Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Approximation of homomorphisms and derivations on non-Archimedean random Lie C^*-algebras via fixed point method

Jung Im Kang[1] and Reza Saadati[2*]

*Correspondence: rsaadati@eml.cc
[2]Department of Mathematics, Iran University Science and Technology, Tehran, Iran
Full list of author information is available at the end of the article

Abstract

In this paper, using fixed point methods, we prove the generalized Hyers-Ulam stability of random homomorphisms in random C^*-algebras and random Lie C^*-algebras and of derivations on non-Archimedean random C^*-algebras and non-Archimedean random Lie C^*-algebras for an m-variable additive functional equation.

MSC: Primary 39A10; 39B52; 39B72; 46L05; 47H10; 46B03

Keywords: additive functional equation; fixed point; non-Archimedean random space; homomorphism in C^*-algebras and Lie C^*-algebras; generalized Hyers-Ulam stability; derivation on C^*-algebras and Lie C^*-algebras

1 Introduction and preliminaries

By a *non-Archimedean field* we mean a field K equipped with a function (valuation) $|\cdot|$ from K into $[0,\infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$ and $|r+s| \le \max\{|r|,|s|\}$ for all $r,s \in K$. Clearly, $|1| = |-1| = 1$ and $|n| \le 1$ for all $n \in \mathbb{N}$. By the trivial valuation we mean the mapping $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$. Let X be a vector space over a field K with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : X \to [0,\infty)$ is called a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) for any $r \in K$, $x \in X$, $\|rx\| = |r|\|x\|$;
(iii) the strong triangle inequality (ultrametric) holds; namely

$$\|x+y\| \le \max\{\|x\|,\|y\|\} \quad (x,y \in X).$$

Then $(X,\|\cdot\|)$ is called a non-Archimedean normed space. From the fact that

$$\|x_n - x_m\| \le \max\{\|x_{j+1} - x_j\| : m \le j \le n-1\} \quad (n > m)$$

holds, a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space. By a complete non-Archimedean normed space, we mean the one in which every Cauchy sequence is convergent.

For any nonzero rational number x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-

Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x,y) = |x-y|_p$ is denoted by $\mathbb{Q}_p$, which is called the p-adic number field.

A non-Archimedean Banach algebra is a complete non-Archimedean algebra $\mathcal{A}$ which satisfies $\|ab\| \le \|a\|\|b\|$ for all $a, b \in \mathcal{A}$. For more detailed definitions of non-Archimedean Banach algebras, we refer the reader to [1, 2].

If $\mathcal{U}$ is a non-Archimedean Banach algebra, then an involution on $\mathcal{U}$ is a mapping $t \to t^*$ from $\mathcal{U}$ into $\mathcal{U}$ which satisfies

(i) $t^{**} = t$ for $t \in \mathcal{U}$;

(ii) $(\alpha s + \beta t)^* = \overline{\alpha} s^* + \overline{\beta} t^*$;

(iii) $(st)^* = t^* s^*$ for $s, t \in \mathcal{U}$.

If, in addition, $\|t^* t\| = \|t\|^2$ for $t \in \mathcal{U}$, then $\mathcal{U}$ is a non-Archimedean C^*-algebra.

The stability problem of functional equations was originated from a question of Ulam [3] concerning the stability of group homomorphisms. Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group (a metric which is defined on a set with a group property) with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x * y), h(x) \diamond h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? If the answer is affirmative, we would say that the equation of a homomorphism $H(x * y) = H(x) \diamond H(y)$ is stable (see also [4–6]).

We recall a fundamental result in fixed point theory. Let Ω be a set. A function $d : \Omega \times \Omega \to [0, \infty]$ is called a *generalized metric* on Ω if d satisfies

(1) $d(x,y) = 0$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$ for all $x, y \in \Omega$;

(3) $d(x,z) \le d(x,y) + d(y,z)$ for all $x, y, z \in \Omega$.

Theorem 1.1 [7] *Let (Ω, d) be a complete generalized metric space, and let $J : \Omega \to \Omega$ be a contractive mapping with the Lipschitz constant $L < 1$. Then for each given element $x \in \Omega$, either $d(J^n x, J^{n+1} x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that*

(1) $d(J^n x, J^{n+1} x) < \infty$, $\forall n \ge n_0$;

(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;

(3) y^* is the unique fixed point of J in the set $\Gamma = \{y \in \Omega \mid d(J^{n_0} x, y) < \infty\}$;

(4) $d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$ for all $y \in \Gamma$.

In this paper, using the fixed point method, we prove the generalized Hyers-Ulam stability of homomorphisms and derivations in non-Archimedean random C^*-algebras and non-Archimedean random Lie C^*-algebras for the following additive functional equation (see [8]):

$$\sum_{i=1}^{m} f\left(mx_i + \sum_{j=1, j\neq i}^{m} x_j\right) + f\left(\sum_{i=1}^{m} x_i\right) = 2f\left(\sum_{i=1}^{m} mx_i\right) \quad (m \in \mathbb{N}, m \ge 2). \tag{1.1}$$

2 Random spaces

In the section, we adopt the usual terminology, notations, and conventions of the theory of random normed spaces as in [9–21]. Throughout this paper, Δ^+ is the space of distribution functions, that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \to [0, 1]$ such that F is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is, $l^-f(x) = \lim_{t\to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, *i.e.*, $F \le G$ if and only if $F(t) \le G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0 & \text{if } t \le 0, \\ 1 & \text{if } t > 0. \end{cases}$$

Definition 2.1 [20] A mapping $T : [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm (briefly, a continuous t-norm) if T satisfies the following conditions:

(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a,1) = a$ for all $a \in [0,1]$;
(d) $T(a,b) \le T(c,d)$ whenever $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$.

Typical examples of continuous t-norms are $T_P(a,b) = ab$, $T_M(a,b) = \min(a,b)$ and $T_L(a,b) = \max(a+b-1,0)$ (the Lukasiewicz t-norm).

Definition 2.2 [21] A *non-Archimedean random normed space* (briefly, NA-RN-space) is a triple (X,μ,T), where X is a vector space, T is a continuous t-norm, and μ is a mapping from X into D^+ such that the following conditions hold:

(RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
(RN2) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X$, $\alpha \ne 0$;
(RN3) $\mu_{x+y}(t) \ge T(\mu_x(t), \mu_y(t))$ for all $x, y \in X$ and all $t \ge 0$.

Every normed space $(X, \|\cdot\|)$ defines a non-Archimedean random normed space (X,μ,T_M), where

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $t > 0$, and T_M is the minimum t-norm. This space is called the induced random normed space.

Definition 2.3 [22] A *non-Archimedean random normed algebra* (X,μ,T,T') is a non-Archimedean random normed space (X,μ,T) with an algebraic structure such that

(RN-4) $\mu_{xy}(t) \ge T'(\mu_x(t), \mu_y(t))$ for all $x, y \in X$ and all $t > 0$, in which T' is a continuous t-norm.

Every non-Archimedean normed algebra $(X, \|\cdot\|)$ defines a non-Archimedean random normed algebra (X,μ,T_M), where

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $t > 0$ iff

$$\|xy\| \le \|x\|\|y\| + t\|y\| + t\|x\| \quad (x, y \in X; t > 0).$$

This space is called an induced non-Archimedean random normed algebra.

Definition 2.4

(1) Let (X, μ, T_M) and (Y, μ, T_M) be non-Archimedean random normed algebras. An $\mathbb{R}$-linear mapping $f : X \to Y$ is called a *homomorphism* if $f(xy) = f(x)f(y)$ for all $x, y \in X$.

(2) An $\mathbb{R}$-linear mapping $f : X \to X$ is called a *derivation* if $f(xy) = f(x)y + xf(y)$ for all $x, y \in X$.

Definition 2.5 Let $(\mathcal{U}, \mu, T, T')$ be a non-Archimedean random Banach algebra, then an involution on $\mathcal{U}$ is a mapping $u \to u^*$ from $\mathcal{U}$ into $\mathcal{U}$ which satisfies

(i) $u^{**} = u$ for $u \in \mathcal{U}$;

(ii) $(\alpha u + \beta v)^* = \overline{\alpha} u^* + \overline{\beta} v^*$;

(iii) $(uv)^* = v^* u^*$ for $u, v \in \mathcal{U}$.

If, in addition, $\mu_{u^*u}(t) = T'(\mu_u(t), \mu_u(t))$ for $u \in \mathcal{U}$ and $t > 0$, then $\mathcal{U}$ is a non-Archimedean random C^*-algebra.

Definition 2.6 Let (X, μ, T) be an NA-RN-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.

(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x_{n+1}}(\epsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.

(3) An RN-space (X, μ, T) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X.

3 Stability of homomorphisms and derivations in non-Archimedean random C^*-algebras

Throughout this section, assume that $\mathcal{A}$ is a non-Archimedean random C^*-algebra with the norm $\mu^{\mathcal{A}}_{\cdot}$ and that $\mathcal{B}$ is a non-Archimedean random C^*-algebra with the norm $\mu^{\mathcal{B}}_{\cdot}$.

For a given mapping $f : \mathcal{A} \to \mathcal{B}$, we define

$$D_\lambda f(x_1, \dots, x_m) := \sum_{i=1}^{m} \lambda f\left(mx_i + \sum_{j=1, j\neq i}^{m} x_j\right) + f\left(\lambda \sum_{i=1}^{m} x_i\right) - 2f\left(\lambda \sum_{i=1}^{m} mx_i\right)$$

for all $\lambda \in \mathbb{T}^1 := \{\nu \in \mathbb{C} : |\nu| = 1\}$ and all $x_1, \dots, x_m \in \mathcal{A}$.

Note that a $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ is called a *homomorphism* in non-Archimedean random C^*-algebras if H satisfies $H(xy) = H(x)H(y)$ and $H(x^*) = H(x)^*$ for all $x, y \in \mathcal{A}$.

We prove the generalized Hyers-Ulam stability of homomorphisms in non-Archimedean random C^*-algebras for the functional equation $D_\lambda f(x_1, \dots, x_m) = 0$.

Theorem 3.1 *Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping for which there are functions $\varphi : \mathcal{A}^m \to D^+$, $\psi : \mathcal{A}^2 \to D^+$, and $\eta : \mathcal{A} \to D^+$ such that $|m| < 1$ is far from zero and*

$$\mu^{\mathcal{B}}_{D_\lambda f(x_1, \dots, x_m)}(t) \geq \varphi_{x_1, \dots, x_m}(t), \tag{3.1}$$

$$\mu^{\mathcal{B}}_{f(xy) - f(x)f(y)}(t) \geq \psi_{x,y}(t), \tag{3.2}$$

$$\mu^{\mathcal{B}}_{f(x^*) - f(x)^*}(t) \geq \eta_x(t), \tag{3.3}$$

for all $\lambda \in \mathbb{T}^1$, *all* $x_1, \ldots, x_m, x, y \in A$ *and* $t > 0$. *If there exists an* $L < 1$ *such that*

$$\varphi_{mx_1,\ldots,mx_m}\big(|m|Lt\big) \geq \varphi_{x_1,\ldots,x_m}(t), \tag{3.4}$$

$$\psi_{mx,my}\big(|m|^2 Lt\big) \geq \psi_{x,y}(t), \tag{3.5}$$

$$\eta_{mx}\big(|m|Lt\big) \geq \eta_x(t), \tag{3.6}$$

for all $x, y, x_1, \ldots, x_m \in \mathcal{A}$ *and* $t > 0$, *then there exists a unique random homomorphism* $H : \mathcal{A} \to \mathcal{B}$ *such that*

$$\mu^{\mathcal{B}}_{f(x)-H(x)}(t) \geq \varphi_{x,0,\ldots,0}\big(\big(|m| - |m|L\big)t\big) \tag{3.7}$$

for all $x \in \mathcal{A}$ *and* $t > 0$.

Proof It follows from (3.4), (3.5), (3.6), and $L < 1$ that

$$\lim_{n\to\infty} \varphi_{m^n x_1,\ldots,m^n x_m}\big(|m|^n t\big) = 1, \tag{3.8}$$

$$\lim_{n\to\infty} \psi_{m^n x, m^n y}\big(|m|^{2n} t\big) = 1, \tag{3.9}$$

$$\lim_{n\to\infty} \eta_{m^n x}\big(|m|^n t\big) = 1, \tag{3.10}$$

for all $x, y, x_1, \ldots, x_m \in \mathcal{A}$ and $t > 0$.

Let us define Ω to be the set of all mappings $g : \mathcal{A} \longrightarrow \mathcal{B}$ and introduce a generalized metric on Ω as follows:

$$d(g,h) = \inf\big\{k \in (0,\infty) : \mu^{\mathcal{B}}_{g(x)-h(x)}(kt) > \phi_{x,0,\ldots,0}(t), \forall x \in \mathcal{A}, t > 0\big\}.$$

It is easy to show that (Ω, d) is a generalized complete metric space (see [23]).

Now, we consider the function $J : \Omega \longrightarrow \Omega$ defined by $Jg(x) = \frac{1}{m}g(mx)$ for all $x \in \mathcal{A}$ and $g \in \Omega$. Note that for all $g, h \in \Omega$, we have

$$\begin{aligned}
d(g,h) < k \quad &\Longrightarrow \quad \mu^{\mathcal{B}}_{g(x)-h(x)}(kt) > \phi_{x,0,\ldots,0}(t) \\
&\Longrightarrow \quad \mu^{\mathcal{B}}_{\frac{1}{m}g(mx)-\frac{1}{m}h(mx)}(kt) > |m|\phi_{mx,0,\ldots,0}\big(|m|t\big) \\
&\Longrightarrow \quad \mu^{\mathcal{B}}_{\frac{1}{m}g(mx)-\frac{1}{m}h(mx)}(kLt) > \phi_{mx,0,\ldots,0}(t) \\
&\Longrightarrow \quad d(Jg, Jh) < kL.
\end{aligned}$$

From this it is easy to see that $d(Jg, Jk) \leq Ld(g,h)$ for all $g, h \in \Omega$, that is, J is a self-function of Ω with the Lipschitz constant L.

Putting $\mu = 1$, $x = x_1$ and $x_2 = x_3 = \cdots = x_m = 0$ in (3.1), we have

$$\mu^{\mathcal{B}}_{f(mx)-mf(x)}(t) \geq \phi_{x,0,\ldots,0}(t)$$

for all $x \in \mathcal{A}$ and $t > 0$. Then

$$\mu^{\mathcal{B}}_{f(x)-\frac{1}{m}f(mx)}(t) \geq \phi_{x,0,\ldots,0}\big(|m|t\big)$$

for all $x \in \mathcal{A}$ and $t > 0$, that is, $d(Jf, f) \leq \frac{1}{|m|} < \infty$. Now, from the fixed point alternative, it follows that there exists a fixed point H of J in Ω such that

$$H(x) = \lim_{n\to\infty} \frac{1}{|m|^n} f\left(m^n x\right) \tag{3.11}$$

for all $x \in \mathcal{A}$ since $\lim_{n\to\infty} d(J^n f, H) = 0$.

On the other hand, it follows from (3.1), (3.8), and (3.11) that

$$\begin{aligned}\mu^{\mathcal{B}}_{D_\lambda H(x_1,\dots,x_m)}(t) &= \lim_{n\to\infty} \mu^{\mathcal{B}}_{\frac{1}{m^n} Df(m^n x_1,\dots,m^n x_m)}(t) \\ &\geq \lim_{n\to\infty} \phi_{m^n x_1,\dots,m^n x_m}\left(|m|^n t\right) = 1.\end{aligned}$$

By a similar method to the above, we get $\lambda H(mx) = H(m\lambda x)$ for all $\lambda \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Thus, one can show that the mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear.

It follows from (3.2), (3.9), and (3.11) that

$$\begin{aligned}\mu^{\mathcal{B}}_{H(xy)-H(x)H(y)}(t) &= \lim_{n\to\infty} \mu^{\mathcal{B}}_{f(m^{2n}xy)-f(m^n x)f(m^n y)}\left(|m|^{2n} t\right) \\ &\geq \lim_{n\to\infty} \psi_{m^n x, m^n y}\left(|m|^{2n} t\right) = 1\end{aligned}$$

for all $x, y \in \mathcal{A}$. So, $H(xy) = H(x)H(y)$ for all $x, y \in \mathcal{A}$. Thus, $H : \mathcal{A} \to \mathcal{B}$ is a homomorphism satisfying (3.7) as desired.

Also by (3.3), (3.10), (3.11) and by a similar method, we have $H(x^*) = H(x)^*$. □

Corollary 3.2 *Let $r > 1$ and θ be nonnegative real numbers, and let $f : \mathcal{A} \to \mathcal{B}$ be a mapping such that*

$$\begin{aligned}&\mu^{\mathcal{B}}_{D_\lambda f(x_1,\dots,x_m)}(t) \geq \frac{t}{t + \theta \cdot (\|x_1\|^r_{\mathcal{A}} + \|x_2\|^r_{\mathcal{A}} + \cdots + \|x_m\|^r_{\mathcal{A}})}, \\ &\mu^{\mathcal{B}}_{f(xy)-f(x)f(y)}(t) \geq \frac{t}{t + \theta \cdot (\|x\|^r_{\mathcal{A}} \cdot \|y\|^r_{\mathcal{A}})}, \\ &\mu^{\mathcal{B}}_{f(x^*)-f(x)^*}(t) \geq \frac{t}{t + \theta \cdot \|x\|^r_{\mathcal{A}}}\end{aligned}$$

for all $\lambda \in \mathbb{T}^1$, all $x_1, \dots, x_m, x, y \in \mathcal{A}$ and $t > 0$. Then there exists a unique homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$\mu^{\mathcal{B}}_{f(x)-H(x)}(t) \geq \frac{(|m| - |m|^r)t}{(|m| - |m|^r)t + \theta|m| - |m|^r \|x\|^r_{\mathcal{A}}}$$

for all $x \in \mathcal{A}$ and $t > 0$.

Proof The proof follows from Theorem 3.1. By taking

$$\begin{aligned}&\varphi_{x_1,\dots,x_m}(t) = \frac{t}{t + \theta \cdot (\|x_1\|^r_{\mathcal{A}} + \|x_2\|^r_{\mathcal{A}} + \cdots + \|x_m\|^r_{\mathcal{A}})}, \\ &\psi_{x,y}(t) := \frac{t}{t + \theta \cdot (\|x\|^r_{\mathcal{A}} \cdot \|y\|^r_{\mathcal{A}})},\end{aligned}$$

$$\eta_x(t) = \frac{t}{t + \theta \cdot \|x\|_{\mathcal{A}}^r}$$

for all $x_1, \ldots, x_m, x, y \in A$, $L = |m|^{r-1}$ and $t > 0$, we get the desired result. □

We prove the generalized Hyers-Ulam stability of derivations on non-Archimedean random C^*-algebras for the functional equation $D_\lambda f(x_1, \ldots, x_m) = 0$.

Theorem 3.3 *Let $f : \mathcal{A} \to \mathcal{A}$ be a mapping for which there are functions $\varphi : \mathcal{A}^m \to D^+$, $\psi : \mathcal{A}^2 \to D^+$, and $\eta : \mathcal{A} \to D^+$ such that $|m| < 1$ is far from zero and*

$$\mu^{\mathcal{A}}_{D_\lambda f(x_1,\ldots,x_m)}(t) \geq \varphi_{x_1,\ldots,x_m}(t),$$
$$\mu^{\mathcal{A}}_{f(xy)-f(x)y-xf(y)}(t) \geq \psi_{x,y}(t),$$
$$\mu^{\mathcal{A}}_{f(x^*)-f(x)^*}(t) \geq \eta_x(t),$$

for all $\lambda \in \mathbb{T}^1$ and all $x_1, \ldots, x_m, x, y \in A$ and $t > 0$. If there exists an $L < 1$ such that (3.4), (3.5), *and* (3.6) *hold, then there exists a unique derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that*

$$\mu^{\mathcal{A}}_{f(x)-\delta(x)}(t) \geq \varphi_{x,0,\ldots,0}\big(\big(|m| - |m|L\big)t\big)$$

for all $x \in \mathcal{A}$ and $t > 0$.

4 Stability of homomorphisms and derivations in non-Archimedean Lie C^*-algebras

A non-Archimedean random C^*-algebra $\mathcal{C}$, endowed with the Lie product

$$[x, y] := \frac{xy - yx}{2}$$

on $\mathcal{C}$, is called a *Lie non-Archimedean random C^*-algebra*.

Definition 4.1 Let $\mathcal{A}$ and $\mathcal{B}$ be random Lie C^*-algebras. A $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ is called a *non-Archimedean Lie C^*-algebra homomorphism* if $H([x, y]) = [H(x), H(y)]$ for all $x, y \in \mathcal{A}$.

Throughout this section, assume that $\mathcal{A}$ is a non-Archimedean random Lie C^*-algebra with the norm $\mu^{\mathcal{A}}$ and that $\mathcal{B}$ is a non-Archimedean random Lie C^*-algebra with the norm $\mu^{\mathcal{B}}$.

We prove the generalized Hyers-Ulam stability of homomorphisms in non-Archimedean random Lie C^*-algebras for the functional equation $D_\lambda f(x_1, \ldots, x_m) = 0$.

Theorem 4.2 *Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping for which there are functions $\varphi : \mathcal{A}^m \to D^+$ and $\psi : \mathcal{A}^2 \to D^+$ such that* (3.1) *and* (3.3) *hold and*

$$\mu^{\mathcal{B}}_{f([x,y])-[f(x),f(y)]}(t) \geq \psi_{x,y}(t) \tag{4.1}$$

for all $\lambda \in \mathbb{T}^1$, all $x, y \in \mathcal{A}$ and $t > 0$. If there exists an $L < 1$ and (3.4), (3.5), *and* (3.6) *hold, then there exists a unique homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that* (3.7) *holds.*

Proof By the same reasoning as in the proof of Theorem 3.1, we can find the mapping $H : \mathcal{A} \to \mathcal{B}$ given by

$$H(x) = \lim_{n\to\infty} \frac{f(m^n x)}{|m|^n} \tag{4.2}$$

for all $x \in \mathcal{A}$. It follows from (3.5) and (4.2) that

$$\begin{aligned}\mu^{\mathcal{B}}_{H([x,y])-[H(x),H(y)]}(t) &= \lim_{n\to\infty} \mu^{\mathcal{B}}_{f(m^{2n}[x,y])-[f(m^n x),f(m^n y)]}\big(|m|^{2n}t\big) \\ &\geq \lim_{n\to\infty} \psi_{m^n x,m^n y}\big(|m|^{2n}t\big) = 1\end{aligned}$$

for all $x, y \in \mathcal{A}$ and $t > 0$, then

$$H\big([x,y]\big) = \big[H(x), H(y)\big]$$

for all $x, y \in \mathcal{A}$. Thus, $H : \mathcal{A} \to \mathcal{B}$ is a Lie C^*-algebra homomorphism satisfying (3.7), as desired. □

Corollary 4.3 *Let $r > 1$ and θ be nonnegative real numbers, and let $f : \mathcal{A} \to \mathcal{B}$ be a mapping such that*

$$\begin{aligned}&\mu^{\mathcal{B}}_{D_\lambda f(x_1,\dots,x_m)}(t) \geq \frac{t}{t + \theta(\|x_1\|^r_A + \|x_2\|^r_A + \cdots + \|x_m\|^r_A)}, \\ &\mu^{\mathcal{B}}_{f([x,y])-[f(x),f(y)]}(t) \geq \frac{t}{t + \theta \cdot \|x\|^r_A \cdot \|y\|^r_A}, \\ &\mu^{\mathcal{B}}_{f(x^*)-f(x)^*}(t) \geq \frac{t}{t + \theta \cdot \|x\|^r_{\mathcal{A}}}\end{aligned}$$

for all $\lambda \in \mathbb{T}^1$, all $x_1, \dots, x_m, x, y \in \mathcal{A}$ and $t > 0$. Then there exists a unique homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$\mu^{\mathcal{B}}_{f(x)-H(x)}(t) \geq \frac{(|m| - |m|^r)t}{(|m| - |m|^r)t + \theta \|x\|^r_{\mathcal{A}}}$$

for all $x \in \mathcal{A}$ and $t > 0$.

Proof The proof follows from Theorem 4.2 and a method similar to Corollary 3.2. □

Definition 4.4 Let $\mathcal{A}$ be a non-Archimedean random Lie C^*-algebra. A $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a *Lie derivation* if $\delta([x,y]) = [\delta(x), y] + [x, \delta(y)]$ for all $x, y \in \mathcal{A}$.

We prove the generalized Hyers-Ulam stability of derivations on non-Archimedean random Lie C^*-algebras for the functional equation $D_\lambda f(x_1, \dots, x_m) = 0$.

Theorem 4.5 *Let $f : \mathcal{A} \to \mathcal{A}$ be a mapping for which there are functions $\varphi : A^m \to D^+$ and $\psi : A^2 \to D^+$ such that* (3.1) *and* (3.3) *hold and*

$$\mu^{\mathcal{A}}_{f([x,y])-[f(x),y]-[x,f(y)]}(t) \geq \psi_{x,y}(t), \tag{4.3}$$

for all $x, y \in \mathcal{A}$. If there exists an $L < 1$ and (3.4), (3.5), *and* (3.6) *hold, then there exists a unique Lie derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that* (3.7) *holds.*

Proof By the same reasoning as the proof of Theorem 4.2, there exists a unique $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ satisfying (3.7); the mapping $\delta : \mathcal{A} \to \mathcal{A}$ is given by

$$\delta(x) = \lim_{n\to\infty} \frac{f(m^n x)}{|m|^n} \tag{4.4}$$

for all $x \in \mathcal{A}$.

It follows from (3.5) and (4.4) that

$$\begin{aligned} &\mu^{\mathcal{A}}_{\delta([x,y])-[\delta(x),y]-[x,\delta(y)]}(t) \\ &\quad = \lim_{n\to\infty} \mu^{\mathcal{A}}_{f(m^{2n}[x,y])-[f(m^n x),\cdot m^n y]-[m^n x, f(m^n y)]}\bigl(|m|^{2n} t\bigr) \\ &\quad \ge \lim_{n\to\infty} \psi_{m^n x, m^n y}\bigl(|m|^{2n} t\bigr) = 1 \end{aligned}$$

for all $x, y \in \mathcal{A}$ and $t > 0$, then

$$\delta\bigl([x,y]\bigr) = \bigl[\delta(x), y\bigr] + \bigl[x, \delta(y)\bigr]$$

for all $x, y \in \mathcal{A}$. Thus, $\delta : \mathcal{A} \to \mathcal{A}$ is a Lie derivation satisfying (3.7). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors carried out the proof. All authors conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

Author details
[1]National Institute for Mathematical Sciences, KT Daeduk 2 Research Center, 463-1 Jeonmin-Dong, Yuseong-Gu, Daejeon, 305-811, Korea. [2]Department of Mathematics, Iran University Science and Technology, Tehran, Iran.

Received: 18 April 2012 Accepted: 15 October 2012 Published: 29 October 2012

References
1. Eshaghi Gordji, M, Alizadeh, Z: Stability and superstability of ring homomorphisms on non-Archimedean Banach algebras. Abstr. Appl. Anal. **2011**, Article ID 123656 (2011). doi:10.1155/2011/123656
2. Shilkret, N: Non-Archimedean Banach algebras. PhD thesis, Polytechnic University, ProQuest LLC (1968)
3. Ulam, SM: A Collection of the Mathematical Problems. Interscience, New York (1960)
4. O'Regan, D, Rassias, JM, Saadati, R: Approximations of ternary Jordan homomorphisms and derivations in multi-C^* ternary algebras. Acta Math. Hung. **134**(12), 99-114 (2012)
5. Rassias, JM: On approximation of approximately linear mappings by linear mappings. J. Funct. Anal. **46**, 126-130 (1982)
6. Rassias, TM: On the stability of the linear mapping in Banach spaces. Proc. Am. Math. Soc. **72**, 297-300 (1978)
7. Diaz, J, Margolis, B: A fixed point theorem of the alternative for contractions on a generalized complete metric space. Bull. Am. Math. Soc. **74**, 305-309 (1968)
8. Zamani Eskandani, G: On the Hyers-Ulam-Rassias stability of an additive functional equation in quasi-Banach spaces. J. Math. Anal. Appl. **345**(1), 405-409 (2008)
9. Chang, SS, Cho, YJ, Kang, SM: Nonlinear Operator Theory in Probabilistic Metric Spaces. Nova Science Publishers, New York (2001)
10. Cho, YJ, Kang, JI, Saadati, R: Fixed points and stability of additive functional equations on the Banach algebras. J. Comput. Anal. Appl. **14**, 1103-1111 (2012)
11. Cho, YJ, Park, C, Saadati, R: Functional inequalities in non-Archimedean in Banach spaces. Appl. Math. Lett. **60**, 1994-2002 (2010)
12. Cho, YJ, Saadati, R, Vahidi, J: Approximation of homomorphisms and derivations on non-Archimedean Lie C^*-algebras via fixed point method. Discrete Dyn. Nat. Soc. **2012**, Article ID 373904 (2012)

13. Azadi Kenary, H, Jang, SY, Park, C: A fixed point approach to the Hyers-Ulam stability of a functional equation in various normed spaces. Fixed Point Theory Appl. **2011**, 67 (2011)
14. Azadi Kenary, H, Rezaei, H, Talebzadeh, S, Park, C: Stability for the Jensen equation in C^*-algebras: a fixed point alternative approach. Adv. Differ. Equ. **2012**, 17 (2012)
15. Miheţ, D: The probabilistic stability for a functional equation in a single variable. Acta Math. Hung. **123**, 249-256 (2009)
16. Miheţ, D: The fixed point method for fuzzy stability of the Jensen functional equation. Fuzzy Sets Syst. **160**, 1663-1667 (2009)
17. Miheţ, D, Radu, V: On the stability of the additive Cauchy functional equation in random normed spaces. J. Math. Anal. Appl. **343**, 567-572 (2008)
18. Najati, A, Cho, YJ: Generalized Hyers-Ulam stability of the pexiderized Cauchy functional equation in non-Archimedean spaces. Fixed Point Theory Appl. **2011**, Article ID 309026 (2011)
19. Park, C, Cho, YJ, Kenary, HA: Orthogonal stability of a generalized quadratic functional equation in non-Archimedean spaces. J. Comput. Anal. Appl. **14**, 526-535 (2012)
20. Schweizer, B, Sklar, A: Probabilistic Metric Spaces. North-Holland, New York (1983)
21. Sherstnev, AN: On the notion of a random normed space. Dokl. Akad. Nauk SSSR **149**, 280-283 (1963) (in Russian)
22. Mirmostafaee, AK: Perturbation of generalized derivations in fuzzy Menger normed algebras. Fuzzy Sets Syst. **195**, 109-117 (2012)
23. Cădariu, L, Radu, V: Fixed points and the stability of Jensen's functional equation. J. Inequal. Pure Appl. Math. **4**(1), Art. 4 (2003)

doi:10.1186/1029-242X-2012-251
Cite this article as: Kang and Saadati: **Approximation of homomorphisms and derivations on non-Archimedean random Lie C^*-algebras via fixed point method.** *Journal of Inequalities and Applications* 2012 **2012**:251.