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The Berry-Esséen bounds for kernel density estimator under dependent sample

Wenzhi Yang and Shuhe Hu*

*Correspondence: hushuhe@263.net
School of Mathematical Science, Anhui University, Hefei, 230039, P.R. China

Abstract

Let $\{X_n\}_{n\geq 1}$ be a φ-mixing sequence with an unknown common probability density function $f(x)$ and the mixing coefficients satisfy $\varphi(n) = O(n^{-18/5})$. By using some inequalities for φ-mixing random variables and selecting some positive bandwidths h_n, we investigate the Berry-Esséen bounds of the estimator $f_n(x)$ for $f(x)$ and its bounds are presented as $O(n^{-1/6} \cdot \log n \cdot \log\log n)$ and $O(n^{-1/6} \cdot \log n \cdot \log\log n) + O(h_n^{\delta}) + O(h_n^{13(1-\delta)/5})$, where $0 < \delta < 1$.

MSC: 62G05; 62G07

Keywords: kernel density estimation; bandwidths h_n; Berry-Esséen bound; φ-mixing sequence

1 Introduction

The most popular nonparametric estimator of a distribution based on a sample of observations is the empirical distribution, and the most popular method of nonparametric density estimation is the kernel method. For an introduction and applications of this field, the books by Prakasa Rao [1] and Silverman [2] provide the basic methods for density estimation. For the nonparametric curve estimation from time series such as φ-mixing, ρ-mixing and α-mixing, Györfi *et al.* [3] studied the density estimator and hazard function estimator for these mixing sequences. It is known that φ-mixing $\Rightarrow$ ρ-mixing $\Rightarrow$ α-mixing, and its converse is not true. Although, φ-mixing is stronger than α-mixing, some properties of φ-mixing such as moment inequality, exponential inequality, *etc.*, are better than those of α-mixing to use. For the properties and examples of mixing, we can read the book of Doukhan [4]. In this paper, we only give the definition of a φ-mixing sequence. For the basic properties of φ-mixing, one can refer to Billingsley [5].

Denote $\mathcal{F}_n^m = \sigma(X_i, n \leq i \leq m)$ and define the coefficients as follows:

$$\varphi(n) = \sup_{m\geq 1} \sup_{A\in\mathcal{F}_1^m, B\in\mathcal{F}_{m+n}^{\infty}, P(A)\neq 0} \bigl|P(B|A) - P(B)\bigr|.$$

If $\varphi(n) \downarrow 0$ as $n \to \infty$, then $\{X_n\}_{n\geq 1}$ is said to be a φ-mixing sequence.

Many works have been done for the kernel density estimation. For example, Masry [6] gave the recursive probability density estimation under a mixing-dependent sample, Fan and Yao [7] summarized the nonparametric and parametric methods including a nonparametric density estimator for nonlinear time series such as φ-mixing, α-mixing, *etc.* For an independent sample, Cao [8] investigated the bootstrap approximations in a nonparametric density estimator and obtained Berry-Esséen bounds for the kernel density estimation.

Under φ-mixing dependence errors, Li *et al.* [9] obtained the asymptotic normality of a wavelet estimator of the regression model. Li *et al.* [10] also gave the Berry-Esséen bound of a wavelet estimator of the regression model. Meanwhile, Yang *et al.* [11] studied the Berry-Esséen bound of sample quantiles for φ-mixing random variables. In this paper, we will investigate the Berry-Esséen bounds for a kernel density estimator under a φ-mixing dependent sample.

Let $\{X_n\}_{n\geq 1}$ be a φ-mixing sequence with an unknown common probability density function $f(x)$ and the mixing coefficients satisfy $\varphi(n) = O(n^{-18/5})$. With the help of techniques of inequalities such as moment inequality, exponential inequality and the Bernstein's big-block and small-block procedure, by selecting some positive bandwidths h_n, which do not depend on the mixing coefficients and the lengths of Bernstein's big-block and small-block, we investigate the Berry-Esséen bounds of the estimator $f_n(x)$ for $f(x)$ and its bounds are presented as $O(n^{-1/6} \cdot \log n \cdot \log\log n)$ and $O(n^{-1/6} \cdot \log n \cdot \log\log n) + O(h_n^{\delta}) + O(h_n^{13(1-\delta)/5})$, where $0 < \delta < 1$. Particularly, if $\delta = 13/18$ and $h_n = n^{-16/69}$, the bound is presented as $O(n^{-1/6} \cdot \log n \cdot \log\log n)$. For details, please see our results in Section 3. Some assumptions and lemmas are presented in Section 2. Regarding the technique of Bernstein's big-block and small-block procedure, the reader can refer to Masry [6, 12], Fan and Yao [7], Roussas [13] and the references therein.

For the kernel density estimator under association and a negatively associated sample, one can refer to Roussas [13] and Liang and Baek [14] obtained for asymptotic normality, Wei [15] for the consistences, Henriques and Oliveira [16] for exponential rates, Liang and Baek [17] for the Berry-Esséen bounds, *etc.* Regarding other works about the Berry-Esséen bounds, we can refer to Chang and Rao [18] for the Kaplan-Meier estimator, Cai and Roussas [19] for the smooth estimator of a distribution function, Yang [20] for the regression weighted estimator, Dedecker and Prieur [21] for some new dependence coefficients, examples and applications to statistics, Yang *et al.* [22] for sample quantiles under negatively associated sample, Herve *et al.* [23] for M-estimators of geometrically ergodic Markov chains, and so on. On the other hand, Härdle *et al.* [24] summarized the Berry-Esséen bounds of partially linear models (see Chapter 5 of Härdle *et al.* [24]).

Throughout the paper, $c, c_1, c_2, \dots, C, M_0$ denote some positive constants not depending on n, which may be different in various places, $\lfloor x \rfloor$ means the largest integer not exceeding x and $I(A)$ is the indicator function of the set A. Let $c(x)$ be some positive constant depending only on x. For convenience, we denote $c = c(x)$ in this paper, whose value may vary at different places.

2 Some assumptions and lemmas

For the unknown common probability density function $f(x)$, we assume that

$$f(x) \in C_{s,\alpha}, \tag{2.1}$$

where α is a positive constant and $C_{s,\alpha}$ is a family of probability density functions having derivatives of sth order, $f^{(s)}(x)$ are continuous and $|f^{(s)}(x)| \leq \alpha$, $s = 0, 1, 2, \dots$.

Let $K(\cdot)$ be a kernel function in R and satisfy the following condition (A_1):

(A_1) Assume that $K(\cdot)$ is a bounded probability density function and $K(\cdot) \in H_s$, where H_s

is a class of functions $K(\cdot)$ with the properties

$$\int_{-\infty}^{\infty} u^r K(u)\,du = 0, \quad r = 1,2,\ldots,s-1, \qquad \int_{-\infty}^{\infty} u^s K(u)\,du = A \neq 0. \tag{2.2}$$

Here A is a finite constant and s is a positive integer for $s \geq 2$.

Obviously, the probability density functions Gaussian kernel $K(x) = (2\pi)^{-\frac{1}{2}} \exp\{-\frac{x^2}{2}\}$ and Epanechnikov kernel $K(x) = \frac{3}{20\sqrt{5}}(5 - x^2)I(|x| \leq \sqrt{5})$ belong to H_2. For more details, one can refer to Chapter 2 of Prakasa Rao [1].

For a fixed x, the kernel-type estimator of $f(x)$ is defined as

$$f_n(x) = \frac{1}{nh_n} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h_n}\right), \tag{2.3}$$

where h_n is a sequence of positive bandwidths tending to zero as $n \to \infty$.

Similar to the proof of Theorem 2.2 of Wei [15], we have, by using Taylor's expansion for $f(x - h_n u)$, that

$$f(x - h_n u) = f(x) - f'(x)h_n u + \cdots + \frac{f^{(s-1)}(x)}{(s-1)!}(-h_n u)^{s-1} + \frac{f^{(s)}(x - \xi h_n u)}{s!}(-h_n u)^s,$$

where $0 < \xi < 1$. By (2.1) and (2.2), it follows

$$\left|Ef_n(x) - f(x)\right| \leq \int_{-\infty}^{\infty} \left|K(u)h_n^s u^s\right| \cdot \left|\frac{f^{(s)}(x - \xi h_n u)}{s!}\right| du \leq ch_n^s,$$

which yields

$$\left|Ef_n(x) - f(x)\right| = O\left(h_n^s\right).$$

For $s \geq 2$, one can get the 'bias' term rate as

$$\sqrt{nh_n}\left|Ef_n(x) - f(x)\right| \leq cn^{1/2}h_n^{(2s+1)/2},$$

by providing $n^{1/2}h_n^{(2s+1)/2} \to 0$.

It can be checked that $K(x) = (2\pi)^{-\frac{1}{2}} \exp\{-\frac{x^2}{2}\}$ and $K(x) = \frac{3}{20\sqrt{5}}(5 - x^2)I(|x| \leq \sqrt{5})$ belong to H_2. So, with $s = 2$, one can see that $h_n = n^{-1/4}$ satisfies the conditions $0 < h_n \to 0$ and $n^{1/2}h_n^{(2s+1)/2} \to 0$ as $n \to \infty$. Consequently, we pay attention to the Berry-Esséen bound of the centered variate as

$$\sqrt{nh_n}\left(f_n(x) - Ef_n(x)\right)$$

in this paper.

Similar to Masry [6] and Roussas [13], we give the following assumption.

(A_2) Assume that $f(x, y, k)$ are the joint p.d.f. of the random variables X_j and X_{j+k}, $j = 1, 2, \ldots$, which satisfy

$$\sup_{x,y}\left|f(x, y, k) - f(x)f(y)\right| \leq M_0, \quad \text{for } k \geq 1.$$

Under the assumption (A_2) and other conditions, Masry [6] gave the asymptotic normality for the density estimator under a mixing dependent sample and Roussas [13] obtained the asymptotic normality for the kernel density estimator under an association sample. Unlike the mixing case, association and negatively associated random variables $X_1, X_2, \ldots, X_n$ are subject to the transformation $K(\frac{x-X_i}{h_n})$, $i = 1, 2, \ldots, n$, losing in the process the association or negatively associated property, *i.e.*, the kernel weights $K(\frac{x-X_i}{h_n})$, $i = 1, 2, \ldots, n$, are not necessarily association or negatively associated random variables (see Roussas [13] and Liang and Baek [14, 17]). In addition, if $K(x) = \frac{1}{2}I\ (-1 \le x \le 1)$, which is a function of bounded variation, then $K(x) = K_1(x) - K_2(x)$, where $K_1(x) = \frac{1}{2}I\ (x \le 1)$ and $K_2(x) = \frac{1}{2}I\ (x < -1)$ are bounded and monotone nonincreasing functions. Although the transformations $\{K_1(\frac{x-X_i}{h_n}), 1 \le i \le n\}$ and $\{K_2(\frac{x-X_i}{h_n}), 1 \le i \le n\}$ are also the association or negatively associated random variables, $K_1(x)$ and $K_2(x)$ are not integrable in R. So, there are some difficulties in investigating the kernel density estimator under these dependent samples. Meanwhile, the nonparametric estimation and nonparametric tests for association and negatively associated random variables can be found in Prakasa Rao [25].

In order to obtain the Berry-Esséen bounds for the kernel density estimator under a φ-mixing sample, we give some useful inequalities such as covariance inequality, moment inequality, characteristic function inequality and exponential inequality for a φ-mixing sequence.

Lemma 2.1 (Billingsley [5], inequality (20.28), p.171) *If* $E|\xi| < \infty$ *and* $P(|\eta| > C) = 0$ (ξ *measurable* $\mathcal{M}_{-\infty}^{k}$ *and* η *measurable* $\mathcal{M}_{k+n}^{\infty}$), *then*

$$\left|E(\xi\eta) - E\xi E\eta\right| \le 2C\varphi(n)E|\xi|.$$

Lemma 2.2 (Yang [26], Lemma 2) *Let* $\{X_n\}_{n\ge1}$ *be a mean zero* φ*-mixing sequence with* $\sum_{n=1}^{\infty}\varphi^{1/2}(n) < \infty$. *Assume that there exists some* $p \ge 2$ *such that* $E|X_n|^p < \infty$ *for all* $n \ge 1$. *Then*

$$E\left|\sum_{i=1}^{n} X_i\right|^p \le C\left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right\}, \quad n \ge 1,$$

where C *is a positive constant depending only on* $\varphi(\cdot)$.

Lemma 2.3 (Li *et al.* [9], Lemma 3.4) *Let* $\{X_n\}_{n\ge1}$ *be a* φ*-mixing sequence. Suppose that* p *and* q *are two positive integers. Set* $\eta_l = \sum_{j=(l-1)(p+q)+1}^{(l-1)(p+q)+p} X_j$ *for* $1 \le l \le k$. *Then*

$$\left|E\exp\left\{it\sum_{l=1}^{k}\eta_l\right\} - \prod_{l=1}^{k}E\exp\{it\eta_l\}\right| \le C|t|\varphi(q)\sum_{l=1}^{k}E|\eta_l|.$$

Lemma 2.4 *Let* X *and* Y *be random variables. Then for any* $a > 0$,

$$\sup_t\left|P(X + Y \le t) - \Phi(t)\right| \le \sup_t\left|P(X \le t) - \Phi(t)\right| + \frac{a}{\sqrt{2\pi}} + P\big(|Y| > a\big).$$

Remark 2.1 Lemma 2.4 is due to Petrov (Petrov [27], Lemma 1.9, p.20 and p.36, lines 19-20). It can also be found in Lemma 2 of Chang and Rao [18].

Lemma 2.5 (Yang *et al.* [11], Corollary A.1) *Let $\{X_n\}_{n\ge 1}$ be a mean zero φ-mixing sequence with $|X_n| \le d < \infty$, a.s., for all $n \ge 1$. For $0 < \lambda < 1$, let $m = \lfloor n^\lambda \rfloor$ and $\Delta_2 = \sum_{i=1}^n EX_i^2$. Then for $\forall\varepsilon > 0$ and $n \ge 2$,*

$$P\left(\left|\sum_{i=1}^n X_i\right| \ge \varepsilon\right) \le 2eC_1 \exp\left\{-\frac{\varepsilon^2}{2C_2(2\Delta_2 + n^\lambda d\varepsilon)}\right\},$$

where $C_1 = \exp\{2en^{1-\lambda}\varphi(m)\}$, $C_2 = 4[1 + 4\sum_{i=1}^{2m} \varphi^{1/2}(i)]$.

3 Main results

Theorem 3.1 *For $s \ge 2$, let the condition (A_1) hold true. Assume that $\{X_n\}_{n\ge 1}$ is a sequence of identically distributed φ-mixing random variables with the mixing coefficients $\varphi(n) = O(n^{-18/5})$. If $h_n^{-1/2} \le cn^{8/69}$, $0 < h_n \to 0$ as $n \to \infty$ and $\liminf_{n\to\infty}\{nh_n \operatorname{Var}(f_n(x))\} = \sigma_1^2(x) > 0$, then*

$$\begin{aligned}&\sup_{-\infty<t<\infty}\left|P\left(\frac{\sqrt{nh_n}(f_n(x) - Ef_n(x))}{\sqrt{\operatorname{Var}(\sqrt{nh_n} f_n(x))}} \le t\right) - \Phi(t)\right| \\ &\quad= O\big(n^{-1/6} \cdot \log n \cdot \log\log n\big), \quad n \to \infty, \end{aligned} \tag{3.1}$$

where $\Phi(\cdot)$ is the standard normal distribution function.

Proof It can be found that

$$\frac{\sqrt{nh_n}(f_n(x) - Ef_n(x))}{\sqrt{\operatorname{Var}(\sqrt{nh_n} f_n(x))}} = \frac{\sum_{i=1}^n Z_{n,i}(x)}{\sqrt{\operatorname{Var}(\sum_{i=1}^n Z_{n,i}(x))}}, \tag{3.2}$$

where $Z_{n,i}(x) = \frac{1}{\sqrt{h_n}}[K(\frac{x-X_i}{h_n}) - EK(\frac{x-X_i}{h_n})]$. We employ the Bernstein's big-block and small-block procedure to prove (3.1). Denote

$$\mu = \mu_n = \lfloor n^{2/3} \rfloor, \qquad \nu = \nu_n = \lfloor n^{1/6} \rfloor, \qquad k = k_n = \left\lfloor \frac{n}{\mu_n + \nu_n} \right\rfloor = \lfloor n^{1/3} \rfloor, \tag{3.3}$$

and $\tilde{Z}_{n,i}(x) = Z_{n,i}(x)/\sqrt{\operatorname{Var}(\sum_{i=1}^n Z_{n,i}(x))}$. Define η_j, ξ_j, ζ_k as follows:

$$\eta_j = \sum_{i=j(\mu+\nu)+1}^{j(\mu+\nu)+\mu} \tilde{Z}_{n,i}(x), \quad 0 \le j \le k-1, \tag{3.4}$$

$$\xi_j = \sum_{i=j(\mu+\nu)+\mu+1}^{(j+1)(\mu+\nu)} \tilde{Z}_{n,i}(x), \quad 0 \le j \le k-1, \tag{3.5}$$

$$\zeta_k = \sum_{i=k(\mu+\nu)+1}^{n} \tilde{Z}_{n,i}(x). \tag{3.6}$$

By (3.2), (3.4), (3.5) and (3.6), one has

$$S_n = \frac{\sum_{i=1}^n Z_{n,i}(x)}{\sqrt{\operatorname{Var}(\sum_{i=1}^n Z_{n,i}(x))}} = \sum_{j=0}^{k-1} \eta_j + \sum_{j=0}^{k-1} \xi_j + \zeta_k = S_n' + S_n'' + S_n'''. \tag{3.7}$$

From (3.5) and (3.7), it follows

$$E\big[S_n''\big]^2 = \operatorname{Var}\left[\sum_{j=0}^{k-1} \xi_j\right] = \sum_{j=0}^{k-1} \operatorname{Var}[\xi_j] + 2 \sum_{0\le i<j\le k-1} \operatorname{Cov}(\xi_i, \xi_j) := I_1 + I_2. \tag{3.8}$$

We have by (2.1) and (A_1) that

$$EZ_{n,i}^2(x) = EZ_{n,1}^2(x) \le c_1 h_n^{-1} EK^2\left(\frac{x - X_1}{h_n}\right) = c_1 h_n^{-1} \int_{-\infty}^{\infty} K^2\left(\frac{x-u}{h_n}\right) f(u)\, du \le c_2.$$

So, by the conditions $\liminf_{n\to\infty}\{nh_n \operatorname{Var}(f_n(x))\} = \liminf_{n\to\infty}\{n^{-1}\operatorname{Var}(\sum_{i=1}^n Z_{n,i}(x))\} = \sigma_1^2(x) > 0$, $\varphi(n) = O(n^{-18/5})$ and $EZ_{n,i}(x) = 0$, we apply Lemma 2.2 with $p = 2$ and obtain that

$$\operatorname{Var}[\xi_j] = E\left[\sum_{i=j(\mu+\nu)+1}^{(j+1)(\mu+\nu)} \tilde{Z}_{n,i}(x)\right]^2 \le \frac{c_3}{n} E\left[\sum_{i=j(\mu+\nu)+1}^{(j+1)(\mu+\nu)} Z_{n,i}(x)\right]^2 \le \frac{c_4}{n}\nu_n.$$

Consequently,

$$I_1 = \sum_{j=0}^{k-1} \operatorname{Var}[\xi_j] \le \frac{c_3 k_n \nu_n}{n} = O\big(n^{-1/2}\big). \tag{3.9}$$

Meanwhile, one has $|\tilde{Z}_{n,i}(x)| \le c_1 n^{-1/2} h_n^{-1/2}$, $E|\tilde{Z}_{n,i}(x)| \le c_2 n^{-1/2} h_n^{1/2}$, $1 \le i \le n$. With $\lambda_j = j(\mu_n + \nu_n) + \mu_n$,

$$I_2 = 2\sum_{0\le i<j\le k-1} \operatorname{Cov}(\xi_i, \xi_j) = 2\sum_{0\le i<j\le k-1} \sum_{l_1=1}^{\nu_n}\sum_{l_2=1}^{\nu_n} \operatorname{Cov}\big[\tilde{Z}_{n,\lambda_i+l_1}(x), \tilde{Z}_{n,\lambda_j+l_2}(y)\big],$$

but since $i \ne j$, $|\lambda_i - \lambda_j + l_1 - l_2| \ge \mu_n$, we have, by applying Lemma 2.1 with $\varphi(n) = O(n^{-18/5})$ and (3.3), that

$$\begin{aligned} |I_2| &\le 2 \sum_{\substack{1\le i<j\le n\\ j-i\ge\mu_n}} \big|\operatorname{Cov}\big[\tilde{Z}_{n,i}(x), \tilde{Z}_{n,j}(x)\big]\big| \le 4c_1c_2 \sum_{\substack{1\le i<j\le n\\ j-i\ge\mu_n}} n^{-1/2}h_n^{-1/2}n^{-1/2}h_n^{1/2}\varphi(j-i) \\ &\le c_3 \sum_{k\ge\mu_n} k^{-18/5} \le c_4 \mu_n^{-13/5} = O\big(n^{-26/15}\big). \end{aligned} \tag{3.10}$$

So, by (3.8), (3.9) and (3.10), one has

$$E\big[S_n''\big]^2 = O\big(n^{-1/2}\big). \tag{3.11}$$

On the other hand, by $\varphi(n) = O(n^{-18/5})$, $EZ_{n,i}(x) = 0$ and Lemma 2.1 with $p = 2$, we obtain that

$$\begin{aligned} E\big[S_n'''\big]^2 &\le \frac{c_7}{n} E\left(\sum_{i=k(\mu+\nu)+1}^{n} Z_{n,i}\right)^2 \le \frac{c_8}{n}\big(n - k_n(\mu_n + \nu_n)\big) \\ &\le \frac{c_9(\mu_n + \nu_n)}{n} = O\big(n^{-1/3}\big). \end{aligned} \tag{3.12}$$

Now, we turn to estimate $\sup_{-\infty<t<\infty} |P(S'_n \le t) - \Phi(t)|$. Define

$$s_n^2 = \sum_{j=0}^{k-1} \operatorname{Var}(\eta_j), \qquad \Gamma_n = \sum_{0\le i<j\le k-1} \operatorname{Cov}(\eta_i, \eta_j).$$

Since $ES_n^2 = 1$, one has

$$E(S'_n)^2 = E\big[S_n - (S''_n + S'''_n)\big]^2 = 1 + E(S''_n + S'''_n)^2 - 2E\big[S_n(S''_n + S'''_n)\big].$$

Combining (3.11) with (3.12), one can check that

$$\begin{aligned}\big|E(S'_n)^2 - 1\big| &= \big|E(S''_n + S'''_n)^2 - 2E\big[S_n(S''_n + S'''_n)\big]\big| \\ &\le E(S''_n)^2 + E(S'''_n)^2 + 2\big[E(S''_n)^2\big]^{1/2}\big[E(S'''_n)^2\big]^{1/2} \\ &\quad + 2\big[E(S_n^2)\big]^{1/2}\big[E(S''_n)^2\big]^{1/2} + 2\big[E(S_n^2)\big]^{1/2}\big[E(S'''_n)^2\big]^{1/2} \\ &= O(n^{-1/4}) + O(n^{-1/6}) = O(n^{-1/6}).\end{aligned} \tag{3.13}$$

With $\lambda_j = j(\mu_n + \nu_n)$, $i \ne j$, $|\lambda_i - \lambda_j + l_1 - l_2| \ge \nu_n$, one has

$$2\Gamma_n = 2\sum_{0\le i<j\le k-1} \operatorname{Cov}(\eta_i, \eta_j) = 2\sum_{0\le i<j\le k-1}\sum_{l_1=1}^{\mu_n}\sum_{l_2=1}^{\mu_n} \operatorname{Cov}\big[\tilde{Z}_{n,\lambda_i+l_1}(x), \tilde{Z}_{n,\lambda_j+l_2}(x)\big].$$

So, similar to the proof of (3.10), by Lemma 2.1 with $\varphi(n) = O(n^{-18/5})$, $|\tilde{Z}_{n,i}(x)| \le c_1 n^{-1/2} h_n^{-1/2}$ and $E|\tilde{Z}_{n,j}(x)| \le c_2 n^{-1/2} h_n^{1/2}$, we have that

$$\begin{aligned}|\Gamma_n| &\le 2\sum_{\substack{1\le i<j\le n\\ j-i\ge\nu_n}} \big|\operatorname{Cov}\big[\tilde{Z}_{n,i}(x), \tilde{Z}_{n,j}(x)\big]\big| \le 4c_1c_2 \sum_{\substack{1\le i<j\le n\\ j-i\ge\nu_n}} n^{-1/2}h_n^{-1/2}n^{-1/2}h_n^{1/2}\varphi(j-i) \\ &\le c_3 \sum_{k\ge\nu_n} k^{-18/5} \le c_4 \nu_n^{-13/5} = O(n^{-13/30}).\end{aligned} \tag{3.14}$$

Obviously,

$$s_n^2 = E\big[S'_n\big]^2 - 2\Gamma_n, \tag{3.15}$$

by (3.13), (3.14) and (3.15), we obtain that

$$\big|s_n^2 - 1\big| = O(n^{-1/6}). \tag{3.16}$$

Let η'_j, $j = 0, 1, \ldots, k-1$, be the independent random variables and η'_j have the same distribution as η_j for $j = 0, 1, \ldots, k-1$. Put $B_n = \sum_{j=0}^{k-1} \eta'_j$. It can be seen that

$$\begin{aligned}\sup_{-\infty<t<\infty} \big|P(S'_n \le t) - \Phi(t)\big| &\le \sup_{-\infty<t<\infty} \big|P(S'_n \le t) - P(B_n \le t)\big| \\ &\quad + \sup_{-\infty<t<\infty} \big|P(B_n \le t) - \Phi(t/s_n)\big| \\ &\quad + \sup_{-\infty<t<\infty} \big|\Phi(t/s_n) - \Phi(t)\big| := F_1 + F_2 + F_3.\end{aligned} \tag{3.17}$$

Denote the characteristic functions of S_n' and B_n by $\varphi(t)$ and $\psi(t)$, respectively. Using the Esséen inequality (Petrov [27], Theorem 5.3), for any $T > 0$, we have

$$F_1 \leq \int_{-T}^{T} \left| \frac{\varphi(t) - \psi(t)}{t} \right| dt + T \sup_{-\infty<t<\infty} \int_{|u|\leq \frac{C}{T}} \left| P(B_n \leq u + t) - P(B_n \leq t) \right| du$$
$$:= F_{1n} + F_{2n}. \tag{3.18}$$

It is a simple fact that

$$E\left|Z_{n,i}(x)\right|^3 \leq c_1 h_n^{-3/2} EK^3\left(\frac{x - X_1}{h_n}\right)$$
$$= c_1 h_n^{-3/2} \int_{-\infty}^{\infty} K^3\left(\frac{x-u}{h_n}\right) f(u)\, du \leq c_2 h_n^{-1/2}, \quad 1 \leq i \leq n$$

and $EZ_{n,i}^2(x) \leq c_3$, $1 \leq i \leq n$. Applying Lemma 2.2 with $p = 3$, we obtain by $h_n^{-1/2} \leq cn^{8/69}$ and $\liminf_{n\to\infty}\{n^{-1}\operatorname{Var}(\sum_{i=1}^n Z_{n,i}(x))\} = \sigma_1^2(x) > 0$ that

$$E|\eta_j|^3 = E\left|\sum_{i=j(\mu+\nu)+1}^{j(\mu+\nu)+\mu} \tilde{Z}_{n,i}\right|^3 \leq \frac{c_1}{n^{3/2}} E\left|\sum_{i=j(\mu+\nu)+1}^{j(\mu+\nu)+\mu} Z_{n,i}(x)\right|^3$$
$$\leq \frac{c_2}{n^{3/2}}\left\{\sum_{i=j(\mu+\nu)+1}^{j(\mu+\nu)+\mu} E\left|Z_{n,i}(x)\right|^3 + \left(\sum_{i=j(\mu+\nu)+1}^{j(\mu+\nu)+\mu} EZ_{n,i}^2(x)\right)^{3/2}\right\}$$
$$\leq \frac{c_3}{n^{3/2}}\left(\mu h_n^{-1/2} + \mu^{3/2}\right) \leq \frac{c_4 n}{n^{3/2}} = O\left(n^{-1/2}\right). \tag{3.19}$$

Consequently, by Lemma 2.3, the Jensen inequality, $\varphi(n) = O(n^{-18/5})$, (3.3), (3.4) and (3.19), one can see that

$$\left|\phi(t) - \psi(t)\right| = \left|E\exp\left(it\sum_{j=0}^{k-1}\eta_j\right) - \prod_{j=0}^{k-1} E\exp(it\eta_j)\right|$$
$$\leq c_1|t|\varphi(\nu)\sum_{j=0}^{k-1} E|\eta_j| \leq c_1|t|\varphi(\nu)\sum_{j=0}^{k-1}\left(E|\eta_j|^3\right)^{1/3}$$
$$\leq c_2|t|kn^{-1/6}\varphi(\nu) \leq c_2|t|n^{-13/30}. \tag{3.20}$$

Combining (3.18) with (3.20), we obtain, by taking $T = n^{13/60}$, that

$$F_{1n} = \int_{-T}^{T}\left|\frac{\varphi(t) - \psi(t)}{t}\right| dt \leq cn^{-13/30} \cdot T = O\left(n^{-13/60}\right). \tag{3.21}$$

From (3.16), it follows $s_n \to 1$. Thus, by the Berry-Esséen inequality (Petrov [27], Theorem 5.7), (3.3) and (3.19), one has that

$$\sup_{-\infty<t<\infty}\left|P(B_n/s_n \leq t) - \Phi(t)\right| \leq \frac{c}{s_n^3}\sum_{j=0}^{k-1} E\left|\eta_j'\right|^3 = \frac{c}{s_n^3}\sum_{j=0}^{k-1} E|\eta_j|^3 = O\left(n^{-1/6}\right), \tag{3.22}$$

which implies

$$\begin{aligned}
&\sup_{-\infty<t<\infty}\left|P(B_n \le u+t) - P(B_n \le t)\right| \\
&\quad\le \sup_{-\infty<t<\infty}\left|P\left(\frac{B_n}{s_n} \le \frac{u+t}{s_n}\right) - \Phi\left(\frac{u+t}{s_n}\right)\right| \\
&\qquad + \sup_{-\infty<t<\infty}\left|P\left(\frac{B_n}{s_n} \le \frac{t}{s_n}\right) - \Phi\left(\frac{t}{s_n}\right)\right| + \sup_{-\infty<t<\infty}\left|\Phi\left(\frac{u+t}{s_n}\right) - \Phi\left(\frac{t}{s_n}\right)\right| \\
&\quad\le 2\sup_{-\infty<t<\infty}\left|P\left(\frac{B_n}{s_n} \le t\right) - \Phi(t)\right| + \sup_{-\infty<t<\infty}\left|\Phi\left(\frac{u+t}{s_n}\right) - \Phi\left(\frac{t}{s_n}\right)\right| \\
&\quad= O\left(n^{-1/6}\right) + O\left(|u|/s_n\right).
\end{aligned} \tag{3.23}$$

By (3.18) and (3.23), take $T = n^{13/60}$, we obtain that

$$F_{2n} = T\sup_{-\infty<t<\infty}\int_{|u|\le C/T}\left|P(B_n \le u+t) - P(B_n \le t)\right| du \le \frac{c_1}{n^{1/6}} + \frac{c_2}{T} = O\left(n^{-1/6}\right). \tag{3.24}$$

Therefore, similar to the proof of (2.28) in Yang *et al.* [11], by (3.16), one has

$$F_3 = \sup_{-\infty<t<\infty}\left|\Phi(t/s_n) - \Phi(t)\right| \le c_1\left|s_n^2 - 1\right| = O\left(n^{-1/6}\right), \tag{3.25}$$

and from (3.22), it follows

$$F_2 = \sup_{-\infty<t<\infty}\left|P(B_n/s_n \le t/s_n) - \Phi(t/s_n)\right| = O\left(n^{-1/6}\right). \tag{3.26}$$

Consequently, by (3.17), (3.18), (3.21), (3.24), (3.25) and (3.26), one has that

$$\sup_{-\infty<t<\infty}\left|P\left(S_n' \le t\right) - \Phi(t)\right| = O\left(n^{-1/6}\right) + O\left(n^{-7/24}\right) = O\left(n^{-1/6}\right). \tag{3.27}$$

On the other hand, let $\varepsilon_n = n^{-1/6} \cdot \log n \cdot \log\log n$. By (3.7), we apply Lemma 2.4 with $a = 2\varepsilon_n$ and obtain that

$$\begin{aligned}
\sup_{-\infty<t<\infty}\left|P(S_n \le t) - \Phi(t)\right| &\le \sup_{-\infty<t<\infty}\left|P\left(S_n' \le t\right) - \Phi(t)\right| + \frac{2\varepsilon_n}{\sqrt{2\pi}} \\
&\quad + P\left(\left|S_n''\right| > \varepsilon_n\right) + P\left(\left|S_n'''\right| > \varepsilon_n\right).
\end{aligned} \tag{3.28}$$

Obviously, by (3.11) and Markov's inequality, we have

$$P\left(\left|S_n''\right| > \varepsilon_n\right) \le n^{1/3}(\log n \cdot \log\log n)^{-2} \cdot E\left[S_n''\right]^2 = O\left(n^{-1/6}(\log n \cdot \log\log n)^{-2}\right). \tag{3.29}$$

It is time to estimate $P(|S_n'''| > \varepsilon_n)$. By $h_n^{-1/2} \le cn^{8/69}$ and (3.12), one has

$$|\tilde{Z}_{n,i}| \le C_3 n^{-1/2} h_n^{-1/2} \le C_4 n^{-53/138}, \qquad \sum_{i=k(\mu+\nu)+1}^{n} E\tilde{Z}_{n,i}^2 \le C_5 n^{-1/3}.$$

So, we have, by Lemma 2.5 with $\lambda = 5/23$ and $m = \lfloor n^{5/23} \rfloor = \lfloor n^{\lambda} \rfloor$, that for n large enough,

$$\begin{aligned} P\left(\left|S_n'''\right| > \varepsilon_n\right) &= P\left(\left|\sum_{i=k(\mu+\nu)+1}^{n} Z_{n,i}\right| > n^{-1/6} \cdot \log n \cdot \log\log n\right) \\ &\le 2eC_1 \exp\left\{-\frac{n^{-1/3} \cdot \log^2 n \cdot (\log\log n)^2}{2C_2(2C_5 n^{-1/3} + n^{5/23} C_4 n^{-53/138} n^{-1/6} \cdot \log n \cdot \log\log n)}\right\} \\ &\le \frac{C_{12}}{n}, \end{aligned} \tag{3.30}$$

where

$$C_1 = \exp\left\{2en^{1-\lambda}\varphi(m)\right\} \le C\exp\left\{2en^{1-\lambda}n^{-18\lambda/5}\right\} \le C\exp\{2e\},$$

$$C_2 = 4\left(1 + 4\sum_{i=1}^{2m}\varphi^{1/2}(i)\right) \le 4\left(1 + 4\sum_{i=1}^{\infty}\varphi^{1/2}(i)\right) < \infty.$$

Finally, the desired result (3.1) follows from (3.2), (3.7), (3.27), (3.28), (3.29) and (3.30) immediately. □

Theorem 3.2 *For $s \ge 2$, let the conditions (A_1) and (A_2) hold true. Assume that $\{X_n\}_{n\ge1}$ is a sequence of identically distributed φ-mixing random variables with the mixing coefficients $\varphi(n) = O(n^{-18/5})$, and $f(x)$ satisfies a Lipschitz condition. If $h_n^{-1/2} \le cn^{8/69}$, $0 < h_n \to 0$, then for any $\delta \in (0,1)$,*

$$\begin{aligned} &\sup_{-\infty<t<\infty}\left|P\left(\frac{\sqrt{nh_n}(f_n(x) - Ef_n(x))}{\sigma(x)} \le t\right) - \Phi(t)\right| \\ &\quad = O\left(n^{-1/6} \cdot \log n \cdot \log\log n\right) + O\left(h_n^{\delta}\right) + O\left(h_n^{13(1-\delta)/5}\right), \quad n \to \infty, \end{aligned} \tag{3.31}$$

where $\sigma^2(x) = f(x)\int_{-\infty}^{\infty} K^2(u)\,du$ with $f(x) > 0$ and $\Phi(\cdot)$ is the standard normal distribution function.

Proof By the condition (A_1), $\int_{-\infty}^{\infty} uK(u)\,du = 0$ implies that $\int_{-\infty}^{\infty} |u|K(u)\,du < \infty$. Thus, by the Lipschitz condition of $f(x)$, we obtain that

$$\begin{aligned} &\left|\frac{1}{h_n}EK^2\left(\frac{x - X_1}{h_n}\right) - \sigma^2(x)\right| \\ &\quad = \left|\frac{1}{h_n}\int_{-\infty}^{\infty} K^2\left(\frac{x-u}{h_n}\right)f(u)\,du - f(x)\int_{-\infty}^{\infty} K^2(u)\,du\right| \\ &\quad \le c_1\int_{-\infty}^{\infty} K(u)\left|f(x - h_n u) - f(x)\right|du \\ &\quad \le c_2 h_n\int_{-\infty}^{\infty} |u|K(u)\,du \le c_3 h_n. \end{aligned} \tag{3.32}$$

Obviously, one has

$$\frac{1}{h_n}\left[EK\left(\frac{x - X_1}{h_n}\right)\right]^2 = \frac{1}{h_n}\left[\int_{-\infty}^{\infty} K\left(\frac{x-u}{h_n}\right)f(u)\,du\right]^2 \le ch_n. \tag{3.33}$$

Thus, we obtain by combining (3.32) with (3.33) that

$$\begin{aligned}\left|\operatorname{Var}\bigl(Z_{n,i}(x)\bigr)-\sigma^2(x)\right| &= \left|\operatorname{Var}\bigl(Z_{n,1}(x)\bigr)-\sigma^2(x)\right| \\ &\le \frac{1}{h_n}\left[EK\left(\frac{x-X_1}{h_n}\right)\right]^2+\left|\frac{1}{h_n}EK^2\left(\frac{x-X_1}{h_n}\right)-\sigma^2(x)\right| \\ &\le c_3h_n,\quad 1\le i\le n. \end{aligned}\tag{3.34}$$

Meanwhile, for $i\neq j$, one has by the condition (A_2) that

$$\begin{aligned}&\left|\operatorname{Cov}\bigl[Z_{n,i}(x),Z_{n,j}(y)\bigr]\right| \\ &\quad=\left|\frac{1}{h_n}\operatorname{Cov}\left[K\left(\frac{x-X_i}{h_n}\right),K\left(\frac{y-X_j}{h_n}\right)\right]\right| \\ &\quad\le h_n\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K(s)K(t)\left|\bigl[f(x-h_ns,y-h_nt,j-i)-f(x-h_ns)f(y-h_nt)\bigr]\right|\,ds\,dt \\ &\quad\le ch_n. \end{aligned}\tag{3.35}$$

By (3.35), we take $r_n=h_n^{\delta-1}$ and obtain that

$$\frac{2}{n}\sum_{\substack{1\le i<j\le n\\1\le j-i\le r_n}}\left|\operatorname{Cov}\bigl[Z_{n,i}(x),Z_{n,j}(y)\bigr]\right|\le c_4h_nr_n=c_4h_n^{\delta}.\tag{3.36}$$

Applying Lemma 2.2 with $|Z_{n,i}(x)|\le c_1h_n^{-1/2}$, $E|Z_{n,j}(x)|\le c_2h_n^{1/2}$ and $\varphi(n)=O(n^{-18/5})$, we obtain that

$$\frac{2}{n}\sum_{\substack{1\le i<j\le n\\ j-i>r_n}}\left|\operatorname{Cov}\bigl[Z_{n,i}(x),Z_{n,j}(y)\bigr]\right|\le c_5\sum_{k>r_n}\varphi(k)\le c_6h_n^{13(1-\delta)/5}.\tag{3.37}$$

Define

$$\sigma_n^2(x)=\operatorname{Var}\left[\sum_{i=1}^nZ_{n,i}(x)\right],\qquad \sigma_{n,0}^2(x)=n\sigma^2(x),\quad n\ge1.\tag{3.38}$$

Consequently, by (3.34), (3.36), (3.37) and (3.38), it can be checked that

$$\begin{aligned}\left|\sigma_n^2(x)-\sigma_{n,0}^2(x)\right| &\le n\left|\operatorname{Var}\bigl(Z_{n,1}(x)\bigr)-\sigma^2(x)\right|+2\sum_{1\le i<j\le n}\left|\operatorname{Cov}\bigl[Z_{n,i}(x),Z_{n,j}(y)\bigr]\right| \\ &\le c_7n\bigl(h_n+h_n^{\delta}+h_n^{13(1-\delta)/5}\bigr). \end{aligned}\tag{3.39}$$

We obtain, by (3.2), (3.31) and (3.38), that

$$\begin{aligned}&\sup_{-\infty<t<\infty}\left|P\left(\frac{\sqrt{nh_n}(f_n(x)-Ef_n(x))}{\sigma(x)}\le t\right)-\Phi(t)\right| \\ &\quad=\sup_{-\infty<t<\infty}\left|P\left(\frac{\sum_{i=1}^nZ_{n,i}(x)}{\sigma_{n,0}(x)}\le t\right)-\Phi(t)\right| \\ &\quad\le\sup_{-\infty<t<\infty}\left|P\left(\frac{\sum_{i=1}^nZ_{n,i}(x)}{\sigma_n(x)}\le\frac{\sigma_{n,0}(x)}{\sigma_n(x)}t\right)-\Phi\left(\frac{\sigma_{n,0}(x)}{\sigma_n(x)}t\right)\right|\end{aligned}$$

$$\begin{aligned} &+ \sup_{-\infty<t<\infty}\left|\Phi\left(\frac{\sigma_{n,0}(x)}{\sigma_n(x)}t\right)-\Phi(t)\right| \\ &:= Q_1 + Q_2. \end{aligned} \tag{3.40}$$

From (3.38) and (3.39), it follows $\lim_{n\to\infty}\sigma_n^2(x)/\sigma_{n,0}^2(x) = 1$, since $h_n \to 0$ as $n \to \infty$ and $\delta \in (0,1)$. Thus, by applying Theorem 3.1, we establish that

$$Q_1 = O\big(n^{-1/6}\cdot\log n\cdot\log\log n\big). \tag{3.41}$$

On the other hand, similar to the proof of (2.34) in Yang *et al.* [11], it follows by (3.39) again that

$$Q_2 \le c_2\left|\frac{\sigma_n^2(x)}{\sigma_{n,0}^2(x)}-1\right| = \frac{c_2}{\sigma_{n,0}^2(x)}\left|\sigma_n^2(x)-\sigma_{n,0}^2(x)\right| = O\big(h_n^\delta\big)+O\big(h_n^{13(1-\delta)/5}\big). \tag{3.42}$$

Finally, by (3.40), (3.41) and (3.42), (3.31) holds true. □

Remark 3.1 Under an independent sample, Cao [8] studied the bootstrap approximations in nonparametric density estimation and obtained Berry-Esséen bounds as $O_p(n^{-1/5})$ and $O_p(n^{-2/9})$ (see Theorem 1 and Theorem 2 of Cao [8]). Under a negatively associated sample, Liang and Baek [17] studied the Berry-Esséen bound and obtained the rate $O((\frac{\log n}{n})^{1/6})$ under some conditions (see Remark 3.1 of Liang and Baek [17]). In our Theorem 3.1 and Theorem 3.2, under the mixing coefficients condition $\varphi(n) = O(n^{-18/5})$ and other simple assumptions, we obtain the Berry-Esséen bounds of the centered variate as $O(n^{-1/6}\cdot\log n\cdot\log\log n)$ and $O(n^{-1/6}\cdot\log n\cdot\log\log n)+O(h_n^\delta)+O(h_n^{13(1-\delta)/5})$, where $0<\delta<1$. Particularly, by taking $\delta = 13/18$ and $h_n = n^{-16/69}$ in Theorem 3.2, the Berry-Esséen bound of the centered variate is presented as

$$\sup_{-\infty<t<\infty}\left|P\left(\frac{\sqrt{nh_n}(f_n(x)-Ef_n(x))}{\sigma(x)}\le t\right)-\Phi(t)\right| = O\big(n^{-1/6}\cdot\log n\cdot\log\log n\big),\quad n\to\infty,$$

where $\sigma(x)$ and $\Phi(\cdot)$ are defined in Theorem 3.2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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