T}_{t_0}$. Then, for all $t > 0$ and $\nu > 0$, we have*

$$\begin{aligned}\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\nabla_q^{-\nu} f^2(t) - \left(\nabla_q^{-\nu} f(t)\right)^2 &= \left(\varphi_2 \frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)} - \nabla_q^{-\nu} f(t)\right)\left(\nabla_q^{-\nu} f(t) - \varphi_1 \frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\right) \\ &\quad - \frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\nabla_q^{-\nu}\left(\varphi_2 - f(t)\right)\left(f(t)-\varphi_1\right). \end{aligned} \tag{1}$$

Proof Let $\varphi_1, \varphi_2 \in \mathbb{R}$ and f be a function defined on $\mathbb{T}_{t_0}$. For any $\tau > 0$ and $\rho > 0$, we have

$$\begin{aligned}&\left(\varphi_2 - f(\rho)\right)\left(f(\tau)-\varphi_1\right) + \left(\varphi_2 - f(\tau)\right)\left(f(\rho)-\varphi_1\right) - \left(\varphi_2 - f(\tau)\right)\left(f(\tau)-\varphi_1\right) \\ &\quad - \left(\varphi_2 - f(\rho)\right)\left(f(\rho)-\varphi_1\right) = f^2(\tau) + f^2(\rho) - 2f(\tau)f(\rho). \end{aligned} \tag{2}$$

Multiplying both sides of (2) by $(t-q\tau)^{\underline{(\nu-1)}}/\Gamma_q(\nu)$ and integrating the resulting identity with respect to τ from 0 to t, we get

$$\begin{aligned}&\left(\varphi_2 - f(\rho)\right)\left(\nabla_q^{-\nu} f(t) - \varphi_1 \frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\right) + \left(\varphi_2 \frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)} - \nabla_q^{-\nu} f(t)\right)\left(f(\rho)-\varphi_1\right) \\ &\quad - \nabla_q^{-\nu}\left(\varphi_2 - f(t)\right)\left(f(t)-\varphi_1\right) - \left(\varphi_2 - f(\rho)\right)\left(f(\rho)-\varphi_1\right)\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)} \\ &= \nabla_q^{-\nu} f^2(t) + f^2(\rho)\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)} - 2f(\rho)\nabla_q^{-\nu} f(t). \end{aligned} \tag{3}$$

Multiplying both sides of (3) by $(t-q\rho)^{\underline{(\nu-1)}}/\Gamma_q(\nu)$ and integrating the resulting identity with respect to ρ from 0 to t, we obtain

$$\begin{aligned}
&\left(\varphi_2\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}-\nabla_q^{-\nu}f(t)\right)\left(\nabla_q^{-\nu}f(t)-\varphi_1\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\right)\\
&\quad+\left(\varphi_2\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}-\nabla_q^{-\nu}f(t)\right)\left(\nabla_q^{-\nu}f(t)-\varphi_1\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\right)\\
&\quad-\nabla_q^{-\nu}\big(\varphi_2-f(t)\big)\big(f(t)-\varphi_1\big)\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}-\nabla_q^{-\nu}\big(\varphi_2-f(t)\big)\big(f(t)-\varphi_1\big)\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\\
&=\nabla_q^{-\nu}f^2(t)\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}+\nabla_q^{-\nu}f^2(t)\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}-2\nabla_q^{-\nu}f(t)\nabla_q^{-\nu}f(t),
\end{aligned}$$

which implies (1). □

Lemma 2 *Let f and g be two functions defined on* $\mathbb{T}_{t_0}$. *Then, for all* $t>0$, $\mu>0$ *and* $\nu>0$, *we have*

$$\begin{aligned}
&\left(\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\nabla_q^{-\mu}(fg)(t)+\frac{t^{\underline{(\mu)}}}{\Gamma_q(\mu+1)}\nabla_q^{-\nu}(fg)(t)-\nabla_q^{-\nu}f(t)\nabla_q^{-\mu}g(t)-\nabla_q^{-\mu}f(t)\nabla_q^{-\nu}g(t)\right)^2\\
&\leq\left(\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\nabla_q^{-\mu}f^2(t)+\frac{t^{\underline{(\mu)}}}{\Gamma_q(\mu+1)}\nabla_q^{-\nu}f^2(t)-2\nabla_q^{-\nu}f(t)\nabla_q^{-\mu}f(t)\right)\\
&\quad\times\left(\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\nabla_q^{-\mu}g^2(t)+\frac{t^{\underline{(\mu)}}}{\Gamma_q(\mu+1)}\nabla_q^{-\nu}g^2(t)-2\nabla_q^{-\nu}g(t)\nabla_q^{-\mu}g(t)\right).
\end{aligned}\tag{4}$$

Proof In order to prove Lemma 2, we firstly prove that the following inequality (*i.e.*, Cauchy-Schwarz inequality for double q-integrals) holds. Let $f(x,y)$, $g(x,y)$ and $h(x,y)$ be three functions defined on $\mathbb{T}_{t_0}^2$ with $h(x,y)\geq 0$. Then we have

$$\begin{aligned}
&\left(\int_0^t\int_0^t h(x,y)f(x,y)g(x,y)\,d_qx\,d_qy\right)^2\\
&\leq\left(\int_0^t\int_0^t h(x,y)f^2(x,y)\,d_qx\,d_qy\right)\left(\int_0^t\int_0^t h(x,y)g^2(x,y)\,d_qx\,d_qy\right).
\end{aligned}$$

According to the definition of q-integral, it is easy to obtain that double q-integral is

$$\begin{aligned}
&\int_0^t\int_0^t f(x,y)\,d_qx\,d_qy\\
&=\int_0^t\left((1-q)t\sum_{i=0}^{\infty}q^if\big(tq^i,y\big)\right)d_qy\\
&=(1-q)t\sum_{j=0}^{\infty}q^j\left((1-q)t\sum_{i=0}^{\infty}q^if\big(tq^i,tq^j\big)\right)\\
&=(1-q)^2t^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}q^{i+j}f\big(tq^i,tq^j\big).
\end{aligned}$$

Due to discrete Cauchy-Schwarz inequality with weight coefficient, we have

$$\begin{aligned}
&\left(\int_0^t\int_0^t h(x,y)f(x,y)g(x,y)\,d_qx\,d_qy\right)^2\\
&=\left((1-q)^2t^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}q^{i+j}h\big(tq^i,tq^j\big)f\big(tq^i,tq^j\big)g\big(tq^i,tq^j\big)\right)^2\\
&\le\left((1-q)^2t^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}q^{i+j}h\big(tq^i,tq^j\big)f^2\big(tq^i,tq^j\big)\right)\\
&\quad\times\left((1-q)^2t^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}q^{i+j}h\big(tq^i,tq^j\big)g^2\big(tq^i,tq^j\big)\right)\\
&=\left(\int_0^t\int_0^t h(x,y)f^2(x,y)\,d_qx\,d_qy\right)\left(\int_0^t\int_0^t h(x,y)g^2(x,y)\,d_qx\,d_qy\right).
\end{aligned}$$

Next, we prove that Lemma 2 holds. Let $H(\tau,\rho)$ be defined by

$$H(\tau,\rho)=\big(f(\tau)-f(\rho)\big)\big(g(\tau)-g(\rho)\big),\quad t>0,\tau>0,\rho>0. \tag{5}$$

Multiplying both sides of (5) by $(t-q\tau)^{\underline{(\nu-1)}}(t-q\rho)^{\underline{(\mu-1)}}/(\Gamma_q(\nu)\Gamma_q(\mu))$ and integrating the resulting identity with respect to τ and ρ from 0 to t, then applying the Cauchy-Schwarz inequality for double q-integrals, we obtain (4). □

Lemma 3 *Let $\varphi_1,\varphi_2\in\mathbb{R}$ and f be a function defined on $\mathbb{T}_{t_0}$. Then, for all $t>0$ and $\nu>0$, we have*

$$\begin{aligned}
&\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\nabla_q^{-\mu}f^2(t)+\frac{t^{\underline{(\mu)}}}{\Gamma_q(\mu+1)}\nabla_q^{-\nu}f^2(t)-2\nabla_q^{-\nu}f(t)\nabla_q^{-\mu}f(t)\\
&=\left(\varphi_2\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}-\nabla_q^{-\nu}f(t)\right)\left(\nabla_q^{-\mu}f(t)-\varphi_1\frac{t^{\underline{(\mu)}}}{\Gamma_q(\mu+1)}\right)\\
&\quad+\left(\varphi_2\frac{t^{\underline{(\mu)}}}{\Gamma_q(\mu+1)}-\nabla_q^{-\mu}f(t)\right)\left(\nabla_q^{-\nu}f(t)-\varphi_1\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\right)\\
&\quad-\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\nabla_q^{-\mu}\big(\varphi_2-f(t)\big)\big(f(t)-\varphi_1\big)\\
&\quad-\frac{t^{\underline{(\mu)}}}{\Gamma_q(\mu+1)}\nabla_q^{-\nu}\big(\varphi_2-f(t)\big)\big(f(t)-\varphi_1\big).
\end{aligned}\tag{6}$$

Proof Multiplying both sides of (3) by $(t-q\rho)^{\underline{(\mu-1)}}/\Gamma_q(\mu)$ and integrating the resulting identity with respect to ρ from 0 to t, we obtain

$$\begin{aligned}
&\left(\varphi_2\frac{t^{\underline{(\mu)}}}{\Gamma_q(\mu+1)}-\nabla_q^{-\mu}f(t)\right)\left(\nabla_q^{-\nu}f(t)-\varphi_1\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\right)\\
&\quad+\left(\varphi_2\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}-\nabla_q^{-\nu}f(t)\right)\left(\nabla_q^{-\mu}f(t)-\varphi_1\frac{t^{\underline{(\mu)}}}{\Gamma_q(\mu+1)}\right)\\
&\quad-\nabla_q^{-\nu}\big(\varphi_2-f(t)\big)\big(f(t)-\varphi_1\big)\frac{t^{\underline{(\mu)}}}{\Gamma_q(\mu+1)}
\end{aligned}$$

$$
\begin{aligned}
&- \nabla_q^{-\mu}\bigl(\varphi_2 - f(t)\bigr)\bigl(f(t) - \varphi_1\bigr)\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)} \\
&= \nabla_q^{-\nu} f^2(t)\frac{t^{\underline{(\mu)}}}{\Gamma_q(\mu+1)} + \nabla_q^{-\mu} f^2(t)\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)} - 2\nabla_q^{-\nu} f(t)\nabla_q^{-\mu} f(t),
\end{aligned}
$$

which implies (6). □

Theorem 1 *Let f and g be two functions defined on $\mathbb{T}_{t_0}$ satisfying* (A). *Then, for all $t > 0$ and $\nu > 0$, we have*

$$
\left|\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\nabla_q^{-\nu}(fg)(t) - \nabla_q^{-\nu} f(t)\nabla_q^{-\nu} g(t)\right| \le \left(\frac{t^{\underline{(\nu)}}}{2\Gamma_q(\nu+1)}\right)^2 (\varphi_2 - \varphi_1)(\psi_2 - \psi_1). \tag{7}
$$

Proof Let f and g be two functions defined on $\mathbb{T}_{t_0}$ satisfying (A). Multiplying both sides of (6) by $(t - q\tau)^{\underline{(\nu-1)}}(t - q\rho)^{\underline{(\nu-1)}}/\Gamma_q^2(\nu)$ and integrating the resulting identity with respect to τ and ρ from 0 to t, we can state that

$$
\begin{aligned}
&\frac{1}{\Gamma_q^2(\mu)}\int_0^t\int_0^t (t - q\tau)^{\underline{(\nu-1)}}(t - q\rho)^{\underline{(\nu-1)}} H(\tau,\rho)\nabla_q\tau\nabla_q\rho \\
&= 2\left(\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\nabla_q^{-\nu}(fg)(t) - \nabla_q^{-\nu} f(t)\nabla_q^{-\nu} g(t)\right).
\end{aligned} \tag{8}
$$

Applying the Cauchy-Schwarz inequality for double q-integrals, we have

$$
\begin{aligned}
&\left(\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\nabla_q^{-\nu}(fg)(t) - \nabla_q^{-\nu} f(t)\nabla_q^{-\nu} g(t)\right)^2 \\
&\le \left(\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\nabla_q^{-\nu} f^2(t) - \bigl(\nabla_q^{-\nu} f(t)\bigr)^2\right)\left(\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\nabla_q^{-\nu} g^2(t) - \bigl(\nabla_q^{-\nu} g(t)\bigr)^2\right).
\end{aligned} \tag{9}
$$

Since $(\varphi_2 - f(x))(f(x) - \varphi_1) \ge 0$ and $(\psi_2 - g(x))(g(x) - \psi_1) \ge 0$, we have

$$
\begin{aligned}
&\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\nabla_q^{-\nu}\bigl(\varphi_2 - f(x)\bigr)\bigl(f(x) - \varphi_1\bigr) \ge 0, \\
&\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\nabla_q^{-\nu}\bigl(\psi_2 - g(x)\bigr)\bigl(g(x) - \psi_1\bigr) \ge 0.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\nabla_q^{-\nu} f^2(t) - \bigl(\nabla_q^{-\nu} f(t)\bigr)^2 \\
&\quad\le \left(\varphi_2\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)} - \nabla_q^{-\nu} f(t)\right)\left(\nabla_q^{-\nu} f(t) - \varphi_1\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\right), \\
&\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\nabla_q^{-\nu} g^2(t) - \bigl(\nabla_q^{-\nu} g(t)\bigr)^2 \\
&\quad\le \left(\psi_2\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)} - \nabla_q^{-\nu} g(t)\right)\left(\nabla_q^{-\nu} g(t) - \psi_1\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\right).
\end{aligned} \tag{10}
$$

Combining (9) and (10), from Lemma 1, we deduce that

$$\begin{aligned}
&\left(\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\nabla_q^{-\nu}(fg)(t)-\nabla_q^{-\nu}f(t)\nabla_q^{-\nu}g(t)\right)^2\\
&\quad\le\left(\varphi_2\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}-\nabla_q^{-\nu}f(t)\right)\left(\nabla_q^{-\nu}f(t)-\varphi_1\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\right)\\
&\qquad\times\left(\psi_2\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}-\nabla_q^{-\nu}g(t)\right)\left(\nabla_q^{-\nu}g(t)-\psi_1\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\right).
\end{aligned}\tag{11}$$

Now by using the elementary inequality $4xy\le(x+y)^2$, $x,y\in\mathbb{R}$, we can state that

$$\begin{aligned}
&4\left(\varphi_2\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}-\nabla_q^{-\nu}f(t)\right)\left(\nabla_q^{-\nu}f(t)-\varphi_1\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\right)\\
&\quad\le\left(\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}(\varphi_2-\varphi_1)\right)^2,\\
&4\left(\psi_2\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}-\nabla_q^{-\nu}g(t)\right)\left(\nabla_q^{-\nu}g(t)-\psi_1\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\right)\\
&\quad\le\left(\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}(\psi_2-\psi_1)\right)^2.
\end{aligned}\tag{12}$$

From (11) and (12), we obtain (7). □

Theorem 2 *Let f and g be two functions defined on $\mathbb{T}_{t_0}$ satisfying* (A). *Then, for all $t>0$, $\mu>0$ and $\nu>0$, we have*

$$\begin{aligned}
&\left(\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\nabla_q^{-\mu}(fg)(t)+\frac{t^{\underline{(\mu)}}}{\Gamma_q(\mu+1)}\nabla_q^{-\nu}(fg)(t)-\nabla_q^{-\nu}f(t)\nabla_q^{-\mu}g(t)-\nabla_q^{-\mu}f(t)\nabla_q^{-\nu}g(t)\right)^2\\
&\quad\le\left(\left(\varphi_2\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}-\nabla_q^{-\nu}f(t)\right)\left(\nabla_q^{-\mu}f(t)-\varphi_1\frac{t^{\underline{(\mu)}}}{\Gamma_q(\mu+1)}\right)+\left(\nabla_q^{-\nu}f(t)-\varphi_1\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\right)\right.\\
&\qquad\times\left.\left(\varphi_2\frac{t^{\underline{(\mu)}}}{\Gamma_q(\mu+1)}-\nabla_q^{-\mu}f(t)\right)\right)\left(\left(\psi_2\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}-\nabla_q^{-\nu}g(t)\right)\right.\\
&\qquad\times\left(\nabla_q^{-\mu}g(t)-\psi_1\frac{t^{\underline{(\mu)}}}{\Gamma_q(\mu+1)}\right)\\
&\qquad+\left.\left(\nabla_q^{-\nu}g(t)-\psi_1\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\right)\left(\psi_2\frac{t^{\underline{(\mu)}}}{\Gamma_q(\mu+1)}-\nabla_q^{-\mu}g(t)\right)\right).
\end{aligned}$$

Proof Since $(\varphi_2-f(x))(f(x)-\varphi_1)\ge 0$ and $(\psi_2-g(x))(g(x)-\psi_1)\ge 0$, then we can write

$$\begin{aligned}
&-\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\nabla_q^{-\mu}\big(\varphi_2-f(x)\big)\big(f(x)-\varphi_1\big)-\frac{t^{\underline{(\mu)}}}{\Gamma_q(\mu+1)}\nabla_q^{-\nu}\big(\varphi_2-f(x)\big)\big(f(x)-\varphi_1\big)\\
&\quad\le 0,\\
&-\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\nabla_q^{-\mu}\big(\psi_2-g(x)\big)\big(g(x)-\psi_1\big)-\frac{t^{\underline{(\mu)}}}{\Gamma_q(\mu+1)}\nabla_q^{-\nu}\big(\psi_2-g(x)\big)\big(g(x)-\psi_1\big)\\
&\quad\le 0.
\end{aligned}\tag{13}$$

Applying Lemma 3 to f and g, then by using Lemma 2 and the formula (13), we obtain Theorem 2. □

4 The other fractional q-integral inequalities

For the sake of simplicity, we always assume that $\nabla_q^\nu \phi$ denotes $\nabla_q^\nu \phi(t)$ and all of fractional q-integrals are finite in this section.

Theorem 3 *Let f and g be two functions defined on $\mathbb{T}_{t_0}$ and $\alpha, \beta > 1$ satisfying $1/\alpha + 1/\beta = 1$. Then the following inequalities hold*:

(a) $\frac{1}{\alpha}\nabla_q^{-\nu}(|f|^\alpha) + \frac{1}{\beta}\nabla_q^{-\nu}(|g|^\beta) \geq \frac{\Gamma_q(\nu+1)}{t^{(\nu)}}\nabla_q^{-\nu}(|f|)\nabla_q^{-\nu}(|g|)$.

(b) $\frac{1}{\alpha}\nabla_q^{-\nu}(|f|^\alpha)\nabla_q^{-\nu}(|g|^\alpha) + \frac{1}{\beta}\nabla_q^{-\nu}(|f|^\beta)\nabla_q^{-\nu}(|g|^\beta) \geq (\nabla_q^{-\nu}(|fg|))^2$.

(c) $\frac{1}{\alpha}\nabla_q^{-\nu}(|f|^\alpha)\nabla_q^{-\nu}(|g|^\beta) + \frac{1}{\beta}\nabla_q^{-\nu}(|f|^\beta)\nabla_q^{-\nu}(|g|^\alpha) \geq \nabla_q^{-\nu}(|f||g|^{\alpha-1})\nabla_q^{-\nu}(|f||g|^{\beta-1})$.

(d) $\nabla_q^{-\nu}(|f|^\alpha)\nabla_q^{-\nu}(|g|^\beta) \geq \nabla_q^{-\nu}(|fg|)\nabla_q^{-\nu}(|f|^{\alpha-1}|g|^{\beta-1})$.

Proof According to the well-known Young inequality,

$$\frac{1}{\alpha}x^\alpha + \frac{1}{\beta}y^\beta \geq xy, \quad \forall x, y \geq 0, \alpha, \beta > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

Putting $x = f(\tau)$ and $y = g(\rho)$, $\tau, \rho > 0$, we have

$$\frac{1}{\alpha}\left|f(\tau)\right|^\alpha + \frac{1}{\beta}\left|g(\rho)\right|^\beta \geq \left|f(\tau)\right|\left|g(\rho)\right|, \quad \forall \tau, \rho > 0. \tag{14}$$

Multiplying both sides of (6) by $(t - q\tau)^{\underline{(\nu-1)}}(t - q\rho)^{\underline{(\nu-1)}}/\Gamma_q^2(\nu)$, we obtain

$$\begin{aligned}&\frac{1}{\alpha}\frac{(t-q\rho)^{\underline{(\nu-1)}}}{\Gamma_q(\nu)}\frac{(t-q\tau)^{\underline{(\nu-1)}}}{\Gamma_q(\nu)}\left|f(\tau)\right|^\alpha + \frac{1}{\beta}\frac{(t-q\tau)^{\underline{(\nu-1)}}}{\Gamma_q(\nu)}\frac{(t-q\rho)^{\underline{(\nu-1)}}}{\Gamma_q(\nu)}\left|g(\rho)\right|^\beta \\ &\quad \geq \frac{(t-q\tau)^{\underline{(\nu-1)}}}{\Gamma_q(\nu)}\left|f(\tau)\right|\frac{(t-q\rho)^{\underline{(\nu-1)}}}{\Gamma_q(\nu)}\left|g(\rho)\right|.\end{aligned}$$

Integrating the preceding identity with respect to τ and ρ from 0 to t, we can state that

$$\frac{1}{\alpha}\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\nabla_q^{-\nu}\left(\left|f(t)\right|^\alpha\right) + \frac{1}{\beta}\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\nabla_q^{-\nu}\left(\left|g(t)\right|^\beta\right) \geq \nabla_q^{-\nu}\left(\left|f(t)\right|\right)\nabla_q^{-\nu}\left(\left|g(t)\right|\right),$$

which implies (a). The rest of inequalities can be proved in the same manner by the next choice of the parameters in the Young inequality:

(b) $x = |f(\tau)||g(\rho)|$, $y = |f(\rho)||g(\tau)|$.

(c) $x = |f(\tau)|/|g(\tau)|$, $y = |f(\rho)|/|g(\rho)|$, $(g(\tau)g(\rho) \neq 0)$.

(d) $x = |f(\rho)|/|f(\tau)|$, $y = |g(\rho)|/|g(\tau)|$, $(f(\tau)g(\rho) \neq 0)$.

Repeating the foregoing arguments, we obtain (b)-(d). □

Theorem 4 *Let f and g be two functions defined on $\mathbb{T}_{t_0}$ and $\alpha, \beta > 1$ satisfying $1/\alpha + 1/\beta = 1$. Then the following inequalities hold*:

(a) $\frac{1}{\alpha}\nabla_q^{-\nu}(|f|^\alpha)\nabla_q^{-\nu}(|g|^2) + \frac{1}{\beta}\nabla_q^{-\nu}(|f|^2)\nabla_q^{-\nu}(|g|^\beta) \geq \nabla_q^{-\nu}(|fg|)\nabla_q^{-\nu}(|f|^{2/\beta}|g|^{2/\alpha})$.

(b) $\frac{1}{\alpha}\nabla_q^{-\nu}(|f|^2)\nabla_q^{-\nu}(|g|^\beta) + \frac{1}{\beta}\nabla_q^{-\nu}(|f|^\beta)\nabla_q^{-\nu}(|g|^2) \geq \nabla_q^{-\nu}(|f|^{2/\alpha}|g|^{2/\beta})\nabla_q^{-\nu}(|f|^{\alpha-1}|g|^{\beta-1})$.

(c) $\nabla_q^{-\nu}(|f|^2)\nabla_q^{-\nu}(\frac{1}{\alpha}|g|^\alpha + \frac{1}{\beta}|g|^\beta) \geq \nabla_q^{-\nu}(|f|^{2/\alpha}|g|)\nabla_q^{-\nu}(|f|^{2/\beta}|g|)$.

Proof As a previous one, the proof is based on the Young inequality with the following appropriate choice of parameters:

(a) $x = |f(\tau)||g(\rho)|^{2/\alpha}$, $y = |f(\rho)|^{2/\beta}|g(\tau)|$.
(b) $x = |f(\tau)|^{2/\alpha}/|f(\rho)|$, $y = |g(\tau)|^{2/\beta}/|g(\rho)|$, $(f(\rho)g(\rho) \neq 0)$.
(c) $x = |f(\tau)|^{2/\alpha}/|g(\rho)|$, $y = |f(\rho)|^{2/\beta}/|g(\tau)|$, $(g(\tau)g(\rho) \neq 0)$. □

Theorem 5 *Let f and g be two positive functions defined on* $\mathbb{T}_{t_0}$ *such that for all* $t > 0$,

$$m = \min_{0\leq\tau\leq t} \frac{f(\tau)}{g(\tau)}, \qquad M = \max_{0\leq\tau\leq t} \frac{f(\tau)}{g(\tau)}. \tag{15}$$

Then the following inequalities hold:

(a) $0 \leq \nabla_q^{-\nu}(f^2)\nabla_q^{-\nu}(g^2) \leq \frac{(m+M)^2}{4mM}(\nabla_q^{-\nu}(fg))^2$.
(b) $0 \leq \sqrt{\nabla_q^{-\nu}(f^2)\nabla_q^{-\nu}(g^2)} - \nabla_q^{-\nu}(fg) \leq \frac{(\sqrt{M}-\sqrt{m})^2}{2\sqrt{mM}}\nabla_q^{-\nu}(fg)$.
(c) $0 \leq \nabla_q^{-\nu}(f^2)\nabla_q^{-\nu}(g^2) - (\nabla_q^{-\nu}(fg))^2 \leq \frac{(M-m)^2}{4mM}(\nabla_q^{-\nu}(fg))^2$.

Proof It follows from (15) and

$$\left(\frac{f(\tau)}{g(\tau)} - m\right)\left(M - \frac{f(\tau)}{g(\tau)}\right)g^2(\tau) \geq 0, \quad 0 \leq \tau \leq t. \tag{16}$$

Multiplying both sides of (15) by $(t - q\tau)^{\underline{(\nu-1)}}/\Gamma_q(\nu)$ and integrating the resulting identity with respect to τ from 0 to t, we can get

$$\nabla_q^{-\nu}(f^2) + mM\nabla_q^{-\nu}(g^2) \leq (m + M)\nabla_q^{-\nu}(fg). \tag{17}$$

On the other hand, it follows from $mM > 0$ and $(\sqrt{\nabla_q^{-\nu}(f^2)} - \sqrt{mM\nabla_q^{-\nu}(g^2)})^2 \geq 0$ that

$$2\sqrt{\nabla_q^{-\nu}(f^2)}\sqrt{mM\nabla_q^{-\nu}(g^2)} \leq \nabla_q^{-\nu}(f^2) + mM\nabla_q^{-\nu}(g^2). \tag{18}$$

According to (17) and (18), we have

$$4mM\nabla_q^{-\nu}(f^2)\nabla_q^{-\nu}(g^2) \leq (m + M)^2\left(\nabla_q^{-\nu}(fg)\right)^2,$$

which implies (a). By a few transformations of (a), similarly, we obtain (b) and (c). □

Corollary 1 *Under the conditions of Theorem* 5, *if* $\alpha, \beta \in (0,1)$, $\alpha + \beta = 1$, *then it follows from the arithmetic-geometric mean inequality that*

$$\left(\frac{1}{\alpha}\nabla_q^{-\nu}(f^2)\right)^\alpha \left(\frac{mM}{\beta}\nabla_q^{-\nu}(g^2)\right)^\beta \leq \nabla_q^{-\nu}(f^2) + mM\nabla_q^{-\nu}(g^2) \leq (m + M)\nabla_q^{-\nu}(fg),$$

which implies that

$$\left(\nabla_q^{-\nu}(f^2)\right)^\alpha\left(\nabla_q^{-\nu}(g^2)\right)^\beta \leq \alpha^\alpha\beta^\beta\frac{m + M}{(mM)^\beta}\nabla_q^{-\nu}(fg).$$

Theorem 6 *Let f and g be two positive functions on $\mathbb{T}_{t_0}$ and*

$$0 < \Phi_1 \le f(\tau) \le \Phi_2 < \infty, \qquad 0 < \Psi_1 \le g(\tau) \le \Psi_2 < \infty. \tag{19}$$

Then the following inequalities hold:

(a) $0 \le \nabla_q^{-\nu}(f^2)\nabla_q^{-\nu}(g^2) \le \frac{(\Phi_1\Psi_1+\Phi_2\Psi_2)^2}{4\Phi_1\Psi_1\Phi_2\Psi_2}(\nabla_q^{-\nu}(fg))^2$.

(b) $0 \le \sqrt{\nabla_q^{-\nu}(f^2)\nabla_q^{-\nu}(g^2)} - \nabla_q^{-\nu}(fg) \le \frac{(\sqrt{\Phi_2\Psi_2}-\sqrt{\Phi_1\Psi_1})^2}{2\sqrt{\Phi_1\Psi_1\Phi_2\Psi_2}}\nabla_q^{-\nu}(fg)$.

(c) $0 \le \nabla_q^{-\nu}(f^2)\nabla_q^{-\nu}(g^2) - (\nabla_q^{-\nu}(fg))^2 \le \frac{(\Phi_2\Psi_2-\Phi_1\Psi_1)^2}{4\Phi_1\Psi_1\Phi_2\Psi_2}(\nabla_q^{-\nu}(fg))^2$.

Proof Under the conditions satisfied by the functions f and g, we have

$$\frac{\Phi_1}{\Psi_2} \le \frac{f(\tau)}{g(\tau)} \le \frac{\Phi_2}{\Psi_1}.$$

Applying Theorem 6, we get the inequality (a) and using it, we have (b) and (c). □

Corollary 2 *Let f be a positive function on $\mathbb{T}_{t_0}$ satisfying* (19). *Then the following inequality holds*:

$$\nabla_q^{-\nu}(f^2) \le \frac{\Gamma_q(\nu+1)(\Phi_1+\Phi_2)^2}{4t^{\underline{(\nu)}}\Phi_1\Phi_2}(\nabla_q^{-\nu}(f))^2.$$

Theorem 7 *Let f and g be two positive functions on $\mathbb{T}_{t_0}$ and*

$$0 < m \le \frac{g(\tau)}{f(\tau)} \le M < \infty \tag{20}$$

and $p \ne 0$ be a real number, then the following inequality holds:

$$\nabla_q^{-\nu}(f^{2-p}g^p) + \frac{mM(M^{p-1}-m^{p-1})}{M-m}\nabla_q^{-\nu}(f^p) \le \frac{M^p-m^p}{M-m}\nabla_q^{-\nu}(fg)$$

for $p \notin (0,1)$, or reverse for $p \in (0,1)$. Especially, for $p = 2$, we have

$$\nabla_q^{-\nu}(g^2) + mM\nabla_q^{-\nu}(f^2) \le (m+M)\nabla_q^{-\nu}(fg).$$

Proof The inequality is based on the Lah-Ribaric inequality [18, p.9] and [19, p.123]. □

Theorem 8 *Let f and g be two positive functions on $\mathbb{T}_{t_0}$ and $p \ne 0$ be a real number. Then the following inequality holds*:

$$(\nabla_q^{-\nu}(fg))^p \le (\nabla_q^{-\nu}(f^2))^{p-1}\nabla_q^{-\nu}(f^{2-p}g^p)$$

for $p \notin (0,1)$, or reverse for $p \in (0,1)$.

Proof The above inequality is obtained via the Jensen inequality for the convex functions. □

Corollary 3 *Let f be a positive function on $\mathbb{T}_{t_0}$ and $p \neq 0$ be a real number. Then the following inequality holds*:

$$\left(\nabla_q^{-\nu}(f)\right)^p \le \left(\frac{t^{(\nu)}}{\Gamma_q(\nu+1)}\right)^{p-1} \nabla_q^{-\nu}\left(f^p\right)$$

for $p \notin (0,1)$, or reverse for $p \in (0,1)$.

Theorem 9 *Let p, f and g be three positive functions on $\mathbb{T}_{t_0}$ satisfying* (19). *If $0 < \alpha \le \beta < 1$, $\alpha + \beta = 1$, then the following inequalities hold*:

$$\left(\nabla_q^{-\nu}(pf)\right)^\beta \left(\nabla_q^{-\nu}\left(\frac{p}{f}\right)\right)^\alpha \le \frac{\alpha\Phi_1 + \beta\Phi_2}{(\Phi_1\Phi_2)^\alpha} \nabla_q^{-\nu}(p), \tag{21}$$

$$\left(\nabla_q^{-\nu}\left(pf^2\right)\right)^\beta \left(\nabla_q^{-\nu}\left(pg^2\right)\right)^\alpha \le \frac{\alpha\Phi_1\Psi_1 + \beta\Phi_2\Psi_2}{(\Phi_1\Phi_2)^\alpha(\Psi_1\Psi_2)^\beta} \nabla_q^{-\nu}(pfg). \tag{22}$$

Proof Since $(\beta f(\tau) - \alpha\Phi_1)(f(\tau) - \Phi_2) \le 0$ on $\mathbb{T}_{t_0}$, we have

$$\beta f^2(\tau) - (\alpha\Phi_1 + \beta\Phi_2)f(\tau) + \alpha\Phi_1\Phi_2 \le 0. \tag{23}$$

Multiplying both sides of (23) by $p(\tau)/f(\tau)$, we get

$$\beta p(\tau)f(\tau) + \alpha\Phi_1\Phi_2\frac{p(\tau)}{f(\tau)} \le (\alpha\Phi_1 + \beta\Phi_2)p(\tau). \tag{24}$$

From (24) and arithmetic-geometric mean inequality, we obtain

$$\begin{aligned}
&\left(\frac{1}{\Gamma_q(\nu)}\int_0^t (t-q\tau)^{\underline{(\nu-1)}} p(\tau)f(\tau)\nabla\tau\right)^\beta \left(\frac{1}{\Gamma_q(\nu)}\int_0^t (t-q\tau)^{\underline{(\nu-1)}} \frac{p(\tau)}{f(\tau)}\nabla\tau\right)^\alpha \\
&\quad = \frac{1}{(\Phi_1\Phi_2)^\alpha}\left(\frac{1}{\Gamma_q(\nu)}\int_0^t (t-q\tau)^{\underline{(\nu-1)}} p(\tau)f(\tau)\nabla\tau\right)^\beta \left(\frac{\Phi_1\Phi_2}{\Gamma_q(\nu)}\int_0^t (t-q\tau)^{\underline{(\nu-1)}} \frac{p(\tau)}{f(\tau)}\nabla\tau\right)^\alpha \\
&\quad \le \frac{1}{(\Phi_1\Phi_2)^\alpha}\left(\frac{\beta}{\Gamma_q(\nu)}\int_0^t (t-q\tau)^{\underline{(\nu-1)}} p(\tau)f(\tau)\nabla\tau + \frac{\alpha\Phi_1\Phi_2}{\Gamma_q(\nu)}\int_0^t (t-q\tau)^{\underline{(\nu-1)}} \frac{p(\tau)}{f(\tau)}\nabla\tau\right) \\
&\quad \le \frac{\alpha\Phi_1 + \beta\Phi_2}{(\Phi_1\Phi_2)^\alpha}\left(\frac{1}{\Gamma_q(\nu)}\int_0^t (t-q\tau)^{\underline{(\nu-1)}} p(\tau)\nabla\tau\right),
\end{aligned} \tag{25}$$

which implies (21).

Replacing p and f by pfg and f/g in (25), respectively, and $\Phi_1/\Psi_2 \le f(\tau)/g(\tau) \le \Phi_2/\Psi_1$, we get

$$\begin{aligned}
&\left(\frac{1}{\Gamma_q(\nu)}\int_0^t (t-q\tau)^{\underline{(\nu-1)}} p(\tau)f(\tau)\nabla\tau\right)^\beta \left(\frac{1}{\Gamma_q(\nu)}\int_0^t (t-q\tau)^{\underline{(\nu-1)}} p(\tau)g(\tau)\nabla\tau\right)^\alpha \\
&\quad \le \frac{\alpha\Phi_1\Psi_1 + \beta\Phi_2\Psi_2}{(\Phi_1\Phi_2)^\alpha(\Psi_1\Psi_2)^\beta}\left(\frac{1}{\Gamma_q(\nu)}\int_0^t (t-q\tau)^{\underline{(\nu-1)}} p(\tau)f(\tau)g(\tau)\nabla\tau\right),
\end{aligned}$$

which implies (22). □

Corollary 4 *Let p, f and g be three positive functions on $\mathbb{T}_{t_0}$ satisfying* (20). *If* $0 < \alpha \leq \beta < 1$, $\alpha + \beta = 1$, *then the following inequality holds*:

$$\alpha \nabla_q^{-\nu}\left(pg^2\right) + \beta m M \nabla_q^{-\nu}\left(pf^2\right) \leq (\alpha m + \beta M)\nabla_q^{-\nu}(pfg). \tag{26}$$

Proof Replacing Φ_1, Φ_2 and $f(\tau)$ by m, M and $g(\tau)/f(\tau)$ in (24), and multiplying both sides by $(t - q\tau)^{\underline{(\nu-1)}}/\Gamma_q(\nu)$ and integrating the resulting identity with respect to τ from 0 to t, we get (25). □

Theorem 10 *Let p, f and g be three functions on $\mathbb{T}_{t_0}$ with $p(\tau) \geq 0$.*

(a) *If there exist four constants $\Phi_1, \Phi_2, \Psi_1, \Psi_2 \in \mathbb{R}$ such that $(\Phi_2 g(\tau) - \Psi_1 f(\tau))(\Psi_2 f(\tau) - \Phi_1 g(\tau)) \geq 0$ for all $\tau > 0$, then*

$$\begin{aligned}\Phi_1\Phi_2\nabla_q^{-\nu}\left(pg^2\right) + \Psi_1\Psi_2\nabla_q^{-\nu}\left(pf^2\right) &\leq (\Phi_1\Psi_1 + \Phi_2\Psi_2)\nabla_q^{-\nu}(pfg) \\ &\leq |\Phi_1\Psi_1 + \Phi_2\Psi_2|\left(\nabla_q^{-\nu}\left(pf^2\right) + \nabla_q^{-\nu}\left(pg^2\right)\right).\end{aligned} \tag{27}$$

Moreover, if $\Phi_1\Phi_2\Psi_1\Psi_2 > 0$, then

$$\sqrt{\frac{\Phi_1\Phi_2}{\Psi_1\Psi_2}}\nabla_q^{-\nu}\left(pg^2\right) + \sqrt{\frac{\Psi_1\Psi_2}{\Phi_1\Phi_2}}\nabla_q^{-\nu}\left(pf^2\right) \leq \left(\sqrt{\frac{\Phi_2\Psi_2}{\Phi_1\Psi_1}} + \sqrt{\frac{\Phi_1\Psi_1}{\Phi_2\Psi_2}}\right)\nabla_q^{-\nu}(pfg), \tag{28}$$

$$\nabla_q^{-\nu}\left(pg^2\right)\nabla_q^{-\nu}\left(pf^2\right) \leq \left(\frac{\Phi_1\Psi_1 + \Phi_2\Psi_2}{2\Phi_1\Psi_1\Phi_2\Psi_2}\right)^2 \nabla_q^{-\nu}(pfg). \tag{29}$$

(b) *If there exist four constants $\Phi_1, \Phi_2, \Psi_1, \Psi_2 \in \mathbb{R}$ such that $(\Phi_2 g(\tau) - \Psi_1 f(\rho))(\Psi_2 f(\rho) - \Phi_1 g(\tau)) \geq 0$ for all $\tau, \rho > 0$, then*

$$\begin{aligned}&\Phi_1\Phi_2\nabla_q^{-\nu}(p)\nabla_q^{-\nu}\left(pg^2\right) + \Psi_1\Psi_2\nabla_q^{-\nu}(p)\nabla_q^{-\nu}\left(pf^2\right) \\ &\quad\leq (\Phi_1\Psi_1 + \Phi_2\Psi_2)\nabla_q^{-\nu}(pf)\nabla_q^{-\nu}(pg).\end{aligned} \tag{30}$$

(c) *If $\Phi_1\Phi_2 > 0$ and $\Psi_1\Psi_2 > 0$, then*

$$\Phi_1\Phi_2\left(\nabla_q^{-\nu}(pg)\right)^2 + \Psi_1\Psi_2\left(\nabla_q^{-\nu}(pf)\right)^2 \leq (\Phi_1\Psi_1 + \Phi_2\Psi_2)\nabla_q^{-\nu}(p)\nabla_q^{-\nu}(pfg). \tag{31}$$

(d) *If $\Phi_1\Phi_2 > 0$ and $\Psi_1\Psi_2 > 0$, then*

$$\Phi_1\Phi_2\left(\nabla_q^{-\nu}(pg)\right)^2 + \Psi_1\Psi_2\left(\nabla_q^{-\nu}(pf)\right)^2 \leq (\Phi_1\Psi_1 + \Phi_2\Psi_2)\nabla_q^{-\nu}(pf)\nabla_q^{-\nu}(pg). \tag{32}$$

Proof Case (a). It follows from the assumption that

$$p(\tau)\left(\Phi_2 g(\tau) - \Psi_1 f(\tau)\right)\left(\Psi_2 f(\tau) - \Phi_1 g(\tau)\right) \geq 0$$

for all $\tau \geq 0$, which implies that

$$\Phi_1\Phi_2 p(\tau)g^2(\tau) + \Psi_1\Psi_2 p(\tau)f^2(\tau) \leq (\Phi_1\Psi_1 + \Phi_2\Psi_2)p(\tau)f(\tau)g(\tau). \tag{33}$$

Multiplying both sides of (33) by $(t-q\tau)^{\underline{(\nu-1)}}/\Gamma_q(\nu)$ and integrating the resulting identity with respect to τ from 0 to t, we obtain the left-hand side of (27). Furthermore, by Cauchy's inequality, we get the right-hand side of (27).

Multiplying both sides of the inequality

$$\Phi_1\Phi_2\nabla_q^{-\nu}(pg^2)+\Psi_1\Psi_2\nabla_q^{-\nu}(pf^2)\le(\Phi_1\Psi_1+\Phi_2\Psi_2)\nabla_q^{-\nu}(pfg)$$

by $1/\sqrt{\Phi_1\Phi_2\Psi_1\Psi_2}$, we get (28).

On the other hand, it follows from $\Phi_1\Phi_2\Psi_1\Psi_2>0$ and $(\sqrt{\Phi_1\Phi_2\nabla_q^{-\nu}(pg^2)}-\sqrt{\Psi_1\Psi_2\nabla_q^{-\nu}(pf^2)})^2\ge 0$ that

$$2\sqrt{\Phi_1\Phi_2\nabla_q^{-\nu}(pg^2)}\sqrt{\Psi_1\Psi_2\nabla_q^{-\nu}(pf^2)}\le\Phi_1\Phi_2\nabla_q^{-\nu}(pg^2)+\Psi_1\Psi_2\nabla_q^{-\nu}(pf^2). \tag{34}$$

According to (27) and (34), we have

$$4\Phi_1\Phi_2\Psi_1\Psi_2\nabla_q^{-\nu}(pg^2)\nabla_q^{-\nu}(pf^2)\le(\Phi_1\Psi_1+\Phi_2\Psi_2)^2\bigl(\nabla_q^{-\nu}(pfg)\bigr)^2,$$

which implies (29).

Case (b). It follows from the assumption that

$$p(\tau)p(\rho)\bigl(\Phi_2g(\tau)-\Psi_1f(\rho)\bigr)\bigl(\Psi_2f(\rho)-\Phi_1g(\tau)\bigr)\ge 0$$

for all $\tau,\rho>0$, which implies that

$$\begin{aligned}&\Phi_1\Phi_2p(\tau)p(\rho)g^2(\tau)+\Psi_1\Psi_2p(\tau)p(\rho)f^2(\rho)\\&\quad\le\Phi_1\Psi_1p(\tau)p(\rho)f(\rho)g(\tau)+\Phi_2\Psi_2p(\tau)p(\rho)f(\rho)g(\tau).\end{aligned} \tag{35}$$

Multiplying both sides of (35) by $(t-q\tau)^{\underline{(\nu-1)}}(t-q\rho)^{\underline{(\nu-1)}}/\Gamma_q^2(\nu)$ and integrating the resulting identity with respect to τ and ρ from 0 to t, respectively, we obtain (30).

Case (c) and (d). It follows from Cauchy's inequality that

$$\bigl(\nabla_q^{-\nu}(pf)\bigr)^2\le\nabla_q^{-\nu}(p)\nabla_q^{-\nu}(pf^2),\qquad\bigl(\nabla_q^{-\nu}(pg)\bigr)^2\le\nabla_q^{-\nu}(p)\nabla_q^{-\nu}(pg^2).$$

Combining (a), (b) and the preceding two inequalities, we see that

$$\begin{aligned}\Phi_1\Phi_2\bigl(\nabla_q^{-\nu}(pg)\bigr)^2+\Psi_1\Psi_2\bigl(\nabla_q^{-\nu}(pf)\bigr)^2&\le\Phi_1\Phi_2\nabla_q^{-\nu}(p)\nabla_q^{-\nu}(pf^2)+\Psi_1\Psi_2\nabla_q^{-\nu}(p)\nabla_q^{-\nu}(pg^2)\\&\le(\Phi_1\Psi_1+\Phi_2\Psi_2)\nabla_q^{-\nu}(p)\nabla_q^{-\nu}(pfg),\end{aligned}$$

which implies (31). Furthermore,

$$\begin{aligned}\Phi_1\Phi_2\bigl(\nabla_q^{-\nu}(pg)\bigr)^2+\Psi_1\Psi_2\bigl(\nabla_q^{-\nu}(pf)\bigr)^2&\le\Phi_1\Phi_2\nabla_q^{-\nu}(p)\nabla_q^{-\nu}(pf^2)+\Psi_1\Psi_2\nabla_q^{-\nu}(p)\nabla_q^{-\nu}(pg^2)\\&\le(\Phi_1\Psi_1+\Phi_2\Psi_2)\nabla_q^{-\nu}(pf)\nabla_q^{-\nu}(pg),\end{aligned}$$

which implies (32). □

Theorem 11 *Let p, f and g be three positive functions on $\mathbb{T}_{t_0}$ with $p(\tau) \geq 0$. Then we have*

$$\begin{aligned}\left(\nabla_q^{-\nu}(p)\nabla_q^{-\nu}(pfg) + \nabla_q^{-\nu}(pf)\nabla_q^{-\nu}(pg)\right)^2 &\leq \left(\nabla_q^{-\nu}(p)\nabla_q^{-\nu}\left(pf^2\right) + \left(\nabla_q^{-\nu}(pf)\right)^2\right) \\ &\quad\times\left(\nabla_q^{-\nu}(p)\nabla_q^{-\nu}\left(pg^2\right) + \left(\nabla_q^{-\nu}(pg)\right)^2\right). \end{aligned} \tag{36}$$

Moreover, under the assumptions of (a) *and* (b) *in Theorem* 10, *the following inequality holds*:

$$\begin{aligned}&4\Phi_1\Psi_1\Phi_2\Psi_2\left(\nabla_q^{-\nu}(p)\nabla_q^{-\nu}\left(pf^2\right) + \left(\nabla_q^{-\nu}(pf)\right)^2\right)\left(\nabla_q^{-\nu}(p)\nabla_q^{-\nu}\left(pg^2\right) + \left(\nabla_q^{-\nu}(pg)\right)^2\right) \\ &\quad\leq (\Phi_1\Psi_1 + \Phi_2\Psi_2)^2\left(\nabla_q^{-\nu}(p)\nabla_q^{-\nu}(pfg) + \nabla_q^{-\nu}(pf)\nabla_q^{-\nu}(pg)\right)^2. \end{aligned} \tag{37}$$

Proof First of all, we give the proof of (36). By Cauchy's inequality and the element inequality $2xy\sqrt{uv} \leq x^2u + y^2v$, for all $x, y, u, v \geq 0$, we have

$$\begin{aligned}&\left(\nabla_q^{-\nu}(p)\nabla_q^{-\nu}(pfg) + \nabla_q^{-\nu}(pf)\nabla_q^{-\nu}(pg)\right)^2 \\ &\quad= \left(\nabla_q^{-\nu}(p)\right)^2\left(\nabla_q^{-\nu}(pfg)\right)^2 + \left(\nabla_q^{-\nu}(pf)\right)^2\left(\nabla_q^{-\nu}(pf)\right)^2 \\ &\qquad+ 2\nabla_q^{-\nu}(p)\nabla_q^{-\nu}(pf)\nabla_q^{-\nu}(pg)\nabla_q^{-\nu}(pfg) \\ &\quad\leq \left(\nabla_q^{-\nu}(p)\right)^2\left(\nabla_q^{-\nu}(pfg)\right)^2 + \left(\nabla_q^{-\nu}(pf)\right)^2\left(\nabla_q^{-\nu}(pf)\right)^2 \\ &\qquad+ 2\nabla_q^{-\nu}(p)\nabla_q^{-\nu}(pf)\nabla_q^{-\nu}(pg)\sqrt{\nabla_q^{-\nu}\left(pf^2\right)\nabla_q^{-\nu}\left(pg^2\right)} \\ &\quad\leq \left(\nabla_q^{-\nu}(p)\right)^2\nabla_q^{-\nu}\left(pf^2\right)\nabla_q^{-\nu}\left(pg^2\right) + \left(\nabla_q^{-\nu}(pf)\right)^2\left(\nabla_q^{-\nu}(pf)\right)^2 \\ &\qquad+ \nabla_q^{-\nu}(p)\left(\nabla_q^{-\nu}\left(pf^2\right)\left(\nabla_q^{-\nu}(pg)\right)^2 + \nabla_q^{-\nu}\left(pg^2\right)\left(\nabla_q^{-\nu}(pf)\right)^2\right) \\ &\quad= \left(\nabla_q^{-\nu}(p)\nabla_q^{-\nu}\left(pf^2\right) + \left(\nabla_q^{-\nu}(pf)\right)^2\right)\left(\nabla_q^{-\nu}(p)\nabla_q^{-\nu}\left(pg^2\right) + \left(\nabla_q^{-\nu}(pg)\right)^2\right), \end{aligned}$$

which implies (36).

Next, we prove that (37) holds. It follows from (a) and (b) in Theorem 10 that

$$\begin{aligned}(\Phi_1\Psi_1 + \Phi_2\Psi_2)\nabla_q^{-\nu}(p)\nabla_q^{-\nu}(pfg) &\geq \Phi_1\Phi_2\nabla_q^{-\nu}(p)\nabla_q^{-\nu}\left(pg^2\right) + \Psi_1\Psi_2\nabla_q^{-\nu}(p)\nabla_q^{-\nu}\left(pf^2\right) \\ &\geq \Phi_1\Phi_2\nabla_q^{-\nu}(p)\nabla_q^{-\nu}\left(pg^2\right) + \Psi_1\Psi_2\left(\nabla_q^{-\nu}(pf)\right)^2, \\ (\Phi_1\Psi_1 + \Phi_2\Psi_2)\nabla_q^{-\nu}(pf)\nabla_q^{-\nu}(pg) &\geq \Phi_1\Phi_2\nabla_q^{-\nu}(p)\nabla_q^{-\nu}\left(pg^2\right) + \Psi_1\Psi_2\nabla_q^{-\nu}(p)\nabla_q^{-\nu}\left(pf^2\right) \\ &\geq \Phi_1\Phi_2\left(\nabla_q^{-\nu}(pg)\right)^2 + \Psi_1\Psi_2\nabla_q^{-\nu}(p)\nabla_q^{-\nu}\left(pf^2\right). \end{aligned}$$

Combining the preceding two inequalities and the element inequality $(x + y)^2 \geq 4xy$, we see that

$$\begin{aligned}&(\Phi_1\Psi_1 + \Phi_2\Psi_2)^2\left(\nabla_q^{-\nu}(p)\nabla_q^{-\nu}(pfg) + \nabla_q^{-\nu}(pf)\nabla_q^{-\nu}(pg)\right)^2 \\ &\quad= \left((\Phi_1\Psi_1 + \Phi_2\Psi_2)\nabla_q^{-\nu}(p)\nabla_q^{-\nu}(pfg) + (\Phi_1\Psi_1 + \Phi_2\Psi_2)\nabla_q^{-\nu}(pf)\nabla_q^{-\nu}(pg)\right)^2 \\ &\quad\geq \left(\Phi_1\Phi_2\nabla_q^{-\nu}(p)\nabla_q^{-\nu}\left(pg^2\right) + \Psi_1\Psi_2\left(\nabla_q^{-\nu}(pf)\right)^2\right. \\ &\qquad\left.+ \Phi_1\Phi_2\left(\nabla_q^{-\nu}(pg)\right)^2 + \Psi_1\Psi_2\nabla_q^{-\nu}(p)\nabla_q^{-\nu}\left(pf^2\right)\right)^2 \end{aligned}$$

$$
\begin{aligned}
&= \big(\Phi_1\Phi_2\big(\nabla_q^{-\nu}(p)\nabla_q^{-\nu}\big(pg^2\big) + \big(\nabla_q^{-\nu}(pg)\big)^2\big) \\
&\quad + \Psi_1\Psi_2\big(\nabla_q^{-\nu}(p)\nabla_q^{-\nu}\big(pf^2\big) + \big(\nabla_q^{-\nu}(pf)\big)^2\big)\big)^2 \\
&\geq 4\Phi_1\Psi_1\Phi_2\Psi_2\big(\nabla_q^{-\nu}(p)\nabla_q^{-\nu}\big(pf^2\big) + \big(\nabla_q^{-\nu}(pf)\big)^2\big)\big(\nabla_q^{-\nu}(p)\nabla_q^{-\nu}\big(pg^2\big) + \big(\nabla_q^{-\nu}(pg)\big)^2\big),
\end{aligned}
$$

which implies (37). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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