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A relation between Hilbert and Carlson inequalities

LE Azar*

*Correspondence:
azar1@aabu.edu.jo
Department of Mathematics, Al al-Bayt University, P.O. Box 130095, Mafraq, Jordan

Abstract

In this paper, we introduce some new inequalities with a best constant factor. As an application, we obtain a sharper form of Hilbert's inequality. Some inequalities of Carlson type are also considered.

MSC: 26D15

Keywords: Hilbert's inequality; Carlson's inequality; Cauchy's inequality; sharper form

1 Introduction

If $f(x), g(x) \geq 0$, $0 < \int_0^\infty f^2(x)\,dx < \infty$, and $0 < \int_0^\infty g^2(x)\,dx < \infty$, then (see [1])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \pi \left\{\int_0^\infty f^2(x)\,dx\right\}^{\frac{1}{2}} \left\{\int_0^\infty g^2(x)\,dx\right\}^{\frac{1}{2}}. \tag{1.1}$$

Inequality (1.1) is called Hilbert's integral inequality, which has been extended by Hardy [1] as: if $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(x) > 0$, $0 < \int_0^\infty f^p(x)\,dx < \infty$, and $0 < \int_0^\infty g^q(x)\,dx < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin\frac{\pi}{p}} \left\{\int_0^\infty f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty g^q(x)\,dx\right\}^{\frac{1}{q}}. \tag{1.2}$$

The corresponding inequalities in the discrete case are ($a_m, b_n > 0$):

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n} < \pi \left\{\sum_{n=1}^\infty a_n^2\right\}^{\frac{1}{2}} \left\{\sum_{n=1}^\infty b_n^2\right\}^{\frac{1}{2}}, \tag{1.3}$$

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin\frac{\pi}{p}} \left\{\sum_{n=1}^\infty a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^\infty b_n^q\right\}^{\frac{1}{q}}, \tag{1.4}$$

provided that the series on the right-hand side of (1.3) and (1.4) are convergent. The constant factor π is the best possible in both (1.1) and (1.3), and the constant $\frac{\pi}{\sin\frac{\pi}{p}}$ is the best possible in (1.2) and (1.4). The following general inequality was given in [2]:

$$\begin{aligned}&\int_0^\infty \int_0^\infty K(x,y) f(x) g(y)\,dx\,dy \\ &\quad < k(pA_2) \left\{\int_0^\infty x^{pqA_1-1} f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty y^{pqA_2-1} g^q(x)\,dx\right\}^{\frac{1}{q}},\end{aligned} \tag{1.5}$$

where $k(pA_2) = \int_0^\infty K(1,t)t^{-pA_2}\,dt$ is the best possible constant, $K(x,y) \geq 0$ is a homogeneous function of degree $-\lambda$ ($\lambda > 0$), $A_1 \in (\frac{1-\lambda}{q}, \frac{1}{q})$, $A_2 \in (\frac{1-\lambda}{p}, \frac{1}{p})$, and $pA_2 + qA_1 = 2 - \lambda$. Moreover, in [3] the following is proved:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} K(m,n)a_m b_n < k(pA_2)\left\{\sum_{n=1}^{\infty} m^{pqA_1-1}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{pqA_2-1}b_n^q\right\}^{\frac{1}{q}}, \tag{1.6}$$

here $K(x,y) \geq 0$ is a homogeneous function of degree $-\lambda$ ($\lambda > 0$) strictly decreasing in both parameters x and y, $A_1 \in (\max\{\frac{1-\lambda}{q}, 0\}, \frac{1}{q})$, $A_2 \in (\max\{\frac{1-\lambda}{p}, 0\}, \frac{1}{p})$, and $pA_2 + qA_1 = 2 - \lambda$. In particular, if we set $K(x,y) = \frac{1}{\alpha x + \beta y}$ and $K(m,n) = \frac{1}{\alpha m + \beta n}$ ($\alpha, \beta > 0$) respectively in (1.5) and (1.6) for $\lambda = 1$, we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\alpha x + \beta y}\,dx\,dy \\
&\quad < \frac{B(pA_2, 1 - pA_2)}{\alpha^{1-qA_1}\beta^{1-pA_2}}\left\{\int_0^\infty x^{pqA_1-1}f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty y^{pqA_2-1}g^q(x)\,dx\right\}^{\frac{1}{q}},
\end{aligned} \tag{1.7}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\alpha m + \beta n} < \frac{B(pA_2, 1 - pA_2)}{\alpha^{1-qA_1}\beta^{1-pA_2}}\left\{\sum_{n=1}^{\infty} m^{pqA_1-1}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{pqA_2-1}b_n^q\right\}^{\frac{1}{q}}, \tag{1.8}$$

where $B(s,t)$ is the beta function, $A_1 \in (0, \frac{1}{q})$, $A_2 \in (0, \frac{1}{p})$, and $pA_2 + qA_1 = 1$. We need the following formula for the beta function:

$$B(s, t+1) = \frac{t}{s+t}B(s,t). \tag{1.9}$$

For the sequence of real numbers (a_n), Carlson's inequality is given as

$$\sum_{n=1}^{\infty} a_n < \sqrt{\pi}\left(\sum_{n=1}^{\infty} a_n^2\right)^{\frac{1}{4}}\left(\sum_{n=1}^{\infty} n^2 a_n^2\right)^{\frac{1}{4}}, \tag{1.10}$$

the constant $\sqrt{\pi}$ is the best possible. The continuous version of (1.10) is

$$\int_0^\infty f(x)\,dx \leq \sqrt{\pi}\left(\int_0^\infty f^2(x)\,dx\right)^{\frac{1}{4}}\left(\int_0^\infty x^2 f^2(x)\,dx\right)^{\frac{1}{4}}, \tag{1.11}$$

the constant $\sqrt{\pi}$ is sharp. Regarding these inequalities and their extensions, we refer the reader to the book [4].

In this paper, we introduce two new inequalities with a best constant factor which gives an upper estimate for the double series $\sum_{n=1}^\infty \sum_{m=1}^\infty \sigma_{n,m}$ and the double integral $\int_0^\infty \int_0^\infty G(x,y)\,dx\,dy$, where $\sigma_{n,m}$ is a double sequence of positive numbers and $G(x,y)$ is a positive function on $(0,\infty) \times (0,\infty)$. As an application, we obtain a sharper form of the Hilbert inequality. Some examples of Carlson type inequalities are also considered. The proof of the inequalities depends on inequalities (1.7), (1.8) and Hardy's idea in proving Carlson's inequality.

2 Discrete case

Theorem 2.1 *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, a_n and b_n be two sequences of positive numbers such that $0 < \sum_{m=1}^{\infty} m^{pqA_1-1} a_m^p < \infty$, $0 < \sum_{n=1}^{\infty} n^{pqA_2-1} b_n^q < \infty$. Let $\sigma_{n,m}$ be a double sequence of positive numbers such that $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{m\sigma_{n,m}^2}{a_m b_n} < \infty$ and $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{n\sigma_{n,m}^2}{a_m b_n} < \infty$, then the following inequality holds*:

$$\begin{aligned}\left[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sigma_{n,m}\right]^2 &< L\left\{\sum_{m=1}^{\infty} m^{pqA_1-1}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{pqA_2-1}b_n^q\right\}^{\frac{1}{q}}\\ &\quad\times\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n\sigma_{n,m}^2}{a_m b_n}\right\}^{qA_1}\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m\,\sigma_{n,m}^2}{a_m b_n}\right\}^{pA_2},\end{aligned} \tag{2.1}$$

here $A_1 \in (0, \frac{1}{q})$, $A_2 \in (0, \frac{1}{p})$, $pA_2 + qA_1 = 1$, and the constant $L = \frac{B(pA_2, 1-pA_2)}{(pA_2)^{pA_2}(qA_1)^{qA_1}}$ is the best possible.

Proof Let $\alpha, \beta > 0$, using Cauchy's inequality and then applying (1.8), we get

$$\begin{aligned}\left[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sigma_{n,m}\right]^2 &= \left[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{\sqrt{a_m b_n}}{\sqrt{\alpha m+\beta n}}\right\}\left\{\frac{\sqrt{\alpha m+\beta n}}{\sqrt{a_m b_n}}\sigma_{n,m}\right\}\right]^2\\ &\le \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\alpha m+\beta n}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\alpha m+\beta n}{a_m b_n}\sigma_{n,m}^2\\ &< \frac{B(pA_2, 1-pA_2)}{\alpha^{1-qA_1}\beta^{1-pA_2}}\left\{\sum_{m=1}^{\infty} m^{pqA_1-1}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{pqA_2-1}b_n^q\right\}^{\frac{1}{q}}\\ &\quad\times\left[\alpha\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m\sigma_{n,m}^2}{a_m b_n} + \beta\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n\sigma_{n,m}^2}{a_m b_n}\right]\\ &= B(pA_2, 1-pA_2)\left\{\sum_{m=1}^{\infty} m^{pqA_1-1}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{pqA_2-1}b_n^q\right\}^{\frac{1}{q}}\\ &\quad\times\left[\left(\frac{\alpha}{\beta}\right)^{qA_1}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m\sigma_{n,m}^2}{a_m b_n} + \left(\frac{\beta}{\alpha}\right)^{pA_2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n\sigma_{n,m}^2}{a_m b_n}\right].\end{aligned}$$

Set $T = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m\sigma_{n,m}^2}{a_m b_n}$, $S = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n\sigma_{n,m}^2}{a_m b_n}$, $t = \frac{\alpha}{\beta}$ and consider the function $h(t) = t^{qA_1}T + t^{-pA_2}S$. Since $h'(t) = \frac{1}{t^{pA_2}}(qA_1 T - \frac{pA_2 S}{t})$, we conclude that the minimum of this function attains for $t = \frac{pA_2 S}{qA_1 T}$. Therefore, if we let $\alpha = pA_2 S$ and $\beta = qA_1 T$, we get (2.1).

It remains to show that the constant L in (2.1) is the best possible. To do that, suppose that there exists a positive constant $C < L$ such that (2.1) is still valid if we replace L by C. For $0 < \varepsilon < q(1-pA_2)$, setting $\widetilde{a}_m$ and $\widetilde{b}_n$ as $\widetilde{a}_m = m^{-qA_1-\frac{\varepsilon}{p}}$, $\widetilde{b}_n = n^{-pA_2-\frac{\varepsilon}{q}}$, and $\widetilde{\sigma}_{n,m} = \frac{\widetilde{a}_m\widetilde{b}_n}{m+n}$, we have $\sum_{m=1}^{\infty} m^{pqA_1-1}\widetilde{a}_m^p = \sum_{m=1}^{\infty}\frac{1}{m^{1+\varepsilon}} = 1 + \sum_{m=2}^{\infty}\frac{1}{m^{1+\varepsilon}} < 1 + \int_1^{\infty} x^{-1-\varepsilon}\,dx = 1 + \frac{1}{\varepsilon}$; similarly, we obtain $\sum_{n=1}^{\infty} n^{pqA_2-1}\widetilde{b}_n^q < 1 + \frac{1}{\varepsilon}$. Moreover,

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n\widetilde{\sigma}_{n,m}^2}{\widetilde{a}_m\widetilde{b}_n} &= \sum_{n=1}^{\infty} n^{1-pA_2-\frac{\varepsilon}{q}}\sum_{m=1}^{\infty}\frac{m^{-qA_1-\frac{\varepsilon}{p}}}{(m+n)^2}\\ &< \sum_{n=1}^{\infty} n^{1-pA_2-\frac{\varepsilon}{q}}\int_0^{\infty}\frac{x^{-qA_1-\frac{\varepsilon}{p}}}{(n+x)^2}\,dx\end{aligned}$$

$$\begin{aligned}&=\sum_{n=1}^{\infty}n^{-1-\varepsilon}\int_0^{\infty}\frac{u^{-qA_1-\frac{\varepsilon}{p}}}{(1+u)^2}\,du=\bigl[B(1-qA_1,1+qA_1)+o(1)\bigr]\sum_{n=1}^{\infty}n^{-1-\varepsilon}\\&<\frac{\varepsilon+1}{\varepsilon}\bigl[B(1-qA_1,1+qA_1)+o(1)\bigr].\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m\widetilde{\sigma}_{n,m}^2}{\widetilde{a}_m\widetilde{b}_n}&=\sum_{m=1}^{\infty}m^{1-qA_1-\frac{\varepsilon}{p}}\sum_{n=1}^{\infty}\frac{m^{-pA_2-\frac{\varepsilon}{q}}}{(m+n)^2}<\sum_{m=1}^{\infty}m^{1-qA_1-\frac{\varepsilon}{p}}\int_0^{\infty}\frac{x^{-pA_2-\frac{\varepsilon}{q}}}{(m+x)^2}\,dx\\&=\sum_{m=1}^{\infty}m^{-1-\varepsilon}\int_0^{\infty}\frac{u^{-pA_2-\frac{\varepsilon}{q}}}{(1+u)^2}\,du=\bigl[B(1-pA_2,1+pA_2)+o(1)\bigr]\sum_{m=1}^{\infty}m^{-1-\varepsilon}\\&<\frac{\varepsilon+1}{\varepsilon}\bigl[B(1-pA_2,1+pA_2)+o(1)\bigr].\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\widetilde{\sigma}_{n,m}&=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m^{-qA_1-\frac{\varepsilon}{p}}n^{-pA_2-\frac{\varepsilon}{q}}}{m+n}>\int_1^{\infty}\int_1^{\infty}\frac{x^{-qA_1-\frac{\varepsilon}{p}}y^{-pA_2-\frac{\varepsilon}{q}}}{x+y}\,dx\,dy\\&>\frac{1}{\varepsilon}\bigl[B(pA_2,1-pA_2)+o(1)\bigr]-O(1).\end{aligned}$$

Substituting the above inequalities in (2.1), we get

$$\begin{aligned}&\left\{\frac{1}{\varepsilon}\bigl[B(pA_2,1-pA_2)+o(1)\bigr]-O(1)\right\}^2\\&\quad<C\frac{\varepsilon+1}{\varepsilon}\left\{\frac{\varepsilon+1}{\varepsilon}\bigl[B(1-qA_1,1+qA_1)+o(1)\bigr]\right\}^{qA_1}\\&\quad\quad\times\left\{\frac{\varepsilon+1}{\varepsilon}\bigl[B(1-pA_2,1+pA_2)+o(1)\bigr]\right\}^{pA_2}.\end{aligned}\tag{2.2}$$

Multiplying inequality (2.2) by ε^2 ($\varepsilon=\varepsilon^{qA_1}\varepsilon^{pA_2}$) and then letting $\varepsilon\to 0^+$, we have

$$B^2(pA_2,1-pA_2)\le CB^{qA_1}(1-qA_1,1+qA_1)B^{pA_2}(1-pA_2,1+pA_2).\tag{2.3}$$

Using (1.9), we find

$$\begin{aligned}B^{qA_1}(1-qA_1,1+qA_1)&=(qA_1)^{qA_1}B^{qA_1}(1-qA_1,qA_1)\\&=(qA_1)^{qA_1}B^{qA_1}(pA_2,1-pA_2),\end{aligned}\tag{2.4}$$

and

$$B^{pA_2}(1-pA_2,1+pA_2)=(pA_2)^{pA_2}B^{pA_2}(1-pA_2,pA_2).\tag{2.5}$$

Substituting (2.4) and (2.5) in (2.3), we obtain the contradiction $C\ge L=\frac{B(pA_2,1-pA_2)}{(pA_2)^{pA_2}(qA_1)^{qA_1}}$. The theorem is proved. □

2.1 Some applications

1. If $\sigma_{n,m} = \frac{a_m b_n}{m+n}$ in (2.1), then we have the following form of Hilbert's inequality:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \sqrt{L}\left\{\sum_{m=1}^{\infty} m^{pqA_1-1}a_m^p\right\}^{\frac{1}{2p}}\left\{\sum_{n=1}^{\infty} n^{pqA_2-1}b_n^q\right\}^{\frac{1}{2q}} \times \left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n a_m b_n}{(m+n)^2}\right\}^{\frac{qA_1}{2}}\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m a_m b_n}{(m+n)^2}\right\}^{\frac{pA_2}{2}}. \tag{2.6}$$

Inequality (2.6) is a sharper form of (1.8). To see that, let us rewrite (2.6) in the following form:

$$\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}\right\}^2 < \frac{B(pA_2,1-pA_2)}{(pA_2)^{pA_2}(qA_1)^{qA_1}}\left\{\sum_{m=1}^{\infty} m^{pqA_1-1}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{pqA_2-1}b_n^q\right\}^{\frac{1}{q}} S^{qA_1}T^{pA_2},$$

where $S = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n a_m b_n}{(m+n)^2}$ and $T = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m a_m b_n}{(m+n)^2}$. Since we may write $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} = T + S$, dividing both sides of the last inequality by $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}$, we get

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < B(pA_2,1-pA_2)\left\{\sum_{m=1}^{\infty} m^{pqA_1-1}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{pqA_2-1}b_n^q\right\}^{\frac{1}{q}} \times \frac{(\frac{S}{qA_1})^{qA_1}(\frac{T}{pA_2})^{pA_2}}{T+S}. \tag{2.7}$$

Applying Young's inequality ($x^\theta y^\mu \le \theta x + \mu y$, $\theta + \mu = 1$) to the product $(\frac{S}{qA_1})^{qA_1}(\frac{T}{pA_2})^{pA_2}$ with $x = \frac{S}{qA_1}$ and $y = \frac{T}{pA_2}$, we obtain

$$\frac{(\frac{S}{qA_1})^{qA_1}(\frac{T}{pA_2})^{pA_2}}{T+S} \le \frac{T+S}{T+S} = 1.$$

Therefore, inequality (2.6) is a sharper form of (1.8). In particular, if we set $p = q = 2$, $A_1 = \frac{1}{4} = A_2$ in (2.6), we obtain the following sharper form of the classical Hilbert inequality (1.3):

$$\left[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}\right]^2 < 2\pi\left\{\sum_{m=1}^{\infty} a_m^2\right\}^{\frac{1}{2}}\left\{\sum_{n=1}^{\infty} b_n^2\right\}^{\frac{1}{2}} \times\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n a_m b_n}{(m+n)^2}\right\}^{\frac{1}{2}}\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m a_m b_n}{(m+n)^2}\right\}^{\frac{1}{2}}. \tag{2.8}$$

Note that we may obtain the Hilbert inequality from (2.8) by applying the AG inequality $\sqrt{ST} \le \frac{S+T}{2}$ to the right-hand side of (2.8).

2. The more accurate Hilbert inequality is given as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n+1} < \frac{\pi}{\sin\frac{\pi}{p}}\left\{\sum_{n=1}^{\infty} a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}}. \tag{2.9}$$

If we put $\sigma_{n,m} = \frac{a_m b_n}{m+n+\mu}$ ($\mu > 0$) in (2.1), then we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n+\mu} \\
&\quad < \sqrt{L}\left\{\sum_{m=1}^{\infty} m^{pqA_1-1}a_m^p\right\}^{\frac{1}{2p}}\left\{\sum_{n=1}^{\infty} n^{pqA_2-1}b_n^q\right\}^{\frac{1}{2q}}\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{na_m b_n}{(m+n+\mu)^2}\right\}^{\frac{qA_1}{2}} \\
&\qquad \times\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{ma_m b_n}{(m+n+\mu)^2}\right\}^{\frac{pA_2}{2}}.
\end{aligned}$$

Since

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n+\mu} \\
&\quad = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{ma_m b_n}{(m+n+\mu)^2}+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{na_m b_n}{(m+n+\mu)^2}+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\mu a_m b_n}{(m+n+\mu)^2} \\
&\quad = S_1 + S_2 + S_3,
\end{aligned}$$

we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n+\mu} &< B(pA_2, 1-pA_2)\left\{\sum_{m=1}^{\infty} m^{pqA_1-1}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{pqA_2-1}b_n^q\right\}^{\frac{1}{q}} \\
&\quad \times\frac{(\frac{S_1}{qA_1})^{qA_1}(\frac{S_2}{pA_2})^{pA_2}}{S_1+S_2+S_3}.
\end{aligned}$$

Applying Young's inequality, we get $\frac{(\frac{S_1}{qA_1})^{qA_1}(\frac{S_2}{pA_2})^{pA_2}}{S_1+S_2+S_3} \leq \frac{S_1+S_2}{S_1+S_2+S_3} < 1$, thus we arrive at

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n+\mu} < B(pA_2, 1-pA_2)\left\{\sum_{m=1}^{\infty} m^{pqA_1-1}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{pqA_2-1}b_n^q\right\}^{\frac{1}{q}}.$$

In particular, if $\mu = 1$, $A_1 = \frac{1}{pq} = A_2$, then we have (2.9).

3. For $\alpha > 1$, set $a_m = m^{-\frac{\alpha}{p}}$, $b_n = n^{-\frac{\alpha}{q}}$, and $A_1 = A_2 = \frac{1}{pq}$ in (2.1), then we get

$$\begin{aligned}
&\left[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sigma_{n,m}\right]^2 \\
&\quad < \frac{p^{\frac{1}{p}}q^{\frac{1}{q}}\pi\zeta(\alpha)}{\sin\frac{\pi}{p}}\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m^{\frac{\alpha}{p}}n^{\frac{\alpha}{q}+1}\sigma_{n,m}^2\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} n^{\frac{\alpha}{q}}m^{\frac{\alpha}{p}+1}\sigma_{n,m}^2\right\}^{\frac{1}{q}},
\end{aligned}$$

where $\zeta(\alpha)$ is the Riemann zeta function. In particular, if $\alpha = 2$ and if $\sigma_{n,m} = c_m c_n$, then we get the following Carlson type inequality:

$$\sum_{m=1}^{\infty} c_m < \left(\frac{\pi^3}{3}\right)^{\frac{1}{4}}\left\{\sum_{m=1}^{\infty} mc_m^2\right\}^{\frac{1}{4}}\left\{\sum_{m=1}^{\infty} m^2c_m^2\right\}^{\frac{1}{4}}.$$

4. If we let $\sigma_{n,m} = a_m a_n$ and $b_n = a_n$, $p = q = 2$, $A_1 = A_2 = \frac{1}{4}$, we get

$$\sum_{m=1}^{\infty} a_m < (2\pi)^{\frac{1}{3}} \left\{ \sum_{m=1}^{\infty} m a_m \right\}^{\frac{1}{3}} \left\{ \sum_{m=1}^{\infty} a_m^2 \right\}^{\frac{1}{3}}.$$

3 Integral case

Theorem 3.1 *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(x) > 0$ such that $0 < \int_0^\infty x^{pqA_1-1} f^p(x)\,dx < \infty$, $0 < \int_0^\infty x^{pqA_1-1} g^q(x)\,dx < \infty$, suppose also that the function $G(x,y)$ is positive on $(0,\infty) \times (0,\infty)$ and that $\int_0^\infty \int_0^\infty \frac{xG^2(x,y)}{f(x)g(y)}\,dx\,dy < \infty$, $\int_0^\infty \int_0^\infty \frac{yG^2(x,y)}{f(x)g(y)}\,dx\,dy < \infty$, then the following inequality holds*:

$$\begin{aligned}
&\left[\int_0^\infty \int_0^\infty G(x,y)\,dx\,dy \right]^2 \\
&\quad < L \left(\int_0^\infty x^{pqA_1-1} f^p(x)\,dx \right)^{\frac{1}{p}} \left(\int_0^\infty y^{pqA_2-1} g^q(y)\,dy \right)^{\frac{1}{q}} \\
&\qquad \times \left\{ \int_0^\infty \int_0^\infty \frac{xG^2(x,y)}{f(x)g(y)}\,dx\,dy \right\}^{pA_2} \left\{ \int_0^\infty \int_0^\infty \frac{yG^2(x,y)}{f(x)g(y)}\,dx\,dy \right\}^{qA_1},
\end{aligned} \tag{3.1}$$

where $A_1 \in (0, \frac{1}{q})$, $A_2 \in (0, \frac{1}{p})$, $pA_2 + qA_1 = 1$, and the constant $L = \frac{B(pA_2, 1-pA_2)}{(pA_2)^{pA_2}(qA_1)^{qA_1}}$ is the best possible.

Proof By Cauchy's inequality, taking into account (1.7), we get $(\alpha, \beta > 0)$

$$\begin{aligned}
&\left[\int_0^\infty \int_0^\infty G(x,y)\,dx\,dy \right]^2 \\
&\quad = \left[\int_0^\infty \int_0^\infty \frac{\sqrt{f(x)g(y)}}{\sqrt{\alpha x + \beta y}} \frac{\sqrt{\alpha x + \beta y}}{\sqrt{f(x)g(y)}} G(x,y)\,dx\,dy \right]^2 \\
&\quad \le \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\alpha x + \beta y}\,dx\,dy \int_0^\infty \int_0^\infty \frac{\alpha x + \beta y}{f(x)g(y)} G^2(x,y)\,dx\,dy \\
&\quad < \frac{B(pA_2, 1-pA_2)}{\alpha^{1-qA_1}\beta^{1-pA_2}} \left(\int_0^\infty x^{pqA_1-1} f^p(x)\,dx \right)^{\frac{1}{p}} \left(\int_0^\infty y^{pqA_2-1} g^q(y)\,dy \right)^{\frac{1}{q}} \\
&\qquad \times \left[\alpha \int_0^\infty \int_0^\infty \frac{xG^2(x,y)}{f(x)g(y)}\,dx\,dy + \beta \int_0^\infty \int_0^\infty \frac{yG^2(x,y)}{f(x)g(y)}\,dx\,dy \right] \\
&\quad = B(pA_2, 1-pA_2) \left(\int_0^\infty x^{pqA_1-1} f^p(x)\,dx \right)^{\frac{1}{p}} \left(\int_0^\infty y^{pqA_2-1} g^q(x)\,dx \right)^{\frac{1}{q}} \\
&\qquad \times \left[\left(\frac{\alpha}{\beta} \right)^{qA_1} \int_0^\infty \int_0^\infty \frac{xG^2(x,y)}{f(x)g(y)}\,dx\,dy + \left(\frac{\beta}{\alpha} \right)^{pA_2} \int_0^\infty \int_0^\infty \frac{yG^2(x,y)}{f(x)g(y)}\,dx\,dy \right].
\end{aligned}$$

As in the proof of Theorem 2.1, if we set $I_1 = \int_0^\infty \int_0^\infty \frac{xG^2(x,y)}{f(x)g(y)}\,dx\,dy$, $I_2 = \int_0^\infty \int_0^\infty \frac{yG^2(x,y)}{f(x)g(y)}\,dx\,dy$, $t = \frac{\alpha}{\beta}$ and consider the function $h(t) = t^{qA_1} I_1 + t^{-pA_2} I_2$, we find that the minimum of this function attains for $t = \frac{pA_2 I_2}{qA_1 I_1}$. Thus, if we let $\alpha = pA_2 I_2$ and $\beta = qA_1 I_1$, we get (3.1). If the constant factor L is not the best possible, then there exists a positive constant M (with $M < L$), thus (3.1) is still valid if we replace L by M. For $0 < \varepsilon < q(1 - pA_2)$, setting $\widetilde{f}$ and

$\widetilde{g}$ as $\widetilde{f}(x) = \widetilde{g}(x) = 0$ for $x \in (0,1)$, $\widetilde{f}(x) = x^{-qA_1-\frac{\varepsilon}{p}}$, $\widetilde{g}(x) = x^{-pA_2-\frac{\varepsilon}{q}}$ for $x \in [1,\infty)$, and setting $\widetilde{G}(x,y) = \frac{\widetilde{f}(x)\widetilde{g}(y)}{x+y}$, we have

$$\begin{aligned}
I^2 &:= \left[\int_1^\infty \int_1^\infty \widetilde{G}(x,y)\,dx\,dy\right]^2 \\
&< M\left(\int_1^\infty x^{-1-\varepsilon}\,dx\right)^{\frac{1}{p}}\left(\int_1^\infty y^{-1-\varepsilon}\,dy\right)^{\frac{1}{q}}\left\{\int_1^\infty \int_1^\infty \frac{x^{1-qA_1-\frac{\varepsilon}{p}}y^{-pA_2-\frac{\varepsilon}{q}}}{(x+y)^2}\,dx\,dy\right\}^{pA_2} \\
&\quad \times \left\{\int_1^\infty \int_1^\infty \frac{y^{1-pA_2-\frac{\varepsilon}{q}}x^{-qA_1-\frac{\varepsilon}{p}}}{(x+y)^2}\,dx\,dy\right\}^{qA_1} \\
&= \frac{M}{\varepsilon}(I_1)^{pA_2}(I_2)^{qA_1}.
\end{aligned} \tag{3.2}$$

Let $y = ux$, then we have

$$\begin{aligned}
I_1 &= \int_1^\infty x^{-1-\varepsilon}\int_{1/x}^\infty \frac{u^{-pA_2-\frac{\varepsilon}{q}}}{(1+u)^2}\,du\,dx < \int_1^\infty x^{-1-\varepsilon}\int_0^\infty \frac{u^{-pA_2-\frac{\varepsilon}{q}}}{(1+u)^2}\,du\,dx \\
&= \frac{1}{\varepsilon}\left[B(1-pA_2, 1+pA_2) + o(1)\right].
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_2 &= \int_1^\infty x^{-1-\varepsilon}\int_{1/x}^\infty \frac{u^{1-pA_2-\frac{\varepsilon}{q}}}{(1+u)^2}\,du\,dy < \int_1^\infty x^{-1-\varepsilon}\int_0^\infty \frac{u^{1-pA_2-\frac{\varepsilon}{q}}}{(1+u)^2}\,du\,dx \\
&= \frac{1}{\varepsilon}\left[B(pA_2, 2-pA_2) + o(1)\right].
\end{aligned}$$

Finally,

$$\begin{aligned}
I &= \int_1^\infty x^{-1-\varepsilon}\int_{1/x}^\infty \frac{u^{-pA_2-\frac{\varepsilon}{q}}}{1+u}\,du\,dx \\
&> \frac{1}{\varepsilon}\left[B(1-pA_2, pA_2) + o(1)\right] - O(1).
\end{aligned}$$

Substituting the above estimates in (3.2) and then letting $\varepsilon \to 0^+$, we find $M \geq \frac{B(pA_2,1-pA_2)}{\alpha^{1-qA_1}\beta^{1-pA_2}}$. The theorem is proved. □

3.1 Some applications

1. Putting $G(x,y) = \frac{f(x)g(y)}{x+y}$ in inequality (3.1), we get

$$\begin{aligned}
&\left[\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy\right]^2 \\
&\quad < L\left(\int_0^\infty x^{pqA_1-1}f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{pqA_2-1}g^q(y)\,dy\right)^{\frac{1}{q}} \\
&\quad\quad \times \left\{\int_0^\infty \int_0^\infty \frac{xf(x)g(y)}{(x+y)^2}\,dx\,dy\right\}^{pA_2}\left\{\int_0^\infty \int_0^\infty \frac{yf(x)g(y)}{(x+y)^2}\,dx\,dy\right\}^{qA_1}.
\end{aligned} \tag{3.3}$$

We may prove that inequality (3.3) is sharper than inequality (1.7) as we did in the discrete case, or we may do that by using (1.5) in the following way: set $K(x,y) = \frac{x}{(x+y)^2}$ and

$K(x,y) = \frac{y}{(x+y)^2}$ respectively in (1.5), then we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{xf(x)g(y)}{(x+y)^2}\,dx\,dy \\
&\quad < B(1-pA_2, 1+pA_2)\left(\int_0^\infty x^{pqA_1-1}f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{pqA_2-1}g^q(y)\,dy\right)^{\frac{1}{q}} \\
&\quad = pA_2B(1-pA_2, pA_2)\left(\int_0^\infty x^{pqA_1-1}f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{pqA_2-1}g^q(y)\,dy\right)^{\frac{1}{q}},
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{yf(x)g(y)}{(x+y)^2}\,dx\,dy \\
&\quad < qA_1B(pA_2, 1-pA_2)\left(\int_0^\infty x^{pqA_1-1}f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{pqA_2-1}g^q(y)\,dy\right)^{\frac{1}{q}}.
\end{aligned}$$

Using these two inequalities in (3.3), we get (1.7). In particular, if we set $p = q = 2$, $A_1 = \frac{1}{4} = A_2$ in (3.3), we obtain the following sharper form of the Hilbert inequality (1.1):

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy \\
&\quad < \sqrt{2\pi}\left(\int_0^\infty f^2(x)\,dx\right)^{\frac{1}{4}}\left(\int_0^\infty g^2(y)\,dy\right)^{\frac{1}{4}} \\
&\qquad \times \left\{\int_0^\infty \int_0^\infty \frac{xf(x)g(y)}{(x+y)^2}\,dx\,dy\right\}^{\frac{1}{4}}\left\{\int_0^\infty \int_0^\infty \frac{yf(x)g(y)}{(x+y)^2}\,dx\,dy\right\}^{\frac{1}{4}}.
\end{aligned}$$

2. Assuming $G(x,y) = \frac{f(x)g(y)}{x+y+\mu}$ ($\mu > 0$) in inequality (3.1), we get

$$\begin{aligned}
&\left[\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\mu}\,dx\,dy\right]^2 \\
&\quad < L\left(\int_0^\infty x^{pqA_1-1}f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{pqA_2-1}g^q(y)\,dy\right)^{\frac{1}{q}} \\
&\qquad \times \left\{\int_0^\infty \int_0^\infty \frac{xf(x)g(y)}{(x+y+\mu)^2}\,dx\,dy\right\}^{pA_2}\left\{\int_0^\infty \int_0^\infty \frac{yf(x)g(y)}{(x+y+\mu)^2}\,dx\,dy\right\}^{qA_1}.
\end{aligned}$$

Since we may write $\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\mu}\,dx\,dy = \int_0^\infty \int_0^\infty \frac{xf(x)g(y)}{(x+y+\mu)^2}\,dx\,dy + \int_0^\infty \int_0^\infty \frac{yf(x)g(y)}{(x+y+\mu)^2}\,dx\,dy + \int_0^\infty \int_0^\infty \frac{\mu f(x)g(y)}{(x+y+\mu)^2}\,dx\,dy = I_1 + I_2 + I_3$,

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\mu}\,dx\,dy \\
&\quad < B(1-pA_2, pA_2)\left(\int_0^\infty x^{pqA_1-1}f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{pqA_2-1}g^q(y)\,dy\right)^{\frac{1}{q}} \\
&\qquad \times \frac{(\frac{I_1}{qA_1})^{qA_1}(\frac{I_2}{pA_2})^{pA_2}}{I_1 + I_2 + I_3},
\end{aligned}$$

the quotient $\frac{(\frac{I_1}{qA_1})^{qA_1}(\frac{I_2}{pA_2})^{pA_2}}{I_1+I_2+I_3} \leq 1$, thus we have the following inequality:

$$\begin{aligned}&\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\mu}\,dx\,dy\\&\quad < B(1-pA_2, pA_2)\left(\int_0^\infty x^{pqA_1-1}f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{pqA_2-1}g^q(y)\,dy\right)^{\frac{1}{q}}.\end{aligned}$$

3. Put $f(x) = e^{-x}$ and $g(y) = e^{-y}$ in (3.1), $A_1 = \frac{1}{pq} = A_2$, then we get the following Carlson type inequality for functions of two variables:

$$\begin{aligned}&\left[\int_0^\infty \int_0^\infty G(x,y)\,dx\,dy\right]^2\\&\quad < \frac{\pi}{\sin\frac{\pi}{p}}\left\{\int_0^\infty \int_0^\infty xe^{x+y}G^2(x,y)\,dx\,dy\right\}^{\frac{1}{q}}\left\{\int_0^\infty \int_0^\infty ye^{x+y}G^2(x,y)\,dx\,dy\right\}^{\frac{1}{p}}.\end{aligned} \tag{3.4}$$

Moreover, if we assume $G(x,y) = \varphi(x)\varphi(y)$ in (3.4), we get ($p = 2$)

$$\int_0^\infty \varphi(x)\,dx < \pi^{\frac{1}{4}}\left\{\int_0^\infty xe^x\varphi^2(x)\,dx\right\}^{\frac{1}{4}}\left\{\int_0^\infty e^x\varphi^2(x)\,dx\right\}^{\frac{1}{4}},$$

which is a sharper form of Carlson's inequality (1.11).

4. Let $G(x,y) = f(x)f(y)$, $f(x) = g(x)$ in (3.1), then we obtain the following Carlson type inequality:

$$\int_0^\infty f(x)\,dx < (2\pi)^{\frac{1}{3}}\left\{\int_0^\infty xf(x)\,dx\right\}^{\frac{1}{3}}\left\{\int_0^\infty f^2(x)\,dx\right\}^{\frac{1}{3}}.$$

Competing interests
The author declares that he has no competing interests.

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