Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

The hybrid steepest descent method for solving variational inequality over triple hierarchical problems

Nopparat Wairojjana[1], Thanyarat Jitpeera[2] and Poom Kumam[1*]

*Correspondence: poom.kum@kmutt.ac.th
[1]Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bangmod, Thung Khru, Bangkok, 10140, Thailand
Full list of author information is available at the end of the article

Abstract

An explicit algorithm is introduced to solve the monotone variational inequality over a triple hierarchical problem. The strong convergence for the proposed algorithm to the solution is guaranteed under some assumptions. Our results extend those of Iiduka (Nonlinear Anal. 71:e1292-e1297, 2009), Marino and Xu (J. Optim. Theory Appl. 149(1):61-78, 2011), Yao *et al.* (Fixed Point Theory Appl. 2011:794203, 2011) and other authors.

MSC: 46C05; 47H06; 47H09; 47H10; 47J20; 47J25; 65J15

Keywords: nonexpansive mapping; strongly monotone mapping; strongly positive linear bounded operator; Lipchitz continuous; variational inequality; hierarchical fixed point

1 Introduction

Let C be a closed convex subset of a real Hilbert space H with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. We denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\to$, respectively. A mapping A is a nonlinear mapping. The *Hartmann-Stampacchia variational inequality* [1] for finding $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $VI(C, A)$. The variational inequality has been extensively studied in the literature [2, 3].

Let $f : C \to C$ be called a *ρ-contraction* if there exists a constant $\rho \in [0, 1)$ such that

$$\|f(x) - f(y)\| \leq \rho \|x - y\|, \quad \forall x, y \in C.$$

A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. If C is bounded closed convex and T is a nonexpansive mapping of C into itself, then $F(T)$ is nonempty [4].

We discuss the following variational inequality problem over the fixed point set of a nonexpansive mapping (see [5–13]), which is called a *hierarchical problem*. Consider a monotone, continuous mapping $A : H \to H$ and a nonexpansive mapping $T : H \to H$. Find $x^* \in VI(F(T), A) = \{x^* \in F(T) : \langle Ax^*, x - x^* \rangle \geq 0, \forall x \in F(T)\}$, where $F(T) \neq \emptyset$. This solution set is denoted by Ξ.

We introduce the following variational inequality problem over the solution set of the variational inequality problem over the fixed point set of a nonexpansive mapping (see [14–18]), which is called a *triple hierarchical problem*. Consider an inverse-strongly monotone $A : H \to H$, a strongly monotone and Lipschitz continuous $B : H \to H$ and a nonexpansive mapping $T : H \to H$. Find $x^* \in VI(\Xi, B) = \{x^* \in \Xi : \langle Bx^*, x - x^* \rangle \geq 0, \forall x \in \Xi\}$, where $\Xi := VI(F(T), A) \neq \emptyset$.

A mapping $A : H \to H$ is said to be *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in H.$$

A mapping $A : H \to H$ is said to be α*-strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in H.$$

A mapping $A : H \to H$ is said to be β*-inverse-strongly monotone* if there exists a positive real number β such that

$$\langle Ax - Ay, x - y \rangle \geq \beta \|Ax - Ay\|^2, \quad \forall x, y \in H.$$

A mapping $A : H \to H$ is said to be *L-Lipschitz continuous* if there exists a positive real number L such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in H.$$

A linear bounded operator A is said to be *strongly positive* on H if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2, \quad \forall x \in H.$$

In 2009, Iiduka [14] introduced an iterative algorithm for the following triple hierarchical constrained optimization problem. The sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_1 \in H$, is chosen arbitrarily,

$$\begin{cases} y_n = T(x_n - \lambda_n A_1 x_n), \\ x_{n+1} = y_n - \mu \alpha_n A_2 y_n, \quad \forall n \geq 0, \end{cases} \tag{1.2}$$

where $\alpha_n \in (0, 1]$ and $\lambda_n \in (0, 2\alpha]$ satisfy certain conditions. Let $A_1 : H \to H$ be an inverse-strongly monotone, $A_2 : H \to H$ be a strongly monotone and Lipschitz continuous and $T : H \to H$ be a nonexpansive mapping, then the sequence converges strongly to the set solution of (1.2).

In 2011, Yao *et al.* [19] studied new algorithms. For $x_0 \in C$ chosen arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by

$$x_{n+1} = \beta_n x_n + (1-\beta_n)TP_C\big[I - \alpha_n(A-\gamma f)\big]x_n, \quad \forall n \geq 0,$$

where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$. Then $\{x_n\}$ converges strongly to $x^* \in F(T)$ which is the unique solution of the variational inequality:

$$\text{Find } x^* \in F(T) \text{ such that } \big\langle (A-\gamma f)x^*, x - x^*\big\rangle \geq 0, \quad \forall x \in F(T). \tag{1.3}$$

Let $A : C \to H$ be a strongly positive linear bounded operator, $f : C \to H$ be a ρ-contraction and $T : C \to C$ be a nonexpansive mapping satisfying some conditions. The solution set of (1.3) is denoted by $\Omega := VI(F(T), A-\gamma f)$.

Very recently, Marino and Xu [20] generated a sequence $\{x_n\}$ through the recursive formula

$$x_{n+1} = \lambda_n f x_n + (1-\lambda_n)\big(\alpha_n V x_n + (1-\alpha_n)Tx_n\big), \quad \forall n \geq 0,$$

where f is a contraction on C, the initial guess $x_0 \in C$ is arbitrary and $\{\lambda_n\}$, $\{\alpha_n\}$ are two sequences in $(0,1)$ and $T, V : C \to C$ are two nonexpansive self mappings. Strong convergence of the algorithm is proved under different circumstances of parameter selection.

In this paper, we introduce iterative algorithms and prove a strong convergence theorem for the following variational inequality over the triple hierarchical problem (1.4) below. Let $B : C \to C$ be a β-strongly monotone and L-Lipschitz continuous. Find $x^* \in \Omega$ such that

$$\big\langle Bx^*, x - x^*\big\rangle \geq 0, \quad \forall x \in \Omega, \tag{1.4}$$

where $\Omega := VI(F(T), A-\gamma f) \neq \emptyset$, T is a nonexpansive mapping, $A : C \to H$ is a strongly positive linear bounded operator and $f : C \to H$ is a ρ-contraction. This solution set of (1.4) is denoted by $\Upsilon := VI(\Omega, B) := VI(VI(F(T), A-\gamma f), B)$. Then the sequence $\{x_n\}$ strongly converges to the unique solution of (3.2) in Section 3 and we shall denote the set of such solutions by $\Theta := VI(\Upsilon, I-\phi)$.

2 Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Recall that the (nearest point) projection P_C from H onto C assigns, to each $x \in H$, the unique point in $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

The following characterizes the projection P_C. We recall some lemmas which will be needed in the rest of this paper.

Lemma 2.1 *The function $u \in C$ is a solution of the variational inequality* (1.1) *if and only if $u \in C$ satisfies the relation $u = P_C(u - \lambda Au)$ for all $\lambda > 0$.*

Lemma 2.2 *For a given* $z \in H$, $u \in C$, $u = P_C z \Leftrightarrow \langle u - z, v - u\rangle \geq 0$, $\forall v \in C$.

It is well known that P_C *is a firmly nonexpansive mapping of* H *onto* C *and satisfies*

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H. \tag{2.1}$$

Moreover, $P_C x$ *is characterized by the following properties:* $P_C x \in C$ *and for all* $x \in H$, $y \in C$,

$$\langle x - P_C x, y - P_C x\rangle \leq 0. \tag{2.2}$$

Lemma 2.3 *The following inequality holds in an inner product space* H:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$$

Lemma 2.4 [21] *Let* C *be a closed convex subset of a real Hilbert space* H *and let* $T : C \to C$ *be a nonexpansive mapping. Then* $I - T$ *is demiclosed at zero, that is,* $x_n \rightharpoonup x$ *and* $x_n - Tx_n \to 0$ *imply* $x = Tx$.

Lemma 2.5 [22] *Assume* A *is a strongly positive linear bounded operator on a Hilbert space* H *with the coefficient* $\bar{\gamma} > 0$ *and* $0 < \rho \leq \|A\|^{-1}$, *then* $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.

Lemma 2.6 [23] *Each Hilbert space* H *satisfies Opial's condition, that is, for any sequence* $\{x_n\} \subset H$ *with* $x_n \rightharpoonup x$, *the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for each $y \in H$ *with* $y \neq x$.

Lemma 2.7 [24] *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose* $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.8 [10] *Let* $B : H \to H$ *be* β*-strongly monotone and* L*-Lipschitz continuous and let* $\mu \in (0, \frac{2\beta}{L^2})$. *For* $\lambda \in [0,1]$, *define* $T_\lambda : H \to H$ *by* $T_\lambda(x) := x - \lambda\mu B(x)$ *for all* $x \in H$. *Then, for all* $x, y \in H$,

$$\|T_\lambda(x) - T_\lambda(y)\| \leq (1 - \lambda\tau)\|x - y\|$$

holds, where $\tau := 1 - \sqrt{1 - \mu(2\beta - \mu L^2)} \in (0,1]$.

Lemma 2.9 [25] *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad \forall n \geq 0,$$

where $\{\gamma_n\} \subset (0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathcal{R}$ *such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3 Strong convergence theorem

In this section, we introduce an iterative algorithm for solving the monotone variational inequality over a triple hierarchical problem.

Theorem 3.1 *Let H be a real Hilbert space, C be a closed convex subset of H. Let $A : C \to H$ be a strongly positive linear bounded operator, $f : C \to H$ be a ρ-contraction, γ be a positive real number such that $\frac{\bar{\gamma}-1}{\rho} < \gamma < \frac{\bar{\gamma}}{\rho}$. Let $T : C \to C$ be a nonexpansive mapping, $B : C \to C$ be a β-strongly monotone and L-Lipschitz continuous. Let $\phi : C \to C$ be a k-contraction mapping with $k \in [0,1)$. Assume that $\Upsilon := VI(VI(F(T), A - \gamma f), B)$ is nonempty set. Suppose $\{x_n\}$ is a sequence generated by the following algorithm $x_0 \in C$ arbitrarily*:

$$\begin{cases} z_n = TP_C[I - \delta_n(A - \gamma f)]x_n, \\ y_n = (I - \mu\beta_n B)z_n, \\ x_{n+1} = \alpha_n \phi(x_n) + (1 - \alpha_n)y_n, \quad \forall n \geq 0, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\delta_n\} \subset [0,1]$. If $\mu \in (0, \frac{2\beta}{L^2})$ is used and if $\{\beta_n\} \subset (0,1]$ satisfies the following conditions:

(C1): $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} \delta_n = \infty$;

(C2): $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(C3): $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\lim_{n\to\infty} \alpha_n = 0$;

(C4): $\delta_n \leq \beta_n$ *and* $\beta_n \leq \alpha_n$.

Then $\{x_n\}$ converges strongly to $x^* \in \Upsilon$, which is the unique solution of the variational inequality:

$$\textit{Find } x^* \in \Upsilon \textit{ such that } \langle (I - \phi)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Upsilon. \tag{3.2}$$

Proof We will divide the proof into four steps.

Step 1. We will show $\{x_n\}$ is bounded. For any $q \in \Upsilon$, we have

$$\begin{aligned} \|z_n - q\| &= \big\| TP_C\big[I - \delta_n(A - \gamma f)\big]x_n - TP_C q \big\| \\ &\leq \big\| \big[I - \delta_n(A - \gamma f)\big]x_n - q \big\| \\ &\leq \delta_n \big\| \gamma f(x_n) - \gamma f(q) \big\| + \delta_n \big\| \gamma f(q) - Aq \big\| + \|I - \delta_n A\| \|x_n - q\| \\ &\leq \delta_n \gamma\rho \|x_n - q\| + \delta_n \big\| \gamma f(q) - Aq \big\| + (1 - \delta_n\bar{\gamma})\|x_n - q\| \\ &= \big[1 - (\bar{\gamma} - \gamma\rho)\delta_n\big]\|x_n - q\| + \delta_n \big\| \gamma f(q) - Aq \big\|. \end{aligned} \tag{3.3}$$

By Lemma 2.8, it is found that

$$\begin{aligned} \|y_n - q\| &= \big\| (I - \mu\beta_n B)z_n - (I - \mu\beta_n B)q \big\| \\ &\leq (1 - \beta_n\tau)\|z_n - q\| \\ &\leq (1 - \beta_n\tau)\big\{\big[1 - (\bar{\gamma} - \gamma\rho)\delta_n\big]\|x_n - q\| + \delta_n \big\| \gamma f(q) - Aq \big\|\big\}. \end{aligned} \tag{3.4}$$

From (3.1), we get

$$\begin{aligned}
\|x_{n+1}-q\| &\le \alpha_n\big\|\phi(x_n)-\phi(q)\big\| + \alpha_n\big\|\phi(q)-q\big\| + (1-\alpha_n)\|y_n-q\| \\
&\le \alpha_n k\|x_n-q\| + \alpha_n\big\|\phi(q)-\phi(q)\big\| + (1-\alpha_n)\|y_n-q\| \\
&\le \alpha_n k\|x_n-q\| + (1-\alpha_n)(1-\beta_n\tau) \\
&\quad \times \big\{\big[1-(\bar{\gamma}-\gamma\rho)\delta_n\big]\|x_n-q\| + \delta_n\big\|\gamma f(q)-Aq\big\|\big\} \\
&\le \alpha_n\|x_n-q\| + (1-\alpha_n)(1-\beta_n\tau)\big[1-(\bar{\gamma}-\gamma\rho)\delta_n\big]\|x_n-q\| \\
&\quad + (1-\alpha_n)(1-\beta_n\tau)\delta_n\big\|\gamma f(q)-Aq\big\| \\
&= \alpha_n\|x_n-q\| + (1-\alpha_n)\big[1-(\bar{\gamma}-\gamma\rho)\delta_n-\beta_n\tau+\beta_n\tau(\bar{\gamma}-\gamma\rho)\delta_n\big]\|x_n-q\| \\
&\quad + (1-\alpha_n)(1-\beta_n\tau)\delta_n\big\|\gamma f(q)-Aq\big\| \\
&= \alpha_n\|x_n-q\| + (1-\alpha_n)\big[1-\big\{(\bar{\gamma}-\gamma\rho)\delta_n+\beta_n\tau-\beta_n\tau(\bar{\gamma}-\gamma\rho)\delta_n\big\}\big]\|x_n-q\| \\
&\quad + (1-\alpha_n)(1-\beta_n\tau)\delta_n\big\|\gamma f(q)-Aq\big\| \\
&= \alpha_n\|x_n-q\| \\
&\quad + \big[1-\alpha_n-\big\{(\bar{\gamma}-\gamma\rho)\delta_n+\beta_n\tau-\beta_n\tau(\bar{\gamma}-\gamma\rho)\delta_n\big\}(1-\alpha_n)\big]\|x_n-q\| \\
&\quad + (1-\alpha_n)(1-\beta_n\tau)\delta_n\big\|\gamma f(q)-Aq\big\| \\
&= \big[1-(1-\alpha_n)\big\{(\bar{\gamma}-\gamma\rho)\delta_n+\beta_n\tau-\beta_n\tau(\bar{\gamma}-\gamma\rho)\delta_n\big\}\big]\|x_n-q\| \\
&\quad + (1-\alpha_n)(1-\beta_n\tau)\delta_n\big\|\gamma f(q)-Aq\big\| \\
&= \big[1-(1-\alpha_n)\big\{(\bar{\gamma}-\gamma\rho)\delta_n(1-\beta_n\tau)+\beta_n\tau\big\}\big]\|x_n-q\| \\
&\quad + (1-\alpha_n)(1-\beta_n\tau)\delta_n\big\|\gamma f(q)-Aq\big\| \\
&= \big[1-(1-\alpha_n)(\bar{\gamma}-\gamma\rho)\delta_n(1-\beta_n\tau)-(1-\alpha_n)\beta_n\tau\big]\|x_n-q\| \\
&\quad + (1-\alpha_n)(1-\beta_n\tau)\delta_n\big\|\gamma f(q)-Aq\big\| \\
&= \big[1-(1-\alpha_n)(\bar{\gamma}-\gamma\rho)\delta_n(1-\beta_n\tau)\big]\|x_n-q\| - (1-\alpha_n)\beta_n\tau\|x_n-q\| \\
&\quad + (1-\alpha_n)(1-\beta_n\tau)\delta_n\big\|\gamma f(q)-Aq\big\| \\
&\le \big[1-(\bar{\gamma}-\gamma\rho)(1-\alpha_n)(1-\beta_n\tau)\delta_n\big]\|x_n-q\| \\
&\quad + (\bar{\gamma}-\gamma\rho)(1-\alpha_n)(1-\beta_n\tau)\delta_n\frac{\|\gamma f(q)-Aq\|}{\bar{\gamma}-\gamma\rho} \\
&= (1-\sigma_n)\|x_n-q\| + \sigma_n\frac{\|\gamma f(q)-Aq\|}{\bar{\gamma}-\gamma\rho},
\end{aligned}$$

where $\sigma_n := (\bar{\gamma}-\gamma\rho)(1-\alpha_n)(1-\beta_n\tau)\delta_n$. Then the mathematical induction implies that

$$\|x_n-q\| \le \max\left\{\|x_0-q\|, \frac{\|\gamma f(q)-Aq\|}{\bar{\gamma}-\gamma\rho}\right\}, \quad \forall n \ge 0.$$

Therefore, $\{x_n\}$ is bounded and so are $\{y_n\}$, $\{z_n\}$, $\{Ax_n\}$, $\{Bx_n\}$, $\{\phi(x_n)\}$ and $\{f(x_n)\}$.

Step 2. We claim that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$ and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. From (3.1), we have

$$\begin{aligned}
\|z_{n+1}-z_n\| &= \big\|TP_C\big[I-\delta_{n+1}(A-\gamma f)\big]x_{n+1} - TP_C\big[I-\delta_n(A-\gamma f)\big]x_n\big\| \\
&\le \big\|P_C\big[I-\delta_{n+1}(A-\gamma f)\big]x_{n+1} - P_C\big[I-\delta_n(A-\gamma f)\big]x_n\big\| \\
&\le \big\|\big[I-\delta_{n+1}(A-\gamma f)\big]x_{n+1} - \big[I-\delta_n(A-\gamma f)\big]x_n\big\| \\
&= \big\|\delta_{n+1}\big(\gamma f(x_{n+1}) - \gamma f(x_n)\big) + (\delta_{n+1}-\delta_n)\gamma f(x_n) + (I-\delta_{n+1}A)(x_{n+1}-x_n) \\
&\quad + (\delta_n - \delta_{n+1})Ax_n\big\| \\
&\le \delta_{n+1}\gamma\big\|f(x_{n+1}) - f(x_n)\big\| + (1-\delta_{n+1}\bar{\gamma})\|x_{n+1}-x_n\| \\
&\quad + |\delta_{n+1}-\delta_n|\big(\big\|\gamma f(x_n)\big\| + \|Ax_n\|\big) \\
&\le \delta_{n+1}\gamma\rho\|x_{n+1}-x_n\| + (1-\delta_{n+1}\bar{\gamma})\|x_{n+1}-x_n\| \\
&\quad + |\delta_{n+1}-\delta_n|\big(\big\|\gamma f(x_n)\big\| + \|Ax_n\|\big) \\
&= \big[1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big]\|x_{n+1}-x_n\| + |\delta_{n+1}-\delta_n|\big(\big\|\gamma f(x_n)\big\| + \|Ax_n\|\big)
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \big\|(I-\mu\beta_{n+1}B)z_{n+1} - (I-\mu\beta_n B)z_n\big\| \\
&\le \big\|(I-\mu\beta_{n+1}B)z_{n+1} - (I-\mu\beta_{n+1}B)z_n\big\| \\
&\quad + \big\|(I-\mu\beta_{n+1}B)z_n - (I-\mu\beta_n B)z_n\big\| \\
&\le (1-\beta_n\tau)\|z_{n+1}-z_n\| + \mu|\beta_{n+1}-\beta_n|\|Bz_n\|.
\end{aligned} \tag{3.6}$$

Using (3.5) and (3.6), we get

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &= \big\|\alpha_{n+1}\phi(x_{n+1}) + (1-\alpha_{n+1})y_{n+1} - \alpha_n\phi(x_n) - (1-\alpha_n)y_n\big\| \\
&\le \alpha_{n+1}\big\|\phi(x_{n+1}) - \phi(x_n)\big\| + |\alpha_{n+1}-\alpha_n|\big\|\phi(x_{n+1})\big\| + (1-\alpha_{n+1})\|y_{n+1}-y_n\| \\
&\quad + |\alpha_{n+1}-\alpha_n|\|y_n\| \\
&\le \alpha_{n+1}k\|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|\big(\big\|\phi(x_{n+1})\big\| + \|y_n\|\big) \\
&\quad + (1-\alpha_{n+1})\|y_{n+1}-y_n\| \\
&\le \alpha_{n+1}k\|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|\big(\big\|\phi(x_{n+1})\big\| + \|y_n\|\big) \\
&\quad + (1-\alpha_{n+1})\big\{(1-\beta_n\tau)\|z_{n+1}-z_n\| + \mu|\beta_{n+1}-\beta_n|\|Bz_n\|\big\} \\
&= \alpha_{n+1}k\|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|\big(\big\|\phi(x_{n+1})\big\| + \|y_n\|\big) \\
&\quad + (1-\alpha_{n+1})(1-\beta_n\tau)\|z_{n+1}-z_n\| + (1-\alpha_{n+1})\mu|\beta_{n+1}-\beta_n|\|Bz_n\| \\
&\le \alpha_{n+1}\|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|\big(\big\|\phi(x_{n+1})\big\| + \|y_n\|\big) \\
&\quad + (1-\alpha_{n+1})\mu|\beta_{n+1}-\beta_n|\|Bz_n\| \\
&\quad + (1-\alpha_{n+1})\big\{\big[1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big]\|x_{n+1}-x_n\| \\
&\quad + |\delta_{n+1}-\delta_n|\big(\big\|\gamma f(x_n)\big\| + \|Ax_n\|\big)\big\}
\end{aligned}$$

$$\begin{aligned}
&\le \alpha_{n+1}\|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|\big(\|\phi(x_{n+1})\| + \|y_n\|\big) \\
&\quad + (1-\alpha_{n+1})\mu|\beta_{n+1}-\beta_n|\|Bz_n\| \\
&\quad + (1-\alpha_{n+1})\big[1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big]\|x_{n+1}-x_n\| \\
&\quad + (1-\alpha_{n+1})|\delta_{n+1}-\delta_n|\big(\|\gamma f(x_n)\| + \|Ax_n\|\big) \\
&\le \big[1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}(1-\alpha_{n+1})\big]\|x_{n+1}-x_n\| \\
&\quad + |\alpha_{n+1}-\alpha_n|\big(\|\phi(x_{n+1})\| + \|y_n\|\big) \\
&\quad + \mu|\beta_{n+1}-\beta_n|\|Bz_n\| + |\delta_{n+1}-\delta_n|\big(\|\gamma f(x_n)\| + \|Ax_n\|\big) \\
&\le \big[1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}(1-\alpha_{n+1})\big]\|x_{n+1}-x_n\| \\
&\quad + \big\{|\alpha_{n+1}-\alpha_n| + |\beta_{n+1}-\beta_n| + |\delta_{n+1}-\delta_n|\big\}M,
\end{aligned}$$

where M is some constant such that

$$\sup_{n\ge 0}\big\{\|\phi(x_n)\| + \|y_n\|, \mu\|Bz_n\|, \|\gamma f(x_n)\| + \|Ax_n\|\big\} \le M.$$

From (C1)-(C3) and the boundedness of $\{x_n\}$, $\{y_n\}$, $\{Ax_n\}$, $\{Bz_n\}$, $\{\phi(x_n)\}$ and $\{f(x_n)\}$, by Lemma 2.9, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.7}$$

On the other hand, we note that

$$\begin{aligned}
\|z_n - Tx_n\| &= \big\|TP_C\big[I-\delta_n(A-\gamma f)\big]x_n - Tx_n\big\| \\
&= \big\|TP_C\big[I-\delta_n(A-\gamma f)\big]x_n - TP_Cx_n\big\| \\
&\le \big\|\big[I-\delta_n(A-\gamma f)\big]x_n - x_n\big\| \\
&\le \delta_n\big\|(A-\gamma f)]x_n\big\|,
\end{aligned}$$

by (C3)-(C4) and it follows that

$$\lim_{n\to\infty}\|z_n - Tx_n\| = 0. \tag{3.8}$$

From (3.1), we compute

$$\begin{aligned}
\|x_{n+1}-z_n\| &= \big\|\alpha_n\phi(x_n) + (1-\alpha_n)y_n - z_n\big\| \\
&= \big\|\alpha_n\phi(x_n) + (1-\alpha_n)(I-\mu\beta_nB)z_n - z_n\big\| \\
&\le \alpha_n\big\|\phi(x_n)-z_n\big\| + (1-\alpha_n)\big\|(I-\mu\beta_nB)z_n - z_n\big\| \\
&\le \alpha_nk\|x_n-z_n\| + \alpha_n\big\|\phi(z_n)-z_n\big\| + (1-\alpha_n)\mu\beta_n\|Bz_n\|.
\end{aligned}$$

By (C3) and (C4), it follows that

$$\lim_{n\to\infty}\|x_{n+1}-z_n\| = 0. \tag{3.9}$$

Since

$$\|x_n - Tx_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| + \|z_n - Tx_n\|.$$

By (3.7), (3.8) and (3.9), we obtain

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.10}$$

From (3.1), we compute

$$\begin{aligned} \|x_{n+1} - y_n\| &= \left\|\alpha_n \phi(x_n) + (1-\alpha_n)y_n - y_n\right\| \\ &= \left\|\alpha_n \phi(x_n) + y_n - \alpha_n y_n - y_n\right\| \\ &\le \alpha_n \left\|\phi(x_n) - y_n\right\|. \end{aligned} \tag{3.11}$$

By (C3), it follows that

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{3.12}$$

Since

$$\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|.$$

From (3.7) and (3.12), we obtain

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.13}$$

Step 3. First, $\limsup_{n\to\infty} \langle u_n - x^*, \gamma f(x^*) - Ax^*\rangle \le 0$ is proven. Choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle x_n - x^*, \gamma f(x^*) - Ax^*\rangle = \lim_{i\to\infty} \langle x_{n_i} - x^*, \gamma f(x^*) - Ax^*\rangle.$$

The boundedness of $\{x_{n_i}\}$ implies the existences of a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ and a point $\hat{x} \in H$ such that $\{x_{n_{i_j}}\}$ converges weakly to $\hat{x}$. We may assume, without loss of generality, that $\lim_{i\to\infty} \langle x_{n_i}, w\rangle = \langle \hat{x}, w\rangle$, $w \in H$. Assume $\hat{x} \neq T(\hat{x})$. $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ with $F(T) \neq \emptyset$ guarantees that

$$\begin{aligned} &\liminf_{i\to\infty} \|x_{n_i} - \hat{x}\| \\ &\quad < \liminf_{i\to\infty} \left\|x_{n_i} - T(\hat{x})\right\| \\ &\quad = \liminf_{i\to\infty} \left\|x_{n_i} - T(x_{n_i}) + T(x_{n_i}) - T(\hat{x})\right\| \\ &\quad = \liminf_{i\to\infty} \left\|T(x_{n_i}) - T(\hat{x})\right\| \\ &\quad \le \liminf_{i\to\infty} \|x_{n_i} - \hat{x}\|, \end{aligned}$$

which is a contradiction. Therefore, $\hat{x} \in F(T)$. From $x^* \in VI(F(T), A - \gamma f)$, we find

$$\begin{aligned}\limsup_{n\to\infty}\langle x_n - x^*, \gamma f(x^*) - Ax^*\rangle &= \lim_{i\to\infty}\langle x_{n_i} - x^*, \gamma f(x^*) - Ax^*\rangle \\ &= \langle \hat{x} - x^*, \gamma f(x^*) - Ax^*\rangle \\ &\le 0.\end{aligned}$$

Setting $u_n = [I - \delta_n(A - \gamma f)]x_n$, by (C3)-(C4), we notice that

$$\|u_n - x_n\| \le \delta_n \|(A - \gamma f)\| \to 0.$$

Hence, we get

$$\limsup_{n\to\infty}\langle u_n - x^*, \gamma f(x^*) - Ax^*\rangle \le 0. \tag{3.14}$$

Second, $\limsup_{n\to\infty}\langle x^* - x_{n+1}, Bx^*\rangle \le 0$ is proven. $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ guarantees the existences of a subsequence $\{x_{n_k+1}\}$ of $\{x_{n_k}\}$ and a point $\bar{x} \in H$ such that $\limsup_{n\to\infty}\langle x^* - x_{n+1}, Bx^*\rangle = \lim_{k\to\infty}\langle x^* - x_{n_k+1}, Bx^*\rangle$ and $\lim_{k\to\infty}\langle x_{n_k}, w\rangle = \lim_{k\to\infty}\langle x_{n_k+1}, w\rangle = \langle \bar{x}, w\rangle$, $w \in H$. By the same discussion as in the proof of $\hat{x} \in F(T)$, we have $\bar{x} \in F(T)$. Let $y \in F(T)$ be fixed arbitrarily. Then it follows that $T : C \to C$ is a nonexpansive mapping with $F(T) \neq \emptyset$, $A : C \to H$ is a strongly positive linear bounded operator and $f : C \to H$ is a contraction for all $n \in \mathbb{N}$. From (3.1),

$$\begin{aligned}\|z_n - y\| &= \|TP_C u_n - TP_C y\| \\ &\le \|u_n - y\|.\end{aligned} \tag{3.15}$$

By (C3)-(C4), we observe that

$$\begin{aligned}\|u_n - y\| &= \|[I - \delta_n(A - \gamma f)]x_n - y\| \\ &\le \|x_n - y\| + \delta_n\|(A - \gamma f)x_n\| \\ &\le \|x_n - y\|.\end{aligned} \tag{3.16}$$

Using (3.15) and (3.16),

$$\begin{aligned}\|u_n - y\|^2 &= \|[I - \delta_n(A - \gamma f)]x_n - y\|^2 \\ &= \|\delta_n(\gamma f(x_n) - Ay) + (I - \delta_n A)(x_n - y)\|^2 \\ &\le (1 - \delta_n\bar{\gamma})^2\|x_n - y\|^2 + 2\delta_n\langle \gamma f(x_n) - Ay, u_n - y\rangle \\ &\le (1 - 2\delta_n\bar{\gamma} + \delta_n^2\bar{\gamma}^2)\|x_n - y\|^2 + 2\delta_n\gamma\rho\|x_n - y\|\|u_n - y\| \\ &\quad + 2\delta_n\langle \gamma f(y) - Ay, u_n - y\rangle \\ &\le (1 - 2\delta_n\bar{\gamma} + \delta_n^2\bar{\gamma}^2)\|x_n - y\|^2 + 2\delta_n\gamma\rho\|x_n - y\|^2 + 2\delta_n\langle \gamma f(y) - Ay, u_n - y\rangle \\ &= [1 - 2\delta_n(\bar{\gamma} - \gamma\rho)]\|x_n - y\|^2 + \delta_n^2\bar{\gamma}^2\|x_n - y\|^2 + 2\delta_n\langle \gamma f(y) - Ay, u_n - y\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}
0 &\le \big(\|x_n - y\|^2 - \|u_n - y\|^2\big) - 2\delta_n(\bar{\gamma} - \gamma\rho)\|x_n - y\|^2 + \delta_n^2\bar{\gamma}^2\|x_n - y\|^2 \\
&\quad + 2\delta_n\big\langle \gamma f(y) - Ay, u_n - y\big\rangle \\
&= \big(\|x_n - y\| + \|u_n - y\|\big)\big(\|x_n - y\| - \|u_n - y\|\big) \\
&\quad - 2\delta_n(\bar{\gamma} - \gamma\rho)\|x_n - y\|^2 + \delta_n^2\bar{\gamma}^2\|x_n - y\|^2 \\
&\quad + 2\delta_n\big\langle \gamma f(y) - Ay, u_n - y\big\rangle \\
&\le M_2\|x_n - u_n\| - 2\delta_n(\bar{\gamma} - \gamma\rho)\|x_n - y\|^2 + \delta_n^2\bar{\gamma}^2\|x_n - y\|^2 + 2\delta_n\big\langle \gamma f(y) - Ay, u_n - y\big\rangle,
\end{aligned}$$

where $M_2 := \sup\{\|x_n - y\| + \|u_n - y\| : n \in \mathbb{N}\} < \infty$ for every $n \in \mathbb{N}$. By the weak convergence of $\{u_{n_i}\}$ to $\bar{x} \in F(T)$, $\lim_{n\to\infty}\|u_n - x_n\| = 0$ and (C3)-(C4), we get $\langle(\gamma f - A)y, \bar{x} - y\rangle \le 0$ for all $y \in F(T)$. A mapping A being a strongly positive linear bounded operator and f being a contraction ensure $\langle(\gamma f - A)y, \bar{x} - y\rangle \le 0$ for all $y \in F(T)$, that is, $\bar{x} \in VI(F(T), A - \gamma f)$. Thus, $x^* \in VI(VI(F(T), A - \gamma f), B)$, we have

$$\begin{aligned}
\limsup_{n\to\infty}\big\langle x^* - x_n, Bx^*\big\rangle &= \limsup_{i\to\infty}\big\langle x^* - x_{n_i}, Bx^*\big\rangle \\
&= \big\langle x^* - \bar{x}, Bx^*\big\rangle \\
&\le 0.
\end{aligned}$$

From (3.13), we notice that

$$\limsup_{n\to\infty}\big\langle x^* - y_n, Bx^*\big\rangle \le 0. \tag{3.17}$$

Third, $\limsup_{n\to\infty}\langle x_n - x^*, \phi(x^*) - x^*\rangle \le 0$ is proven. Choose a subsequence $\{x_{n_g}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\big\langle x_n - x^*, \phi\big(x^*\big) - x^*\big\rangle = \lim_{g\to\infty}\big\langle x_{n_g} - x^*, \phi\big(x^*\big) - x^*\big\rangle.$$

The boundedness of $\{x_{n_g}\}$ implies the existence of a subsequence $\{x_{n_{g_h}}\}$ of $\{x_{n_g}\}$ and a point $\tilde{x} \in H$ such that $\{x_{n_{g_h}}\}$ converges weakly to $\tilde{x}$. By $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, we have $\lim_{h\to\infty}\langle x_{n_{g_h}+1}, w\rangle = \langle\tilde{x}, w\rangle$, $w \in H$. We may assume, without loss of generality, that $\lim_{i\to\infty}\langle x_{n_g}, w\rangle = \langle\tilde{x}, w\rangle$, $w \in H$. Assume $\tilde{x} \ne T(\tilde{x})$. $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ with $F(T) \ne \emptyset$ guarantees that

$$\begin{aligned}
\liminf_{g\to\infty}\|x_{n_g} - \tilde{x}\| &< \liminf_{g\to\infty}\big\|x_{n_g} - T(\tilde{x})\big\| \\
&= \liminf_{g\to\infty}\big\|x_{n_g} - T(x_{n_g}) + T(x_{n_g}) - T(\tilde{x})\big\| \\
&= \liminf_{g\to\infty}\big\|T(x_{n_g}) - T(\tilde{x})\big\| \\
&\le \liminf_{g\to\infty}\|x_{n_g} - \tilde{x}\|.
\end{aligned}$$

This is a contradiction, that is, $\tilde{x} \in F(T)$. From $x^* \in VI(VI(VI(F(T), A - \gamma f), B), I - \phi)$, we find

$$\begin{aligned}
&\limsup_{n\to\infty}\langle x_n - x^*, \phi(x^*) - x^*\rangle \\
&\quad = \lim_{g\to\infty}\langle x_{n_g} - x^*, \phi(x^*) - x^*\rangle \\
&\quad = \langle \tilde{x} - x^*, \phi(x^*) - x^*\rangle \\
&\quad \le 0.
\end{aligned} \tag{3.18}$$

Step 4. Finally, we prove $\lim_{n\to\infty}\|x_n - x^*\| = 0$. By Lemma 2.8, we compute

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n\phi(x_n) + (1-\alpha_n)y_n - x^*\|^2 \\
&= \|\alpha_n(\phi(x_n) - \phi(x^*)) + \alpha_n(\phi(x^*) - x^*) + (1-\alpha_n)(y_n - x^*)\|^2 \\
&\le \alpha_n\|\phi(x_n) - \phi(x^*)\|^2 + (1-\alpha_n)\|y_n - x^*\|^2 + 2\alpha_n\langle\phi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&\le \alpha_n k^2\|x_n - x^*\|^2 + (1-\alpha_n)\|(I - \mu\beta_n B)z_n - x^*\|^2 \\
&\quad + 2\alpha_n\langle\phi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&= \alpha_n k^2\|x_n - x^*\|^2 + (1-\alpha_n)\|(z_n - \mu\beta_n Bz_n) - (x^* - \mu\beta_n Bx^*) - \mu\beta_n Bx^*\|^2 \\
&\quad + 2\alpha_n\langle\phi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&\le \alpha_n k^2\|x_n - x^*\|^2 + (1-\alpha_n) \\
&\quad \times \{\|(z_n - \mu\beta_n Bz_n) - (x^* - \mu\beta_n Bx^*)\|^2 + 2\mu\beta_n\langle x^* - y_n, Bx^*\rangle\} \\
&\quad + 2\alpha_n\langle\phi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&\le \alpha_n k^2\|x_n - x^*\|^2 + (1-\alpha_n)(1-\tau\beta_n)^2\|z_n - x^*\|^2 + 2\mu\beta_n\langle x^* - y_n, Bx^*\rangle \\
&\quad + 2\alpha_n\langle\phi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&\le \alpha_n k^2\|x_n - x^*\|^2 + (1-\alpha_n)(1-\tau\beta_n)\|u_n - x^*\|^2 + 2\mu\beta_n\langle x^* - y_n, Bx^*\rangle \\
&\quad + 2\alpha_n\langle\phi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&= \alpha_n k^2\|x_n - x^*\|^2 + (1-\alpha_n)(1-\tau\beta_n)\|[I - \delta_n(A - \gamma f)]x_n - x^*\|^2 \\
&\quad + 2\mu\beta_n\langle x^* - y_n, Bx^*\rangle + 2\alpha_n\langle\phi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&= \alpha_n k^2\|x_n - x^*\|^2 + (1-\alpha_n)(1-\tau\beta_n) \\
&\quad \times \|(I - \delta_n A)(x_n - x^*) + \delta_n(\gamma f(x_n) - Ax^*)\|^2 \\
&\quad + 2\mu\beta_n\langle x^* - y_n, Bx^*\rangle + 2\alpha_n\langle\phi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&\le \alpha_n k^2\|x_n - x^*\|^2 + (1-\alpha_n)(1-\tau\beta_n) \\
&\quad \times \{(1 - \delta_n\bar{\gamma})^2\|x_n - x^*\|^2 + 2\delta_n\langle\gamma f(x_n) - Ax^*, u_n - x^*\rangle\} \\
&\quad + 2\mu\beta_n\langle x^* - y_n, Bx^*\rangle + 2\alpha_n\langle\phi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&\le \alpha_n k\|x_n - x^*\|^2 + (1-\alpha_n)(1-\tau\beta_n)\{(1 - 2\delta_n\bar{\gamma} + \delta_n^2\bar{\gamma}^2)\|x_n - x^*\|^2 \\
&\quad + 2\delta_n\langle\gamma f(x_n) - \gamma f(x^*), u_n - x^*\rangle + 2\delta_n\langle\gamma f(x^*) - Ax^*, u_n - x^*\rangle\}
\end{aligned}$$

$$
\begin{aligned}
&\quad + 2\mu\beta_n\langle x^* - y_n, Bx^*\rangle + 2\alpha_n\langle \phi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&\le \alpha_n k\|x_n - x^*\|^2 + (1-\alpha_n)(1-\tau\beta_n)\{(1-2\delta_n\bar{\gamma})\|x_n - x^*\|^2 \\
&\quad + \delta_n^2\bar{\gamma}^2\|x_n - x^*\|^2 + 2\delta_n\gamma\rho\|x_n - x^*\|\|u_n - x^*\| \\
&\quad + 2\delta_n\langle \gamma f(x^*) - Ax^*, u_n - x^*\rangle\} \\
&\quad + 2\mu\beta_n\langle x^* - y_n, Bx^*\rangle + 2\alpha_n\langle \phi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&\le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)(1-\tau\beta_n)[1-2\delta_n(\bar{\gamma} - \gamma\rho)]\|x_n - x^*\|^2 \\
&\quad + (1-\alpha_n)(1-\tau\beta_n)\delta_n^2\bar{\gamma}^2\|x_n - x^*\|^2 + 2\delta_n\langle \gamma f(x^*) - Ax^*, u_n - x^*\rangle \\
&\quad + 2\mu\beta_n\langle x^* - y_n, Bx^*\rangle + 2\alpha_n\langle \phi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&= \alpha_n\|x_n - x^*\|^2 \\
&\quad + (1-\alpha_n)[1-2\delta_n(\bar{\gamma} - \gamma\rho) - \tau\beta_n + \tau\beta_n 2\delta_n(\bar{\gamma} - \gamma\rho)]\|x_n - x^*\|^2 \\
&\quad + (1-\alpha_n)(1-\tau\beta_n)\delta_n^2\bar{\gamma}^2\|x_n - x^*\|^2 + 2\delta_n\langle \gamma f(x^*) - Ax^*, u_n - x^*\rangle \\
&\quad + 2\mu\beta_n\langle x^* - y_n, Bx^*\rangle + 2\alpha_n\langle \phi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&= \alpha_n\|x_n - x^*\|^2 \\
&\quad + (1-\alpha_n)[1-\{2\delta_n(\bar{\gamma} - \gamma\rho) + \tau\beta_n - \tau\beta_n 2\delta_n(\bar{\gamma} - \gamma\rho)\}]\|x_n - x^*\|^2 \\
&\quad + (1-\alpha_n)(1-\tau\beta_n)\delta_n^2\bar{\gamma}^2\|x_n - x^*\|^2 + 2\delta_n\langle \gamma f(x^*) - Ax^*, u_n - x^*\rangle \\
&\quad + 2\mu\beta_n\langle x^* - y_n, Bx^*\rangle + 2\alpha_n\langle \phi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&= [1-(1-\alpha_n)\{2\delta_n(\bar{\gamma} - \gamma\rho) + \tau\beta_n - \tau\beta_n 2\delta_n(\bar{\gamma} - \gamma\rho)\}]\|x_n - x^*\|^2 \\
&\quad + (1-\alpha_n)(1-\tau\beta_n)\delta_n^2\bar{\gamma}^2\|x_n - x^*\|^2 + 2\delta_n\langle \gamma f(x^*) - Ax^*, u_n - x^*\rangle \\
&\quad + 2\mu\beta_n\langle x^* - y_n, Bx^*\rangle + 2\alpha_n\langle \phi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&= [1-(1-\alpha_n)\{2\delta_n(\bar{\gamma} - \gamma\rho)(1-\tau\beta_n) + \tau\beta_n\}]\|x_n - x^*\|^2 \\
&\quad + (1-\alpha_n)(1-\tau\beta_n)\delta_n^2\bar{\gamma}^2\|x_n - x^*\|^2 + 2\delta_n\langle \gamma f(x^*) - Ax^*, u_n - x^*\rangle \\
&\quad + 2\mu\beta_n\langle x^* - y_n, Bx^*\rangle + 2\alpha_n\langle \phi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&= [1-(1-\alpha_n)2\delta_n(\bar{\gamma} - \gamma\rho)(1-\tau\beta_n)]\|x_n - x^*\|^2 - (1-\alpha_n)\tau\beta_n\|x_n - x^*\|^2 \\
&\quad + (1-\alpha_n)(1-\tau\beta_n)\delta_n^2\bar{\gamma}^2\|x_n - x^*\|^2 + 2\delta_n\langle \gamma f(x^*) - Ax^*, u_n - x^*\rangle \\
&\quad + 2\mu\beta_n\langle x^* - y_n, Bx^*\rangle + 2\alpha_n\langle \phi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&\le [1-2(\bar{\gamma} - \gamma\rho)(1-\alpha_n)(1-\tau\beta_n)\delta_n]\|x_n - x^*\|^2 \\
&\quad + (1-\alpha_n)(1-\tau\beta_n)\delta_n^2\bar{\gamma}^2\|x_n - x^*\|^2 + 2\delta_n\langle \gamma f(x^*) - Ax^*, u_n - x^*\rangle \\
&\quad + 2\mu\beta_n\langle x^* - y_n, Bx^*\rangle + 2\alpha_n\langle \phi(x^*) - x^*, x_{n+1} - x^*\rangle. \qquad (3.19)
\end{aligned}
$$

Since $\{x_n\}$, $\{Ax_n\}$, $\{Bx_n\}$, $\{\phi(x_n)\}$ and $\{f(x_n)\}$ are all bounded, we can choose a constant $M_1 > 0$ such that

$$
\sup_n \frac{1}{\bar{\gamma} - \gamma\rho}\left\{\frac{\delta_n\bar{\gamma}^2}{2}\|x_n - x^*\|^2\right\} \le M_1.
$$

It follows that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \left[1-2(\bar{\gamma}-\gamma\rho)(1-\alpha_n)(1-\tau\beta_n)\delta_n\right]\|x_n-x^*\|^2 \\ &\quad + 2(\bar{\gamma}-\gamma\rho)(1-\alpha_n)(1-\tau\beta_n)\delta_n\varsigma_n, \end{aligned} \tag{3.20}$$

where

$$\begin{aligned}\varsigma_n &= \delta_n M_1 + \frac{1}{(\bar{\gamma}-\gamma\rho)(1-\alpha_n)(1-\tau\beta_n)}\langle \gamma f(x^*) - Ax^*, u_n - x^*\rangle \\ &\quad + \frac{\mu\beta_n}{(\bar{\gamma}-\gamma\rho)(1-\alpha_n)(1-\tau\beta_n)\delta_n}\langle x^* - y_n, Bx^*\rangle \\ &\quad + \frac{\alpha_n}{(\bar{\gamma}-\gamma\rho)(1-\alpha_n)(1-\tau\beta_n)\delta_n}\langle \phi(x^*) - x^*, x_{n+1} - x^*\rangle.\end{aligned}$$

By (3.14), (3.17), (3.18) and (C3)-(C4), we get $\limsup_{n\to\infty}\varsigma_n \le 0$. Applying Lemma 2.9, we can conclude that $x_n \to x^*$. This completes the proof. □

4 An example

Next, the following example shows that all the conditions of Theorem 3.1 are satisfied.

Example 4.1 For instance, let $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{1}{2n}$ and $\delta_n = \frac{1}{3n}$. We will show that the condition (C1) is achieved. Then, clearly, the sequence $\{\delta_n\}$

$$\sum_{n=1}^{\infty}\delta_n = \sum_{n=1}^{\infty}\frac{1}{3n} = \infty$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}|\delta_{n+1}-\delta_n| &= \sum_{n=1}^{\infty}\left|\frac{1}{3(n+1)} - \frac{1}{3n}\right| \\ &\le \left|\frac{1}{3\cdot 1} - \frac{1}{3\cdot 2}\right| + \left|\frac{1}{3\cdot 2} - \frac{1}{3\cdot 3}\right| + \left|\frac{1}{3\cdot 3} - \frac{1}{3\cdot 4}\right| + \cdots \\ &= \frac{1}{3}.\end{aligned}$$

The sequence $\{\delta_n\}$ satisfies the condition (C1).

Next, we will show that the condition (C2) is achieved. We compute

$$\begin{aligned}\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| &= \sum_{n=1}^{\infty}\left|\frac{1}{2(n+1)} - \frac{1}{2n}\right| \\ &\le \left|\frac{1}{2\cdot 1} - \frac{1}{2\cdot 2}\right| + \left|\frac{1}{2\cdot 2} - \frac{1}{2\cdot 3}\right| + \left|\frac{1}{2\cdot 3} - \frac{1}{2\cdot 4}\right| + \cdots \\ &= \frac{1}{2}.\end{aligned}$$

The sequence $\{\beta_n\}$ satisfies the condition (C2).

Next, we will show that the condition (C3) is achieved. We compute

$$\begin{aligned}\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| &= \sum_{n=1}^{\infty}\left|\frac{1}{n+1}-\frac{1}{n}\right| \\ &\le \left|\frac{1}{1}-\frac{1}{2}\right|+\left|\frac{1}{2}-\frac{1}{3}\right|+\left|\frac{1}{3}-\frac{1}{4}\right|+\cdots \\ &= 1\end{aligned}$$

and

$$\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\frac{1}{n} = 0.$$

The sequence $\{\alpha_n\}$ satisfies the condition (C3).

Finally, we will show that the condition (C4) is achieved.

$$\frac{1}{3n} < \frac{1}{2n} \quad \text{and} \quad \frac{1}{2n} < \frac{1}{n}.$$

Corollary 4.2 *Let H be a real Hilbert space, C be a closed convex subset of H. Let $A : C \to H$ be inverse-strongly monotone. Let $T : C \to C$ be a nonexpansive mapping. Let $B : C \to C$ be β-strongly monotone and L-Lipschitz continuous. Assume that $VI(F(T),A)$ is nonempty set. Suppose $\{x_n\}$ is a sequence generated by the following algorithm $x_0 \in C$ arbitrarily*:

$$\begin{cases} z_n = T(I-\delta_n A)x_n, \\ y_n = (I-\mu\beta_n B)z_n, \\ x_{n+1} = (1-\alpha_n)y_n, \quad \forall n \ge 0, \end{cases} \tag{4.1}$$

$\{\alpha_n\}, \{\delta_n\} \subset [0,1]$. If $\mu \in (0, \frac{2\beta}{L^2})$ is used and if $\{\beta_n\} \subset (0,1]$ satisfies the following conditions:

(C1): $\sum_{n=1}^{\infty}|\delta_{n+1}-\delta_n| < \infty$, $\sum_{n=1}^{\infty}\delta_n = \infty$;

(C2): $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty$;

(C3): $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$, $\lim_{n\to\infty}\alpha_n = 0$;

(C4): $\delta_n \le \beta_n$ *and* $\beta_n \le \alpha_n$.

Then $\{x_n\}$ converges strongly to $x^* \in VI(F(T),A)$, which is the unique solution of the variational inequality:

$$\textit{Find } x^* \in VI\big(F(T),A\big) \textit{ such that } \big\langle Bx^*, x-x^*\big\rangle \ge 0, \quad \forall x \in VI\big(F(T),A\big). \tag{4.2}$$

Proof Putting P_C is the identity and $f, \phi \equiv 0$ in Theorem 3.1, we can obtain the desired conclusion immediately. □

Remark 4.3 Corollary 4.2 generalizes and improves the results of Iiduka [14].

Corollary 4.4 *Let H be a real Hilbert space, C be a closed convex subset of H. Let $A : C \to H$ be a strongly positive linear bounded operator, $f : C \to H$ be a ρ-contraction, γ be a positive real number such that $\frac{\bar{\gamma}-1}{\rho} < \gamma < \frac{\bar{\gamma}}{\rho}$. Let $T : C \to C$ be a nonexpansive mapping. Assume that Ω is nonempty set. Suppose $\{x_n\}$ is a sequence generated by the following* *algorithm* $x_0 \in C$ *arbitrarily*:

$$\begin{cases} z_n = TP_C[I - \delta_n(A - \gamma f)]x_n, \\ y_n = (I - \mu\beta_n B)z_n, \\ x_{n+1} = \alpha_n(x_n) + (1 - \alpha_n)y_n, \quad \forall n \geq 0, \end{cases} \tag{4.3}$$

where $\{\alpha_n\}, \{\delta_n\} \subset [0,1]$. *If* $\mu \in (0, \frac{2\beta}{L^2})$ *is used and if* $\{\beta_n\} \subset (0,1]$ *satisfies the following conditions*:

(C1): $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} \delta_n = \infty$;
(C2): $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;
(C3): $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\lim_{n\to\infty} \alpha_n = 0$;
(C4): $\delta_n \leq \beta_n$ *and* $\beta_n \leq \alpha_n$.

Then $\{x_n\}$ *converges strongly to* $x^* \in \Omega$, *which is the unique solution of the variational inequality*:

$$\text{Find } x^* \in \Omega \text{ such that } \langle Bx^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega. \tag{4.4}$$

Proof Putting ϕ is the identity in Theorem 3.1, we can obtain the desired conclusion immediately. □

Remark 4.5 Corollary 4.4 generalizes and improves the results of Marino and Xu [20].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bangmod, Thung Khru, Bangkok, 10140, Thailand. [2]Department of Mathematics, Faculty of Science and Agricultural Technology, Rajamangala University of Technology Lanna, Phan, Chiang Rai, 57120, Thailand.

Acknowledgements
The authors thank to the Higher Education Research Promotion and National Research University Project of Thailand, the Office of the Higher Education Commission for financial support during the preparation of this paper.

Received: 10 May 2012 Accepted: 13 November 2012 Published: 3 December 2012

References

1. Hartman, P, Stampacchia, G: On some nonlinear elliptic differential functional equations. Acta Math. **115**, 271-310 (1966)
2. Cianciaruso, F, Marino, G, Muglia, L, Yao, Y: On a two-step algorithm for hierarchical fixed point problems and variational inequalities. J. Inequal. Appl. **2009**, Article ID 208692 (2009)
3. Yao, JC, Chadli, O: Pseudomonotone complementarity problems and variational inequalities. In: Crouzeix, JP, Haddjissas, N, Schaible, S (eds.) Handbook of Generalized Convexity and Monotonicity, pp. 501-558 (2005)
4. Kirk, WA: Fixed point theorem for mappings which do not increase distance. Am. Math. Mon. **72**, 1004-1006 (1965)
5. Combettes, PL: A block-iterative surrogate constraint splitting method for quadratic signal recovery. IEEE Trans. Signal Process. **51**(7), 1771-1782 (2003)
6. Hirstoaga, SA: Iterative selection method for common fixed point problems. J. Math. Anal. Appl. **324**, 1020-1035 (2006)
7. Iiduka, H, Yamada, I: A subgradient-type method for the equilibrium problem over the fixed point set and its applications. Optimization **58**(2), 251-261 (2009)
8. Slavakis, K, Yamada, I: Robust wideband beamforming by the hybrid steepest descent method. IEEE Trans. Signal Process. **55**(9), 4511-4522 (2007)
9. Slavakis, K, Yamada, I, Sakaniwa, K: Computation of symmetric positive definite Toeplitz matrices by the hybrid steepest descent method. Signal Process. **83**, 1135-1140 (2003)

10. Yamada, I: The hybrid steepest descent method for the variational inequality problems over the intersection of fixed point sets of nonexpansive mappings. In: Butnariu, D, Censor, Y, Reich, S (eds.) Inherently Paralle Algorithms for Feasibllity and Optimization and Their Applications, pp. 473-504. Elsevier, Amsterdam (2001)
11. Yao, Y, Cho, YJ, Liou, Y-C: Iterative algorithms for hierarchical fixed points problems and variational inequalities. Math. Comput. Model. **52**, 1697-1705 (2010)
12. Yamada, I, Ogura, N, Shirakawa, N: A numerically robust hybrid steepest descent method for the convexly constrained generalized inverse problems. In: Nashed, Z, Scherzer, O (eds.) Inverse Problems, Image Analysis, and Medical Imaging. Contemp. Math., vol. 313, pp. 269-305. Amer. Math. Soc., Providence (2002)
13. Yamada, I, Ogura, N: Hybrid steepest descent method for variational inequality problem over the fixed point set of certain quasi-nonexpansive mapping. Numer. Funct. Anal. Optim. **25**, 619-655 (2004)
14. Iiduka, H: Strong convergence for an iterative method for the triple-hierarchical constrained optimization problem. Nonlinear Anal. **71**, e1292-e1297 (2009)
15. Iiduka, H: Iterative algorithm for solving triple-hierarchical constrained optimization problem. J. Optim. Theory Appl. **148**(3), 580-592 (2011)
16. Jitpeera, T, Kumam, P: A new explicit triple hierarchical problem over the set of fixed point and generalized mixed equilibrium problem. J. Inequal. Appl. **2012**, 82 (2012). doi:10.1186/1029-242X-2012-82
17. Jitpeera, T, Kumam, P: Algorithms for solving the variational inequality problem over the triple hierarchical problem. Abstr. Appl. Anal. **2012**, Article ID 827156 (2012). doi:10.1155/2012/827156
18. Wairojjana, N, Kumam, P: General iterative algorithms for hierarchical fixed points approach to variational inequalities. J. Appl. Math. **2012**, Article ID 174318 (2012). doi:10.1155/2012/174318
19. Yao, Y, Liou, Y-C, Kang, SM: Algorithms construction for variational inequalities. Fixed Point Theory Appl. **2011**, Article ID 794203 (2011). doi:10.1155/2011/794203
20. Marino, G, Xu, H-K: Explicit hierarchical fixed point approach to variational inequalities. J. Optim. Theory Appl. **149**(1), 61-78 (2011)
21. Browder, FE: Nonlinear operators and nonlinear equations of evolution in Banach spaces. Proc. Symp. Pure Math. **18**, 78-81 (1976)
22. Marino, G, Xu, HK: A general iterative method for nonexpansive mapping in Hilbert space. J. Math. Anal. Appl. **318**, 43-52 (2006)
23. Opial, Z: Weak convergence of the sequence of successive approximations for nonexpansive mappings. Bull. Am. Math. Soc. **73**, 595-597 (1967)
24. Suzuki, T: Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals. J. Math. Anal. Appl. **305**(1), 227-239 (2005)
25. Xu, HK: Iterative algorithms for nonlinear operators. J. Lond. Math. Soc. **66**, 240-256 (2002)

doi:10.1186/1029-242X-2012-280
Cite this article as: Wairojjana et al.: **The hybrid steepest descent method for solving variational inequality over triple hierarchical problems.** *Journal of Inequalities and Applications* 2012 **2012**:280.