Mahmoud et al. Journal of Inequalities and Applications 2012, **2012**:27
http://www.journalofinequalitiesandapplications.com/content/2012/1/27

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

New upper bounds of n!

Mansour Mahmoud[1,3*], Mohammed A Alghamdi[1] and Ravi P Agarwal[2]

* Correspondence: mansour@mans.edu.eg
[1]Mathematics Department, Faculty of Science, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

In this article, we deduce a new family of upper bounds of n! of the form

$$n! < \sqrt{2\pi n}(n/e)^n e^{M_n^{[m]}} \quad n \in \mathbb{N},$$

$$M_n^{[m]} = \frac{1}{2m+3}\left[\frac{1}{4n} + \sum_{k=1}^{m} \frac{2m-2k+2}{2k+1} 2^{-2k} \zeta(2k, n+1/2)\right] \quad m = 1, 2, 3, \ldots.$$

We also proved that the approximation formula $\sqrt{2\pi n}(n/e)^n e^{M_n^{[m]}}$ for big factorials has a speed of convergence equal to n^{-2m-3} for m = 1,2,3,..., which give us a superiority over other known formulas by a suitable choice of m.

Mathematics Subject Classification (2000): 41A60; 41A25; 57Q55; 33B15; 26D07.

Keywords: Stirling?'? formula, Wallis?'? formula, Bernoulli numbers, Riemann Zeta function, speed of convergence

1 Introduction

Stirling' formula

$$n! \sim \sqrt{2n\pi}(n/e)^n \tag{1}$$

is one of the most widely known and used in asymptotics. In other words, we have

$$\lim_{n\to\infty} \frac{n!e^n}{\sqrt{2\pi n}\, n^n} = 1. \tag{2}$$

This formula provides an extremely accurate approximation of n! for large values of n. The first proofs of Stirling's formula was given by De Moivre (1730) [1] and Stirling (1730) [2]. Both used what is now called the Euler-MacLaurin formula to approximate log 2 + log 3 + ... + log n. The first derivation of De Moivre did not explicity determine the constant $\sqrt{2\pi}$. In 1731, Stirling determine this constant using Wallis' formula

$$\lim_{n\to\infty} \frac{2^{2n}(n!)^2}{(2n)!}\frac{1}{\sqrt{n}} = \sqrt{\pi}.$$

Over the years, there have been many different upper and lower bounds for n! by various authors [3-10]. Artin [11] show that $\mu(n) = \ln \frac{n!e^n}{n^n\sqrt{2\pi n}}$ lies between any two successive partial sums of the Stirling's series

$$\frac{B_2}{1.2.n} + \frac{B_4}{3.4.n^3} + \frac{B_6}{5.6.n^5} + \cdots, \tag{3}$$

where the numbers B_i are called the Bernoulli numbers and are defined by

$$B_0 = 1, \quad \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0, \quad n \geq 2. \tag{4}$$

We can't take infinite sum of the series (3) because the series diverges. Also, Impens [12] deduce Artin result with different proof and show that the Bernoulli numbers in this series cannot be improved by any method whatsoever.

The organization of this article is as follows. In Section 2, we deduce a general double inequality of $n!$, which already obtained in [9] with different proof. Section 3 is devoted to getting a new family of upper bounds of $n!$ different from the partial sums of the Stirling's series. In Section 4, we measure the speed of convergence of our approximation formula $\sqrt{2\pi n}(n/e)^n e^{M_n^{[m]}}$ for big factorials. Also, we offer some numerical computations to prove the superiority of our formula over other known formulas.

2 A double inequality of *n*!

In view of the relation (2), we begin with the two sequences K_n and f_n defined by

$$f_n = \frac{n! e^n}{\sqrt{2\pi n}\, n^n} e^{-K_n} \qquad n \geq 1, \tag{5}$$

where

$$\lim_{n\to\infty} K_n = 0. \tag{6}$$

Then we have

$$\lim_{n\to\infty} f_n = 1. \tag{7}$$

Now define the sequence g_n by

$$g_n = \frac{f_{n+1}}{f_n} = e^{K_n - K_{n+1} + 1} \left(1 + \frac{1}{n}\right)^{-n-1/2}, \tag{8}$$

which satisfies

$$\lim_{n\to\infty} g_n = 1 \tag{9}$$

and

$$g_n' = g_n \frac{d}{dn} \left(K_n - K_{n+1} - (n + 1/2) \ln\left(1 + \frac{1}{n}\right)\right). \tag{10}$$

There are two cases. The first case if $K_n = a_n$ such that

$$\frac{d}{dn} \left(a_n - a_{n+1} - (n + 1/2) \ln\left(1 + \frac{1}{n}\right)\right) > 0 \tag{11}$$

then we get $g_n' > 0$ and hence g_n is strictly increasing function. But $g_n \to 1$ as $n \to \infty$, then $g_n < 1$. Hence $\frac{f_{n+1}}{f_n} < 1$ which give us that $f_{n+1} < f_n$. Then f_n is strictly decreasing function. Also, $f_n \to 1$ as $n \to \infty$, then we obtain $f_n > 1$. Then

$$\frac{n!\, e^n}{\sqrt{2\pi n}\, n^n} e^{-a_n} > 1 \quad n \geq 1 \tag{12}$$

or

$$\sqrt{2\pi n}\, (n/e)^n e^{a_n} < n! \quad n \geq 1. \tag{13}$$

The condition (11) means that the function a_n - a_{n+1} -(n+ 1/2) ln $\left(1 + \frac{1}{n}\right)$ is strictly increasing function also it tends to -1 as $n \to \infty$. Then

$$a_n - a_{n+1} - (n + 1/2) \ln\left(1 + \frac{1}{n}\right) < -1$$

or

$$a_n - a_{n+1} < (n + 1/2) \ln\left(1 + \frac{1}{n}\right) - 1. \tag{14}$$

The second case if $K_n = b_n$ such that

$$\frac{d}{dn}\left(b_n - b_{n+1} - (n + 1/2) \ln\left(1 + \frac{1}{n}\right)\right) < 0. \tag{15}$$

Similarly, we can prove that $f_n < 1$. Then

$$\frac{n! e^n}{\sqrt{2\pi n}\, n^n} e^{-b_n} < 1 \quad n \geq 1 \tag{16}$$

or

$$n! < \sqrt{2\pi n}(n/e)^n e^{b_n} \quad n \geq 1. \tag{17}$$

The condition (15) means that the function b_n - b_{n+1} - (n + 1/2) ln $\left(1 + \frac{1}{n}\right)$ is strictly decreasing function also it tends to -1 as $n \to \infty$. Then

$$b_n - b_{n+1} - (n + 1/2) \ln\left(1 + \frac{1}{n}\right) > -1$$

or

$$(n + 1/2) \ln\left(1 + \frac{1}{n}\right) - 1 < b_n - b_{n+1}. \tag{18}$$

From the well-known expansion

$$\frac{1}{2} \ln\left(\frac{1 + y}{1 - y}\right) = \sum_{k=1}^{\infty} \frac{y^{2k-1}}{2k - 1} \qquad |y| < 1$$

in which we substitute $\gamma = \frac{1}{2n+1}$, so $\frac{1+\gamma}{1-\gamma} = 1 + \frac{1}{n}$. Then

$$(n+1/2)\ln\left(1+\frac{1}{n}\right) - 1 = \sum_{k=1}^{\infty} \frac{1}{2k+1}\frac{1}{(2n+1)^{2k}}. \tag{19}$$

Now we obtain the following result

Theorem 1.

$$\sqrt{2\pi n}\,(n/e)^n e^{a_n} < n! < \sqrt{2\pi n}\,(n/e)^n e^{b_n} \qquad n \geq 1, \tag{20}$$

where the two sequences a_n, $b_n \to 0$ *as* $n \to \infty$ *and satisfy*

$$a_n - a_{n+1} < \sum_{k=1}^{\infty} \frac{1}{2k+1}\frac{1}{(2n+1)^{2k}} < b_n - b_{n+1}. \tag{21}$$

A q-analog of the inequality (20) was introduced in [13].

3 A new family of upper bounds of $n!$

By manipulating the series $\sum_{k=1}^{\infty} \frac{1}{2k+1}\frac{1}{(2n+1)^{2k}}$ to find upper bounds, we get

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{1}{2k+1}\frac{1}{(2n+1)^{2k}} \\
&\quad < \sum_{k=1}^{m} \frac{1}{(2k+1)(2n+1)^{2k}} + \frac{1}{2m+3}\sum_{k=m+1}^{\infty} \frac{1}{(2n+1)^{2k}} \\
&\quad < \sum_{k=1}^{\infty} \frac{1}{(2k+1)(2n+1)^{2k}} + \frac{1}{4(2m+3)n(n+1)(2n+1)^{2m}}, \qquad m = 1, 2, 3, \ldots.
\end{aligned}$$

Let's solve the following recurrence relation w.r.t n

$$M_n^{[m]} - M_{n+1}^{[m]} = \sum_{k=1}^{m} \frac{1}{(2k+1)(2n+1)^{2k}} + \frac{1}{4(2m+3)n(n+1)(2n+1)^{2m}}, \qquad m = 1, 2, 3, \ldots \tag{22}$$

which give us

$$\begin{aligned}
M_n^{[m]} &= M_0^{[m]} - \sum_{i=1}^{n-1}\left(\sum_{k=1}^{m} \frac{1}{(2k+1)(2i+1)^{2k}} + \frac{1}{4(2m+3)i(i+1)(2i+1)^{2m}}\right) \\
&= M_0^{[m]} - \sum_{k=1}^{m} \frac{1}{2k+1}\left(\sum_{i=1}^{n-1} \frac{1}{(2i+1)^{2k}}\right) - \frac{1}{4(2m+3)}\sum_{i=1}^{n-1} \frac{1}{i(i+1)(2i+1)^{2m}}.
\end{aligned}$$

But

$$\begin{aligned}
\sum_{i=1}^{n-1} \frac{1}{i(i+1)(2i+1)^{2m}} &= \sum_{i=1}^{n-1}\left(\frac{1}{i} - \frac{1}{i+1} - \frac{4}{(2i+1)^2} - \frac{4}{(2i+1)^4} - \cdots - \frac{4}{(2i+1)^{2m}}\right) \\
&= \frac{n-1}{n} - 4\sum_{k=1}^{m}\sum_{i=1}^{n-1} \frac{1}{(2i+1)^{2k}}.
\end{aligned}$$

Then

$$M_n^{[m]} = M_0^{[m]} - \frac{n-1}{4n(2m+3)} - \sum_{k=1}^{m} \left(\frac{1}{2k+1} - \frac{1}{2m+3} \right) \left(\sum_{i=1}^{n-1} \frac{1}{(2i+1)^{2k}} \right).$$

The series

$$\sum_{i=1}^{\infty} \frac{1}{(2i+1)^{2k}} = \zeta(2k)(1-2^{-2k}) - 1,$$

where $\zeta(x)$ is the Riemann Zeta function. By using the relation [14]

$$\zeta(2k) = \frac{(-1)^{k-1} 2^{2k-1}}{(2k)!} B_{2k} \pi^{2k},$$

where $B_r's$ are Bernoulli's numbers. Then

$$\sum_{i=1}^{\infty} \frac{1}{(2i+1)^{2k}} = \frac{(-1)^{k-1}(2^{2k}-1)}{2(2k)!} B_{2k} \pi^{2k} - 1.$$

Hence, we can choose

$$M_0^{[m]} = \frac{1}{4(2m+3)} + \sum_{k=1}^{m} \left(\frac{1}{2k+1} - \frac{1}{2m+3} \right) \left(\frac{(-1)^{k-1}(2^{2k}-1)}{2(2k)!} B_{2k} \pi^{2k} - 1 \right), \tag{23}$$

which satisfies

$$\lim_{n\to\infty} M_n^{[m]} = 0, \quad m = 1, 2, 3, \ldots.$$

Then

$$M_n^{[m]} = \frac{1}{2m+3} \left[\frac{1}{4n} + \sum_{k=1}^{m} \frac{2m-2k+2}{2k+1} \left(\frac{(-1)^{k-1}(2^{2k}-1)}{2(2k)!} B_{2k} \pi^{2k} - 1 - \sum_{i=1}^{n-1} \frac{1}{(2i+1)^{2k}} \right) \right]. \tag{24}$$

By using the relation

$$\begin{aligned} \sum_{i=1}^{n-1} \frac{1}{(2i+1)^{2k}} &= -1 - (2-^{2k} - 1)\zeta(2k) - 2^{-2k} \zeta(2k, n+1/2) \\ &= -1 - \frac{(-1)^{k-1}(1-2^{2k})}{2(2k)!} B_{2k} \pi^{2k} - 2^{-2k} \zeta(2k, n+1/2), \end{aligned}$$

we get

$$M_n^{[m]} = \frac{1}{2m+3} \left[\frac{1}{4n} + \sum_{k=1}^{m} \frac{2m-2k+2}{2k+1} 2^{-2k} \zeta(2k, n+1/2) \right]. \tag{25}$$

In the following Lemma, we will see that the upper bound M_n^m will improved with increasing the value of m.

Lemma 3.1.

$$M_n^{[m+1]} < M_n^{[m]}, \qquad m, n = 1, 2, 3, \ldots. \tag{26}$$

Proof. From (25), we get

$$M_n^{[m+1]} = \frac{1}{2m+5}\left[\frac{1}{4n} + \sum_{k=1}^{m} \frac{2m-2k+2}{2k+1} 2^{-2k}\zeta(2k, n+1/2)\right] + \frac{1}{2m+5}\sum_{k=1}^{m+1}\frac{1}{2k+1}2^{1-2k}\zeta(2k, n+1/2).$$

Then

$$\begin{aligned} M_n^{[m+1]} - M_n^{[m]} &= \frac{2}{(2m+3)(2m+5)}\left[\frac{-1}{4n} + \sum_{k=1}^{m+1} 2^{-2k}\zeta(2k, n+1/2)\right] \\ &< \frac{2}{(2m+3)(2m+5)}\left[\frac{-1}{4n} + \sum_{k=1}^{\infty} 2^{-2k}\zeta(2k, n+1/2)\right]. \end{aligned}$$

Now

$$\begin{aligned} \sum_{k=1}^{\infty} 2^{-2k}\zeta(2k, n+1/2) &= \sum_{k=1}^{\infty}\sum_{r=0}^{\infty}\frac{2^{-2k}}{(n+r+1/2)^{2k}} \\ &= \sum_{r=0}^{\infty}\sum_{k=1}^{\infty}\frac{2^{-2k}}{(n+r+1/2)^{2k}} \\ &= \sum_{r=0}^{\infty}\left[\frac{1}{1-\left(\frac{1}{(2n+2r+1)^2}\right)} - 1\right] \\ &= \sum_{r=0}^{\infty}\frac{1}{4(n+r)(n+r+1)} \\ &= \frac{1}{4}\sum_{r=0}^{\infty}\left[\frac{1}{n+r} - \frac{1}{n+r+1}\right] \\ &= \frac{1}{4n}. \end{aligned}$$

Then

$$\sum_{k=1}^{\infty} 2^{-2k}\zeta(2k, n+1/2) = \frac{1}{4n}. \tag{27}$$

and hence

$$\begin{aligned} M_n^{[m+1]} - M_n^{[m]} &< \frac{2}{(2m+3)(2m+5)}\left[\frac{-1}{4n} + \sum_{k=1}^{\infty} 2^{-2k}\zeta(2k, n+1/2)\right] \\ &< \frac{2}{(2m+3)(2m+5)}\left[\frac{-1}{4n} + \frac{1}{4n}\right] \\ &< 0. \end{aligned}$$

The following Lemma show that $M_n^{[1]}$ is an improvement of the upper bound $\frac{1}{12n}$.

Lemma 3.2.

$$M_n^{[1]} < \frac{1}{12n} \tag{28}$$

Proof. From Eq. (25), we have $M_n^{[1]} = \frac{1}{20n} + \frac{1}{30}\zeta(2, n + 1/2)$. By using (27), we obtain

$$\begin{aligned}\frac{1}{n} &= \sum_{k=1}^{\infty} 2^{2-2k}\zeta(2k, n + 1/2)\\ &= \zeta(2, n + 1/2) + \sum_{k=2}^{\infty} 2^{2-2k}\zeta(2k, n + 1/2).\end{aligned}$$

Then

$$\zeta(2, n + 1/2) < \frac{1}{n}$$

which give us that

$$\frac{1}{30}\zeta(2, n + 1/2) + \frac{1}{20n} < \frac{1}{30n} + \frac{1}{20n} = \frac{1}{12n}.$$

4 The speed of convergence of the approximation formula $\sqrt{2\pi n}\,(n/e)^n e^{M_n^{[m]}}$ for big factorials

In what follows, we need the following result, which represents a powerful tool to measure the rate of convergence.

Lemma 4.1. *If* $(w_n)_{n\geq 1}$ *is convergent to zero and there exists the limit*

$$\lim_{n\to\infty} n^k(w_n - w_{n+1}) = l \in \mathbb{R} \tag{29}$$

with $k > 1$, *then there exists the limit:*

$$\lim_{n\to\infty} n^{k-1} w_n = \frac{l}{k-1}$$

This Lemma was first used by Mortici for constructing asymptotic expansions, or to accelerate some convergences [15-21]. By using Lemma (4.1), clearly the sequence $(w_n)_{n\geq 1}$ converges more quickly when the value of k satisfying (29) is larger.

To measure the accuracy of approximation formula $\sqrt{2\pi n}\,(n/e)^n e^{M_n^{[m]}}$, define the sequence $(w_n)_{n\geq 1}$ by the relation

$$n! = \sqrt{2\pi n}(n/e)^n e^{M_n^{[m]}} e^{w_n}; n = 1, 2, 3, ... \tag{30}$$

This approximation formula will be better as $(w_n)_{n\geq 1}$ converges faster to zero. Using the relation (30), we get

$$w_n = \ln n! - \ln\sqrt{2\pi} - (n + 1/2)\ln n + n - M_n^{[m]}$$

Then

$$w_n - w_{n+1} = (n+1/2)\ln(1+1/n) - 1 + M_{n+1}^{[m]} - M_n^{[m]}$$

By using the relations (19) and (22), we have

$$\begin{aligned} w_n - w_{n+1} &= \sum_{k=m+1}^{\infty} \frac{1}{2k+1}\frac{1}{(2n+1)^{2k}} - \frac{1}{4(2m+3)n(n+1)(2n+1)^{2m}} \\ &= \sum_{k=m+3}^{\infty} \frac{1}{2k+1}\frac{1}{(2n+1)^{2k}} - \frac{(5+2m+8n+8n^2)(2n+1)^{-4-2m}}{4n(n+1)(15+16m+4m^2)} \end{aligned}$$

Then

$$\lim_{n\to\infty} n^{2(m+2)}(w_n - w_{n+1}) = \frac{-1}{(2m+3)(2m+5)2^{2m+3}}; \quad n, m = 1, 2, 3, \dots . \tag{31}$$

Theorem 2. *The rate of convergence of the sequence w_n is equal to n^{-2m-3}, since*

$$\lim_{n\to\infty} n^{2m+3} w_n = \frac{1}{(2m+3)^2(2m+5)2^{2m+3}}.$$

In 2011, Mortici [22] shows by numerical computations that his formula

$$n! \sim \sqrt{2\pi n}(n/e)^n e^{\frac{1}{12n+\frac{2}{5n}}} = \mu_n \tag{32}$$

is much stronger than other known formulas such as:

$n! \sim \sqrt{2\pi}\left(\frac{n+1/2}{e}\right)^{n+1/2} = \beta_n$	(Burnside [23])
$n! \sim \frac{\sqrt{2\pi}e^{-n}n^{n+1}}{\sqrt{n-1/6}} = \delta_n$	(Batir [24])
$n! \sim \sqrt{(2n+1/3)\pi}\ (n/e)^n = \gamma_n$	(Gosper [25])
$n! \sim \sqrt{\pi}\sqrt[6]{8n^3+4n^2+n+1/30}\ (n/e)^n = \rho_n$	(Ramanujan [26])

The following table shows numerically that our new formula $\lambda_{n,1} = \sqrt{2\pi n}\ (n/e)^n e^{M_n^{[1]}}$ has a superiority over the the Mortici's formula μ_n.

n	$\lvert n! - \mu_n\rvert$	$\lvert n! - \lambda_{n,1}\rvert$
10	0.0252281	0.00641793
25	1.1×10^{15}	2.8×10^{14}
50	6.8×10^{52}	1.7×10^{52}
100	6.5×10^{144}	1.4×10^{144}

Acknowledgements
This project was founded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant No. 448/130/1431. The authors, therefore, acknowledge with thanks DSR support for Scientific Research.

Author details
[1]Mathematics Department, Faculty of Science, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia
[2]Department of Mathematics, Texas A&M University, Kingsville 700 University Blvd, Kingsville, TX 78363-8202, USA
[3]Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

Authors' contributions
All authors carried out the proofs. All authors conceived of the study and participated in its design and coordination. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 15 October 2011 Accepted: 13 February 2012 Published: 13 February 2012

References

1. De Moivre, A: Miscellanea Analytica de Seriebus et Quadraturis. London (1730)
2. Stirling, J: Methodus Differentialis. London (1730)
3. Hummel, PM: A note on Stirling's formoula. Am Math Monthly. **42**(2):97–99 (1940)
4. Cesàro, E: Elements Lehrbuch der algebraischen analysis und der In-finitesimalrechnung. Leipzig, Teubner (1922)
5. Uspensky, JV: Introduction to Mathematical Probability. McGraw Hill, New York (1937)
6. Robbins, H: A remark on Stirling's formula. Am Math Monthly. **62**, 26–29 (1955). doi:10.2307/2308012
7. Nanjundiah, TS: Note on Stirling's formula. Am Math Monthly. **66**, 701–703 (1965)
8. Maria, AJ: A remark on Striling's formula. Am Math Monthly. **72**, 1096–1098 (1965). doi:10.2307/2315957
9. Beesack, PR: Improvement of Stirling's formula by elementary methods. Univ. Beograd, Publ. Elektrotenhn. Fak Ser Mat Fiz 17–21 (1969). No. 274-301
10. Michel, R: On Stirling's formula. Am. Math. Monthly. **109**(4):388–390 (2002)
11. Artin, E: The Gamma Function (translated by M Butler). Holt, Rinehart and Winston. New York (1964)
12. Impens, C: Stirling's series made easy. Am Math Monthly. **110**(8):730–735 (2003). doi:10.2307/3647856
13. Mansour, M, Obaid, MA: Bounds of q-factorial $[n]_q!$ Ars Combinatoria CII. 313–319 (2011)
14. Andrews, GE, Askey, R, Roy, R: Special Functions. Cambridge University Press, Cambridge (1999)
15. Mortici, C: An ultimate extremely accurate formula for approximation of the factorial function. Archiv der Mathematik (Basel). **93**(1):37–45 (2009). doi:10.1007/s00013-009-0008-5
16. Mortici, C: Product approximations via asymptotic integration. Am Math Monthly. **117**, 434–441 (2010). doi:10.4169/000298910X485950
17. Mortici, C: New approximations of the gamma function in terms of the digamma function. Appl Math Lett. **23**, 97–100 (2010). doi:10.1016/j.aml.2009.08.012
18. Mortici, C: New improvements of the Stirling formula. Appl Math Comput. **217**(2):699–704 (2010). doi:10.1016/j.amc.2010.06.007
19. Mortici, C: Best estimates of the generalized Stirling formula. Appl Math Comput. **215**(11):4044–4048 (2010). doi:10.1016/j.amc.2009.12.013
20. Mortici, C: A class of integral approximations for the factorial function. Comput Math Appl. **59**(6):2053–2058 (2010). doi:10.1016/j.camwa.2009.12.010
21. Mortici, C: Ramanujan formula for the generalized Stirling approximation. Appl Math Comput. **217**(6):2579–2585 (2010). doi:10.1016/j.amc.2010.07.070
22. Mortici, C: A new Stirling series as continued fraction. Numer Algorithm. **56**(1):17–26 (2011). doi:10.1007/s11075-010-9370-4
23. Burnside, W: A rapidly convergent series for Log N! Messenger Math. **46**, 157–159 (1917)
24. Batir, N: Sharp inequalities for factorial *n*. Proyecciones. **27**(1):97–102 (2008)
25. Gosper, RW: Decision procedure for indefinite hypergeometric summation. Proc Natl Acad Sci USA. **75**, 40–42 (1978). doi:10.1073/pnas.75.1.40
26. Ramanujan, S: The Lost Notebook and Other Unpublished Papers (Introduced by Andrews, GE). Narosa Publishing House, New Delhi (1988)

doi:10.1186/1029-242X-2012-27
Cite this article as: Mahmoud et al.: New upper bounds of *n*!. *Journal of Inequalities and Applications* 2012 **2012**:27.